COMPLEX ORIENTABLE COHOMOLOGY THEORIES

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1. Complex orientable theory

Let h be a multiplicative cohomology theory, *i.e.* h is a generalized cohomology theory and has a cup product. In particular, $h^*(pt)$ is a ring.

Definition 1.1. *We say h is complex orientable if there is an isomorphism*

$$h^*(\mathbb{C}P^\infty) \simeq h^*(pt)[[t]].$$

This isomorphism is called a complex orientation of h. Or equivalently the map induced by inclusion $h^2(\mathbb{C}P^\infty) \to h^2(\mathbb{C}P^1)$ is surjective.

Note that h might have more than one complex orientations. Once the orientation is fixed, we say h is complex oriented.

Exercise 1.2. *Show that h is complex orientable if and only if for any complex vector bundle ξ over base X we have Thom isomorphism $h^{*+\mathrm{rk}\xi}(M\xi) \simeq h^*(X)$.*

Example 1.3. *Ordinary cohomology (with any coefficient) is complex orientable.*

2. Formal group law

Definition 2.1. *Let R be a commutative ring, a formal group law over R is a formal power series $f(u, v) \in R[[u, v]]$ such that*

(1) $f(u, 0) = u = f(0, u)$
(2) $f(u, f(v, w)) = f(f(u, v), w)$
(3) $f(u, v) = f(v, u)$

Let E be a complex orientable cohomology theory, then the isomorphism $E(\mathbb{C}P^\infty) \simeq E^*(pt)[[t]]$ permits us to define Chern class by pull-back of t combining splitting principle. Then $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ induced by $\mathcal{O} = \pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(1)$ gives a formal group law

$$c_1^E(\mathcal{O}) = f(u = c_1^E(\pi_1^*\mathcal{O}(1)), v = c_1^E(\pi_2^*\mathcal{O}(1))) \in E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq E^*[[u, v]]$$

(last isomorphism by Atiyah-Hirzebruch spectral sequence) over the coefficient ring.

For example, $c_1(L \otimes L') = c_1(L) + c_1(L')$ for ordinary Chern class, hence the formal group law is the additive group law $\mathbb{G}_a$ over $\mathbb{Z}$.

3. K-theory is complex orientable

It suffices to show K-theory admits Thom isomorphism for complex vector bundles.

Let $\xi \to X$ be a $U(n)$-bundle, let $M(\xi)$ be the Thom space of ξ, we are to construct a map $K(X) \to \tilde{K}(M\xi))$ which is an isomorphism, analogous to the Thom isomorphism $H^*(X) \to \tilde{H}^*(M(\xi))$. So similarly, we start by defining a Thom class $T(\xi) \in \tilde{K}(M(\xi))$.

3.1. **Thom class in K-theory.**

- Thom class $T(\xi)$ is a relative class in $K(D(\xi), \partial D(\xi))$.
- The exterior algebra of ξ , $\wedge(\xi) = \wedge^{ev}(\xi) \oplus \wedge^{od}(\xi)$.
- Pull back ξ to a bundle ξ' over $D(\xi)$, then $\wedge(\xi') = \wedge^{ev}(\xi') \oplus \wedge^{od}(\xi')$
- Moreover, there is a map $\phi : \wedge^{od}(\xi') \to \wedge^{ev}(\xi')$,
$$(x, v, Y) \in (X, D^{2n}, \wedge^{od}(\xi)) \mapsto (x, v, (v\wedge +(v\wedge)^*)Y) \in (X, D^{2n}, \wedge^{ev}\xi).$$
Notice that ϕ is an isomorphism away from zero section of $D(\xi)$.
- Thus, we define
$$T(\xi) := (\wedge^{ev}(\xi'), \wedge^{od}(\xi'), \phi) \in K(D(\xi), \partial D(\xi)).$$

3.2. **Thom class in KO-theory.** Let $\xi \to X$ be an $SU(n)$-bundle

- If $n \equiv 0 \bmod 4$, then $\wedge\xi = R(\xi) \oplus R_-(\xi)$ and $R(\xi) = R^{ev}(\xi) \oplus R^{od}(\xi)$. Pull back to $D(\xi)$ get
$$t(\xi) = (R^{ev}(\xi'), R^{od}(\xi'), \phi) \in KO(D(\xi), \partial D(\xi))$$
- If $n \equiv 2 \bmod 4$, then
$$s(\xi) = (\wedge^{ev}(\xi'), \wedge^{od}(\xi'), \phi) \in K\tilde{S}P(M(\xi))$$

3.3. **Thom isomorphism.** To prove the Thom isomorphism in K-theory, we use the following theorem of Dold.

Theorem 3.1. *Suppose h^* is a multiplicative cohomology theory. Let ξ be an $O(n)$-bundle over a finite CW complex X. Let $t \in h^n(D(\xi), \partial D(\xi))$ be such that inclusion $i : (D^n_x, \partial D^n_x) \to (D(\xi), \partial D(\xi))$, where D^n_x is the cell over $x \in X$, has $h^n(D^n_x, \partial D^n_x)$ a free $h^*(pt)$-module with generator $i^*(t)$. Then there is an isomorphism*
$$h^k(X) \simeq h^{k+n}(D(\xi), \partial D(\xi)), a \mapsto \pi^* a \cdot t$$

Sketch of Proof. The proof is basically the same as the proof of standard Thom isomorphism, which is an induction on cell and an application of five lemma. □

Applying this theorem, one only needs to check that $i^*T(\xi)$ is the generator of $K(D^{2n}_x, \partial D^{2n}_x) = \tilde{K}(S^{2n})$, and is the generator of the free $K^*(pt)$-module $K^*(S^{2n})$. By Bott periodicity, splitting principle and the fact that $M(\xi \oplus \eta) = M(\xi) \wedge M(\eta)$, one suffices to check $n = 1$. We then explicitly compute the Thom class of the universal line bundle.

