Four components of d on almost complex manifolds*

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Abstract

The exterior differential d on complex-valued differential forms of complex manifolds decomposes into the Cauchy-Riemann operator and its complex conjugate. Meanwhile on almost complex manifolds, the exterior d in general has two extra components, thus decomposes into four operators. In this talk, I will introduce these operators and discuss the structure of the (graded) associative algebra generated by these four components of d, subject to relations deduced from d squaring to zero. Then I will compare this