Proposition 3.2. *The tautological $U(1)$-bundle ρ_{n-1} over $\mathbb{C}P^{n-1}$ has Thom space $\mathbb{C}P^n$, and $T(\rho_{n-1}) = 1 - \rho_n \in K(\mathbb{C}P^n)$.*

Proof. First of all, the projection
$$\mathbb{C}P^n - [0, 0, \dots, 1] \to \mathbb{C}P^{n-1}, [z_0, \dots, z_{n-1}, z_n] \mapsto [z_0, \dots, z_{n-1}]$$
is the tautological ρ_{n-1} (one can see this by explicitly writing down the bundle transition function), so $M(\rho_{n-1}) \simeq \mathbb{C}P^n$. From the view of Thom isomorphism, we have $H^*(M(\rho_{n-1})) = \wedge(u, ux)/(u^2 - ux) = \wedge(u)$ where u is the Thom class, this agrees with the cohomology of $\mathbb{C}P^n$.

This is a combination of several facts.

- For any $U(1)$-bundle ξ , $M(\xi)$ is canoniaclly isomorphic to $E(\xi) \circ U(1)/U(1)$, where $E(\xi) \circ U(1)$ is the join of $E(\xi)$ and $U(1)$.
- $S^{2n-1} = U(1) \circ \cdots \circ U(1)$, $\mathbb{C}P^{n-1} = S^{2n-1}/U(1) = U(1) \circ \cdots \circ U(1)/U(1)$.

- $M(\rho_{n-1}) = E(\rho_{n-1}) \circ U(1)/U(1)$.

Recall that

$$T(\rho_{n-1}) = (\wedge^{ev}(\rho'_{n-1}), \wedge^{od}(\rho'_{n-1}), \phi) \in K(D(\rho_{n-1}), \partial D(\rho_{n-1})),$$

and notice that since ρ_{n-1} is a line bundle, we have

$$\wedge^{ev}(\rho_{n-1}) = \wedge^0 = \text{trivial bundle},$$

and

$$\wedge^{od}(\rho_{n-1}) = \wedge^1 = \rho_{n-1}.$$

Thus $T(\rho_{n-1}) = (\varepsilon_{\mathbb{C}}, [\pi^*\rho]_{n-1}) = 1 - \pi^*\rho_{n-1}$. We claim that $\pi^*\rho_{n-1}$ on $M(\rho_{n-1}) = \mathbb{C}P^n$ is the tautological bundle ρ_n, indeed one easily sees this by looking at bundle transition function. □

Remark 3.3. *For $n = 1$, $1 - \rho_1 \in \tilde{K}(\mathbb{C}P^1) = \tilde{K}(S^2)$ is exactly the generator.*

Corollary 3.4 (Thom isomorphism in K-theory)**.** *For any $U(n)$-bundle $\pi : \xi \to X$, we have an isomorphism $K(X) \simeq K(D(\xi), \partial D(\xi)) = \tilde{K}(M(\xi))$, $\eta \mapsto \pi^*\eta \otimes T(\xi)$.*

Similarly we have Thom isomorphisms for $SU(4k)$ and $SU(4k+2)$ bundles

- $KO(X) \simeq \tilde{KO}(M(\xi))$ for $SU(4k)$-bundle
- $KO(X) \simeq \tilde{KSp}(M(\xi))$ for $SU(4k+2)$-bundle

4. Complex cobordism theory is complex orientable

This is purely tautologous.

4.1. Computation of universal formal group law on $\Omega^U_*(pt)$.

Suppose

$$F^{\Omega}(u, v) = \sum a_{rs} u^r v^s,$$

then since $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ is the limit of Segre map $\mathbb{C}P^n \times \mathbb{C}P^m \to \mathbb{C}P^{nm+n+m}$, the pull-back of a hyperplane in $\mathbb{C}P^{nm+n+m}$ is Milnor manifold H_{nm}, so we have

$$[H_{nm}] = \sum_{r=0}^{n} \sum_{s=0}^{m} a_{rs} [\mathbb{C}P^{n-r}][\mathbb{C}P^{m-s}].$$

Therefore one has

$$H(u, v) = \sum H_{nm} u^n v^m = F^{\Omega}(u, v) CP(u) CP(v).$$

QUILLEN'S THEOREM ON COMPLEX COBORDISM RING

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1. Formal group law, Lazard ring, logarithm

Let R be a commutative ring with unit, recall that a formal group law over R is a formal power series $f(x, y) = \sum c_{i,j} x^i y^j \in R[[x, y]]$ satisfying

(1) $f(x, y) = f(y, x)$
(2) $f(x, 0) = x, f(0, y) = y$
(3) $f(f(x, y), z) = f(x, f(y, z))$

So assigning a formal group law over R amounts to choosing coefficients c_{ij}'s satisfying certain relations, $c_{ij} = c_{ji}$ for instance.

We let $L = \mathbb{Z}[c_{ij}]/\sim$, called Lazard ring, be the free algebra generated by variables c_{ij} modulo relations so that $F(x, y) = \sum c_{i,j} x^i y^j$ is a formal group law over L. It is clear that (L, F) is universal among the pairs (ring, formal group law) in the sense that for any such a pair (R, f), there exists a unique ring homomorphism $L \to R$ that takes F to f.

Among all the formal group laws, the simplest one is the additive formal group law $\mathbb{G}_a(x, y) = x+y$, and those ones that are isomorphic to $\mathbb{G}_a$. Here f is isomorphic to f' if there exists $g(x) = x + b_1x^2 + b_2x^3 + \cdots \in R[[x]]$ such that $g^{-1}(f'(x, y)) = f(g^{-1}(x), g^{-1}(y))$. We note that since the leading term of g is x, g has formal inverse "function" g^{-1}.

Therefore assigning a formal group law over R that is isomorphic to $\mathbb{G}_a$ amounts to choose those b_i's. Similarly the ring $\mathbb{Z}[b_1, b_2, \dots]$ together with formal group law $g(f(g^{-1}(x), g^{-1}(y)))$ is universal among such kind of pairs.

Since Lazard ring is universal among all formal group laws, there is a natural morphism $\phi : L \to \mathbb{Z}[b_1, b_2, \dots]$. Now ϕ is not an isomorphism, that is there are formal group laws that are not isomorphic to $\mathbb{G}_a$. However it is true that over the rationals every formal group law is isomorphic to $\mathbb{G}_a$. Follow the steps below to show that $\phi \otimes \mathbb{Q}$ is an isomorphism.

(1) consider the differential form $\omega(x) = \frac{\mathrm{d}x}{F_2(x,0)}$ where $F_2(x, 0) = \frac{\partial}{\partial y}|_{y=0}F(x, y)$, show that ω is invariant under translation of F, namely
$$F^*\omega = \omega(x) + \omega(y)$$
(2) define *logarithm* of the formal group law $l(x)$ to be the unique solution to equation
$$l'(x)\mathrm{d}x = \omega(x), l(0) = 0$$
(3) check that the logarithm for $\mathbb{G}_a(x, y) = x+y$ and $\mathbb{G}_m = x+y+xy$ are x and $\log(1 + x)$ respectively. We remark that $\log(1 + x)$ has rational coefficients.
(4) use (1)(2) to show that $l(F(x, y)) = l(x) + l(y)$, thus every formal group law is isomorphic to additive formal group law after $\otimes\mathbb{Q}$.

Recall that over the complex cobordism ring MU^*, we have the formal group law

$$F^U(x,y) = \frac{H(x,y)}{CP(x)CP(y)}$$

where $CP(x) = 1 + [\mathbb{C}P^1]x + [\mathbb{C}P^2]x^2 + \dots$ and $H(x,y) = \sum_{n,m\geq 0}[H_{n,m}]x^n y^m$.

It is then an easy exercise (hint: use $[H_{1n}] = [\mathbb{C}P^1 \times \mathbb{C}P^{n-1}]$) to show its logarithm satisfies

$$l'(x) = CP(x).$$

2. MU^* is isomorphic to L

We consider the morphism $h : L \to MU^*$ classifying F^U. To show this is an isomorphism we need the following facts.

(1) (Thom) $MU^* \otimes \mathbb{Q} \simeq \mathbb{Q}[\mathbb{C}P^1, \mathbb{C}P^2, \mathbb{C}P^3, \dots]$.
(2) (Milnor) MU^* is torsion free and integrally generated by Milnor hypersurfaces H_{nm}.
(3) (Lazard) Lazard ring is isomorphic to a polynomial ring over $\mathbb{Z}$ with infinitely many generators, in particular L is torsion free.
(4) (Exercise) If we write the logarithm of universal formal group law in the form $\sum_{n\geq 0} p_n x^{n+1}/(n+1)$ for some $p_n \in L\otimes\mathbb{Q}$, then $p_n \in L$.(Hint: p_n is the coefficient of invariant differential form.) Moreover, $L \otimes \mathbb{Q} \simeq \mathbb{Q}[p_1, p_2, \dots]$. Note $p_n \neq b_n$.

Then Quillen argues (effortlessly) as following

(1) It is clear that h takes p_n to $[\mathbb{C}P^n]$ thus $h \otimes Q$ is an isomorphism.
(2) Since L is torsion free, h is injective. Meanwhile (exercise) Milnor manifolds are in the image of h, so h is surjective as well.

3. MU is universal among complex oriented cohomology theories

Now that MU^* is the Lazard ring, it follows that for any complex oriented cohomology theory E, we have a map $MU^* \to E^* = \pi_* E$. It turns out we can enhance this map to a ring spectrum map $MU \to E$. One is suggested to think about how this is achieved, and discuss with the author during tea.

4. Proof of facts in section 2

Thom's result follows from the observation that the Hurewicz homomorphism $\pi_* MU \to H_* MU$ is a rational isomorphism, and one can (with a little but not much effort) use Thom isomorphism to calculate $H_* MU$.

Milnor's result follows from the Adams spectral sequence computation on MU, I wish someday we can learn this together.

Lazard's theorem on the structure of Lazard ring is purely algebraic. For a complete proof, check Lurie's lecture notes on chromatic homotopy theory, lecture 2 and 3.

HERE ENTERS ELLIPTIC OBJECTS

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1. K-theory, Todd genus and Conner-Floyd theorem

Recall that MU is the universal complex orientable theory, with universal formal group law. KU is complex orientable, with multiplicative formal group law $\mathbb{G}_m$.

Thanks to Quillen, there is a ring homomorphism $\mu_C : MU^* \to K^0 \simeq \mathbb{Z}$ classifying $\mathbb{G}_m$. To understand this map, we work over rationals, *i.e.* consider $\mu_c \otimes \mathbb{Q} : \mathbb{Q}[\mathbb{C}P^1, \mathbb{C}P^2, \dots] \to \mathbb{Q}$. We see last time that logarithm of $\mathbb{G}_m$ is

$$l(x) = \log(1+x) = \sum (-1)^n \frac{1}{n+1} x^{n+1}.$$

This implies $\mu_C([\mathbb{C}P^n]) = (-1)^n = (-1)^n Td([\mathbb{C}P^n])$. So $\mu_C = \pm Td$.

Now we consider the functor $MU^*(-) \otimes_{L^*} K^*$ where K^* is treated as L^*-module via μ_C. There is a natural transformation $MU^*(-) \otimes_{L^*} K^* \to K^*(-)$.

Theorem 1.1 (Conner-Floyd)**.** $MU^*(-) \otimes_{L^*} K^* \simeq K^*(-)$.

Instead of following Conner and Floyd's original proof, we appeal to Landweber exactness theorem.

2. Heights of formal group law, Landweber exactness theorem

Let R be a ring and $f(x,y) \in R[[x,y]]$ be a formal group law.

Definition 2.1. *For every $n \in \mathbb{N}$, we define the n-series $[n](t) \in R[[t]]$ inductively to be $[n](t) = f([n-1](t), t)$.*

Since $f(x,y) = x + y + \dots$, it immediately follows that $[n](t) = nt + O(t^2)$, so of $p = 0$ in R, then $[p](t) = ct^k + O(t^{k+1})$ for some $k > 1$. It is an algebraic exercise to show

Proposition 2.2. *If $p = 0$ in R, then either $[p](t) = 0$ or $[p](t) = \lambda t^{p^n} + O(t^{p^n+1})$ for some $n > 0$.*

Definition 2.3. *Let f be a formal group law over R and fix a prime p. Let v_n be the coefficient of t^{p^n} in p-series $[p](t)$. We say f has height $\geq n$ if $v_i = 0$ for $i < n$. We say f has height exactly n if f has height $\geq n$ and v_n is invertible in R.*

Remark 2.4. *We always have $v_0 = p$, so f has height ≥ 1 if and only if $p = 0$ in R.*

Exercise 2.5. *Over $\mathbb{Z}/p\mathbb{Z}$, $\mathbb{G}_m$ has height exactly 1, $\mathbb{G}_a$ has height ∞.*

Height is an invariant of formal group laws. In fact, a quite strong one.

Theorem 2.6 (Lazard)**.** *Let k be an algebraically closed field of characteristic p, then two formal group laws are isomorphic if and only if they have the same heights.*

And almost all formal group laws are not isomorphic to the additive formal group law.

Proposition 2.7. *Let f be a formal group law of infinite height over R (necessarily $p = 0$), then f is isomorphic to additive formal group law.*

It is clear there are universal elements $v_0 = p, v_1, v_2, \cdots \in L$.

Theorem 2.8 (Landweber exactness)**.** *Let M be a (graded) module over the Lazard ring L, then $MU^*(-) \otimes_L M$ is exact if and only if for each prime p, the sequence $v_0, v_1, v_2, \ldots$ is a regular sequence for M.*

Exercise 2.9. *Let $R = \mathbb{Z}[\beta, \beta^{-1}]$ with formal group law $x+y+\beta xy$. Then $\mathbb{Z}[\beta, \beta^{-1}]$ is Landweber exact and gives complex K-theory. This proves Conner-Floyd theorem.*

3. Elliptic cohomology

We follow Landweber, Ravenel and Stong's original construction.

3.1. **Formal group law of Jacobi quartic.** Let δ, ε be indeterminates of weight 2 and 4, respectively, and introduce the graded polynomial ring

$$M_* = \mathbb{Z}[\frac{1}{2}][\delta, \varepsilon].$$

The weight refers to the weight of modular forms, as $M_*(\Gamma_0(2)) = \mathbb{C}[\delta, \varepsilon]$. An algebraic topologist would assign degree $2k$ to an element of weight k. Introduce the differential

$$\omega = (1 - 2\delta x^2 + \varepsilon x^4)^{-\frac{1}{2}} dx = R(x)^{-\frac{1}{2}} dx$$

on the Jacobi quartic

$$y^2 = 1 - 2\delta x^2 + \varepsilon^4,$$

and the corresponding logarithm

$$g(x) = \int_0^x (1 - 2\delta t^2 + \varepsilon t^4)^{-\frac{1}{2}} dt \in M_* \otimes \mathbb{Q}[[x]].$$

A formal group law over M_* is defined by $F_E(x, y) = g^{-1}(g(x) + g(y))$, and one has Euler's explicit formula

$$F_E(x, y) = \frac{x\sqrt{R(y)} + y\sqrt{R(x)}}{1 - \varepsilon x^2 y^2}.$$

The appropriate determinant, of weight 12, is

$$\Delta = \varepsilon(\delta^2 - \varepsilon)^2.$$

It is well-known that formal group laws of elliptic curves are of height 1 or 2.

3.2. **Elliptic genera and elliptic cohomology.** A multiplicative genus, in Hirzebruch's sense is a ring homomorphism

$$\varphi : \Omega_*^{SO} \to R$$

to a commutative algebra over $\mathbb{Q}$, with $\varphi(1) = 1$. The best examples are the signature and $\hat{A}$-genus. Evidently, a multiplicative genus is uniquely determined by its logarithm

$$g(x) = \sum_{n \geq 0} \frac{\varphi(\mathbb{C}P^{2n})}{2n+1} x^{2n+1},$$

which can be any odd series in $R[[x]]$ with linear term x.

By an elliptic genus, in Ochanine's sense, is meant a multiplicative genus φ for which there are elements $\delta, \varepsilon \in R$ so that the logarithm of φ has the form

$$g(x) = \int_0^x (1 - 2\delta t^2 + \varepsilon t^4)^{-\frac{1}{2}} \, dt.$$

Exercise 3.1. *$\delta = \varphi(\mathbb{C}P^2)$ and $\varepsilon = \varphi(\mathbb{H}P^2)$.*

Exercise 3.2. *One obtains the signature by taking $\delta = \varepsilon = 1$, and $\hat{A}$-genus by taking $\delta = -\frac{1}{8}$ and $\varepsilon = 0$. Note these are the only two cases, up to scaling, where $\Delta = 0$.*

It is evident φ maps Ω_*^{SO} into $\mathbb{Q}[\delta, \varepsilon]$, but by Euler's formula φ even takes value in $M_* = \mathbb{Z}[\frac{1}{2}][\delta, \varepsilon]$.

Theorem 3.3 (Landweber, Ravenel, Stong)**.** *The functor*

$$\Omega_*^{SO}(-) \otimes_{\Omega_*^{SO}} M_*[\Delta^{-1}]$$

is Landweber exact.

QUANTUM INVARIANTS OF MANIFOLDS

JIAHAO HU

1. Equivariant index formula for circle action

1.1. **Atiyah-Singer index theorem.** Arguably this is the best theorem in the past century.

Theorem 1.1 (Atiyah-Singer)**.** *Let X be a closed oriented smooth manifold of dimension $2n$ and $D = (D_i : \Gamma E_i \to \Gamma E_{i+1})$ an elliptic complex $(i = 0, \dots, m-1)$, associated to the tangent bundle. Then the index of this complex is determined by the following formula*

$$\mathrm{ind}(D) = (-1)^n \left(\Big(\frac{1}{e(T^*X)} \sum_{i=0}^{m} (-1)^i ch(E_i) \Big) \cdot td(TX \otimes \mathbb{C}) \right) [X].$$

Recall that $\mathrm{ind}(D)$ is defined to be the Euler characteristic of the elliptic complex D. Since each D_i is elliptic, $\mathrm{coker}(D_i)$ is finite dimensional, so Euler characteristic is well-defined. The proof of Atiyah-Singer index theorem is trivial analysis plus trivial topology. The topological essence of index theorem is the formula

$$ch(U_K) = \pm U_H / Td$$

which we established last time.

For convenience, if we write $x_1, \dots x_n$ to be the formal Chern roots of TX, then

$$\mathrm{ind}(D) = \left(\Big(\sum_{i=0}^{m} (-1)^i ch(E_i) \Big) \Big(\prod_{j=1}^{n} \frac{x_j}{1 - e^{-x_j}} \cdot \frac{1}{1 - e^{x_j}} \Big) \right) [X].$$

One has to use this formula with caution, since the formula is neither symmetric in x_j^2 nor an invertible power series.

Now let's discuss several important examples.

1.2. **The de Rham complex.** Let X be compact smooth of dimension k and T its tangent bundle. Let $E_i = \Lambda^i(T^* \otimes \mathbb{C})$. The exterior derivative d yields an elliptic complex. From definition, $\mathrm{ind}(d) = \sum_i \dim_{\mathbb{C}} H^i_{dR} = e(X)$. On the other hand, we have

$$ch\Big(\sum_{i=0}^{n} (\Lambda^i (T^* \otimes \mathbb{C})) \cdot y^i \Big) = \prod_{j=1}^{n} \Big((1 + y e^{x_j})(1 + y e^{-x_j}) \Big).$$

1.3. **The Dolbeault complex.** Let X be a complex n-dimensional manifold, T the holomorphic tangent bundle, *i.e.* $T = T^{0,1}$. With notation $A^{p,q} = \Gamma(\Lambda^p T^* \otimes \Lambda^q \overline{T}^*)$. The index of $\bar\partial$ for fixed p is denoted by χ^p, namely

$$\chi^p = \sum_{q=0}^{n} (-1)^q \cdot h^{p,q}.$$

It then follows from index formula that

$$\chi_y := \sum_{p=0}^{n} \chi^p \cdot y^p = \prod_{j=1}^{n} \Big((1 + ye^{-x_j}) \frac{x_j}{1 - e^{-x_j}} \Big) [X].$$

If we replace x by $x(1+y)$ then an easy exercise shows χ_y is the genus belonging to the power series

$$Q(x) = \frac{x(1 + ye^{-x(1+y)})}{1 - e^{-x(1+y)}}.$$

In particular, χ_0 is Todd genus or sometimes called arithmetic genus, χ_{-1} is the Euler characteristic, and χ_1 is the signature.

1.4. **The signature as an index.** Let X be compact oriented $4k$-dimensional smooth manifold, let T be its tangent bundle provided with a Riemannian metric. Recall that Hodge star operator has the property that $\star^2 = (-1)^{i(n-i)} \cdot \mathrm{id}$ on $\Gamma^i(T^* \otimes \mathbb{C})$. For this, we define $\tau = (-1)^{i(i-1)/2+k} \star$, then $\tau^2 = \mathrm{id}$ holds. Hence τ decompose $\Lambda(T^* \otimes \mathbb{C})$ into direct sum of sub-bundles E_+ with eigenvalue $+1$ and E_- with eigenvalue -1.

Note that $\tau(d + d^*) = -(d + d^*)\tau$, one can consider the elliptic operator $d + d^* : \Gamma E_+ \to \Gamma E_-$. One can show

$$ch(E_+) - ch(E_-) = \prod_{j=1}^{2k} (e^{x_j} - e^{-x_j}).$$

It then follows that $\mathrm{ind}(d + d^*) = \mathrm{sign}(X)$.

1.5. **The equivariant index.** Let X be a compact complex manifold of complex dimension n, and as before $D = (D_i : \Gamma E_i \to \Gamma E_{i+1})$ be an elliptic complex. Let G be a compact topological group acting on X by holomorphic maps. In addition, assume G acts on the elliptic complex, *i.e.* G also acts on the bundle E_i (e.g. if all bundles E_i are associated to tangent bundle of X) and this action commutes with the differential operators D_i. Then G also acts on the cohomology groups H^i of the complex.

Definition 1.2. *The equivariant index* $\mathrm{ind}(g, D)$ *of* D *is defined to be*

$$\mathrm{ind}(g, D) := \sum_{i=0}^{m} (-1)^i \cdot \mathrm{tr}(g, H^i).$$

Example 1.3. $\mathrm{ind}(D) = \mathrm{ind}(id, D)$.

Now we want to compute equivariant index using topological information of X and the group action. Let $X^g = x \in X : g \cdot x = x$ denote the fixed point set of g. This is a complex submanifold of X which is not necessarily connected. We furthermore decompose $X = \cup X^g_\nu$ into connected components. The X^g_ν are connected submanifolds of X of possibly different dimensions.

Let's fix such a component $Y = X^g_\nu$. For a point $p \in Y$, g acts linearly on the tangent space T_pX. There exists a Hermitian metric on $TX|_Y$ so that g acts unitarily on $TX|_Y$. Therefore T_pX decomposes into the direct sum of eigenspaces $N_{p,\lambda}$ for eigenvalues λ of modulus one. Since G acts continuously on X, one further obtains an eigen bundle N_λ over Y. Indeed, N_1 is precisely the tangent bundle TY of Y. Under the variation of points p in Y, the eigenvalues cannot change since depend only on the isomorphism type of representation on T_pX of the subgroup generated by g, and representation ring of this subgroup is discrete. With $d_\lambda = \mathrm{rk} N_\lambda$ we therefore have:

$$TX|_Y = \oplus_\lambda N_\lambda, \quad c(N_\lambda) = \prod_{i=1}^{d_\lambda}(1 + x_i^{(\lambda)}).$$

Now we are in position to write down an index formula for $\mathrm{ind}(g, D)$, as a sum of contributions $a(X_\nu^g)$ corresponding to each components of X_ν^g. We recall the original index formula

$$\mathrm{ind}(D) = \left(\Big(\sum_{i=0}^{m}(-1)^i ch(E_i)\Big)e(X)\Big(\prod_{j=1}^{n}\frac{1}{1-e^{-x_j}}\cdot\frac{1}{1-e^{x_j}}\Big)\right)[X].$$

To obtain the contribution from $Y = X_\nu^g$, we replace, in the above formula, X by Y, e^{x_i} by $\lambda^{-1}\cdot e^{x_i}$ and e^{-x_i} by $\lambda \cdot e^{-x_i}$ (where x_i) belongs to the eigenvalue λ. We apply the same process to the terms $ch(E_i)$. This can obviously be done if the E_i are associated to the tangent bundle of X. For $\lambda = 1$, the term $1 - \lambda^{-1}\cdot e^{x_i}$ is not invertibe; but those x_i, which belong to the eigenvalue 1, originate from the tangent bundle of Y and so cancel with the factor $e(Y)$.

1.6. **The equivariant χ_y-genus for S^1-actions.** Applying the recipe to the χ_y-genus, we obtain from

$$\chi_y(X) = \sum_{p=0}^{n}\chi^p(X)\cdot y^p = \prod_{i=1}^{n}\Big((1+ye^{-x_i})\frac{x_i}{1-e^{-x_i}}\Big)[X]$$

the equivariant formula

$$\chi_y(g, X) = \sum_{p=0}^{n}\chi^p(g, X)\cdot y^p = \sum_\nu a(X_\nu^g),$$

where we have for each fixed point component $Y = X_\nu^g$:

$$a(Y) = \Big(\prod_{\lambda\neq 1}\prod_{i=1}^{d_\lambda}\big(\frac{1+y\cdot\lambda e^{-x_i^{(\lambda)}}}{1-\lambda e^{-x_i^{(\lambda)}}}\big)\cdot\prod_{i=1}^{d_1}\big(1+ye^{-x_i^{(1)}}\big)\frac{x_i^{(1)}}{1-e^{-x_i^{(1)}}}\Big)[Y].$$

Now let $G = S^1$ and let $q \in S^1$ be a topological generator. Then we have $X^q = X^{S^1}$. In this situation, the whole group S^1 acts on the restriction of the tangent bundle of X to X^q. Recall if S^1 acts on a vector space, then one writes

$$V = \sum_{k\in\mathbb{Z}} q^k V_k$$

which means that $V = \oplus_k V_k$ and q acts on V_k as multiplication by q^k. This way, we have a splitting

$$TX|_Y = \sum_{k=-\infty}^{\infty} q^k N_k.$$

The eigenvalues λ of q are now all integral powers q^k of q, the equivariant χ_y-genus is then ($d_k = \mathrm{rk} N_k$)

$$\chi_y(q, X) = \sum_\nu a(X_\nu^{S^1})$$

with

$$a(X_\nu^{S^1}) = \Big(\prod_{i=1}^{d_0} x_i^{(1)} \frac{1+ye^{-x_i^{(1)}}}{1-e^{-x_i^{(1)}}} \prod_{k\neq 0}\prod_{j=1}^{d_k}\big(\frac{1+y\cdot q^k e^{-x_j^{(q^k)}}}{1-q^k e^{-x_j^{(q^k)}}}\big)\Big)[X_\nu^{S^1}].$$

Note that $\chi_y(q,X)$ is a finite Laurent series in q, and it is easy to see

$$a(X_\nu^{S^1})_{q=0} = \chi_y(X^{S^1_\nu})\cdot(-y)^{\sum_{k<0} d_k}, \quad a(X_\nu^{S^1})_{q=\infty} = \chi_y(X_\nu^{S^1})\cdot(-y)^{\sum_{k>0} d_k}.$$

Hence $\chi_y(q,X)$ is constant in q, and $\chi_y(q,X)\equiv\chi_y(id,X)=\chi_y(X)$.

1.7. **The equivariant signature for S^1-actions.** Now let X be a $2n$-dimensional compact, oriented, differentiable manifold with S^1-action. Note that TX is a real vector bundle, we have splitting

$$TX|_{X_\nu^{S^1}} = \sum_{k\geq 0} q^k N_k = TX_\nu^{S^1}\oplus\sum_{k>0} q^k N_k.$$

The bundle N_k are real bundles, which for $k>0$ obtain a complex structure through the identification with a complex vector bundle. This identification yields a unique orientation on N_k, in view of the condition $k>0$. We orient $N_0 = TX_\nu^{S^1}$ so that all orientations taken together yield the orientation of X.

Having made this splitting, the equivariant index theorem yields $\mathrm{sign}(q,X) = \sum_\nu a(X_\nu^{S^1})$ with

$$a(X_\nu^{S^1}) = \Big(\prod_{i=1}^{d_0} x_i^{(1)} \frac{1+e^{-x_i^{(1)}}}{1-e^{-x_i^{(1)}}} \prod_{k>0}\prod_{j=1}^{d_k}\big(\frac{1+q^k e^{-x_j^{(q^k)}}}{1-q^k e^{-x_j^{(q^k)}}}\big)\Big)[X_\nu^{S^1}].$$

Since S^1 is connected, it acts trivially on cohomology, therefore the equivariant signature does not depend on q. We can put $q=0$ and have

$$\mathrm{sign}(X) = \sum_\nu \mathrm{sign}(X_\nu^{S^1}).$$

2. Quantization of classical genera

Let X be a compact, oriented manifold of dimension $4k$. The the free loop space of X is the infinite dimensional manifold

$$\mathcal{L}X = \{g : S^1\to X : g \text{ differentiable}\}.$$

There is a canonical action of S^1 on $\mathcal{L}X$, with $(\mathcal{L}X)^{S^1} = X$ as constant loops. It is not hard to show that at a constant loop $g\equiv p\in X$, $T_p(\mathcal{L}X)\simeq\mathcal{L}(T_pX)$ where the S^1 acts canonically on $\mathcal{L}(T_pX)$. We shall denote $TX\otimes\mathbb{C}$ by $T_\mathbb{C}$, we have a splitting

$$T(\mathcal{L}X)|_X = TX\oplus\sum_{n>0} q^n T_\mathbb{C}.$$

2.1. **Quantization of signature-the construction.** With the standard notation

$$p(X) = 1+p_1+\cdots+p_k = (1+x_1^2)\cdots(1+x_{2k}^2)$$
$$c(T_\mathbb{C}) = (1+x_1)\cdots(1+x_{2k})(1-x_1)\cdots(1-x_{2k})$$

The recipe of the S^1-equivariant signature formula on free loop space yields

Definition 2.1. *The quantum signature of X is*

$$\mathrm{sign}(q,\mathcal{L}X) := \prod_{i=1}^{2k}\Big(x_i\frac{1+e^{-x_i}}{1-e^{-x_i}}\cdot\prod_{n=1}^{\infty}\Big(\frac{(1+q^ne^{-x_i})(1+q^ne^{x_i})}{(1-q^ne^{-x_i})(1-q^ne^{x_i})}\Big)\Big)[X].$$

This power series is symmetric in x_i^2, and after evaluation on X is a power series in q with rational coefficients. Recall that

Definition 2.2 (Twisted signature)**.** *Let X be an oriented manifold of dimension $2k$ and W a complex vector bundle over X. Then the signature of X with values in the vector bundle W is defined as*

$$\mathrm{sign}(X,W) := \Big(\prod_{i=1}^{k}x_i\frac{1+e^{-x_i}}{1-e^{-x_i}}\cdot ch(W)\Big)[X].$$

Definition 2.3 (Twisted $\hat{A}$-genus)**.** *Let X be an oriented manifold of dimension $2k$ and W a complex vector bundle over X. Then the $\hat{A}$-genus of X with values in the vector bundle W is defined as*

$$\hat{A}(X,W) := \Big(\prod_{i=1}^{k}\frac{x_i/2}{\sinh(x_i/2)}\cdot ch(W)\Big)[X].$$

Then with an practiced eye, one observes

Theorem 2.4. *We have*

$$\mathrm{sign}(q,\mathcal{L}X) = \mathrm{sign}(X,\bigotimes_{n=1}^{\infty}S_{q^n}T_{\mathbb{C}}\otimes\bigotimes_{n=1}^{\infty}\Lambda_{q^n}T_{\mathbb{C}}).$$

Therefore $\mathrm{sign}(q,\mathcal{L}X)$ *is a power series in q with integral coefficients and constant term (the coefficient of q^0)* $\mathrm{sign}(X)$.

It is well-known that a product $\prod(1+u_i)$ converges absolutely provided the series $\sum|u_i|$ converges, hence in our case only for $|q|<1$. We have therefore defined quantum signature formally, but it is meaningful as a power series.

2.2. **Quantization of signature-modularity.** Notice that quantum signature is not normalized, in the sense that it has constant term

$$\Big(2\cdot\prod_{n=1}^{\infty}\frac{(1+q^n)^2}{(1-q^n)^2}\Big)^{2k}.$$

Lemma 2.5. *The infinite product*

$$\Big(2\cdot\prod_{n=1}^{\infty}\frac{(1+q^n)^2}{(1-q^n)^2}\Big)^{-4}$$

is a modular form of weight 4 on $\Gamma_0(2)$, and is indeed precisely our modular form ε.

Thus we can rewrite the quantum signature as

$$\mathrm{sign}(q,\mathcal{L}X) = \varphi(X)\cdot\varepsilon^{-k/2}$$

where $\varphi(X)$ is the genus associated to

$$\frac{x}{f(x)} = \frac{1}{2}\frac{x}{\tanh(x/2)}\cdot\prod_{n=1}^{\infty}\Big(\frac{(1+q^ne^{-x_i})(1+q^ne^{x_i})}{(1-q^ne^{-x_i})(1-q^ne^{x_i})}:\frac{(1+q^n)^2}{(1-q^n)^2}\Big).$$

It turns out this is precisely the expansion of $x\sqrt{\wp(x)-e_1}$. Recall that a genus is elliptic if and only if $f(x)=x/Q(x)$ satisfies $(f')^2=1-2\delta f^2+\varepsilon f^4$. One can then check

Corollary 2.6. *$\varphi(X)$ is an elliptic genus of X, hence a modular form of weight $2k$ on $\Gamma_0(2)$. The quantum signature* $\operatorname{sign}(q,\mathcal{L}X)=\varphi(X)\cdot\varepsilon^{-k/2}$ *is a modular function on $\Gamma_0(2)$ for $\dim X=4k\equiv 0(8)$.*

2.3. **Quantization of signature-expansion at cusp 0.** The modular curve of $\Gamma_0(2)$ has two cusps, namely 0 and ∞. Both $\varphi(X)$ and ε are written in the local coordinate q at the cusp ∞. If we write them in local coordinate $\tilde{q}$ at the cusp 0, then we have

Proposition 2.7.

$$\tilde{\varphi}(X)\cdot\tilde{\varepsilon}^{-k/2}=\tilde{q}^{-k/2}\cdot\hat{A}(X,\bigotimes_{n=2m+1}\Lambda_{-\tilde{q}^n}T_{\mathbb{C}}\otimes\bigotimes_{n=2m+2}S_{\tilde{q}^n}T_{\mathbb{C}}).$$

2.4. **The Witten genus.** Let $L\subset\mathbb{C}$ be a lattice, and let

$$\sigma_L(x)=x\cdot\prod_{\omega\in L\setminus\{0\}}\Big((1-\frac{x}{\omega})\exp(\frac{x}{\omega}+\frac{x^2}{2\omega^2})\Big).$$

Definition 2.8 (Witten genus)**.** *For a compact, orientable smooth manifold X^{4k} we define its Witten genus to be*

$$\varphi_W(X)=\Big(\prod_{i=1}^{2k}\frac{x_i}{\sigma_L(x_i)}\Big).$$

It turns out Witten genus is elliptic, and the power series $Q(x)=\frac{x}{\sigma_L(x)}$ has a beautiful product representation

$$Q(x)=\frac{x/2}{\sinh(x/2)}\cdot\prod_{n=1}^{\infty}\frac{(1-q^n)^2}{(1-q^ne^x)(1-q^ne^{-x})}e^{-G_2(\tau)\cdot x^2}.$$

Here $G_2(\tau)=\frac{1}{2}\sum_{\omega\in L'}\frac{1}{\omega^2}=-\frac{1}{24}+q+3q^2+\dots$ is the Eisenstein series of weight 2. If we use our standard notation $p(X)=\prod(1+x_i^2)$, then we obtain for Witten genus

Lemma 2.9.

$$\varphi_W(X)=\Big(\prod_{i=1}^{2k}(\frac{x_i/2}{\sinh(x_i/2)}\prod_{n=1}^{\infty}\frac{(1-q^n)^2}{(1-q^ne^{x_i})(1-q^ne^{-x_i})})e^{-G_2(\tau)\cdot\sum_{i=1}^{2k}x_i^2}\Big)[X]$$

Corollary 2.10. *If X is a string manifold, then $p_1(X)=\sum x_i^2=0$, hence the Witten genus*

$$\varphi_W(X)=\hat{A}(X,\bigotimes_{n=1}^{\infty}S_{q^n}T_{\mathbb{C}})\cdot\prod_{n=1}^{\infty}(1-q^n)^{4k}$$

is a modular form of weight $2k$ with integral Fourier expansion.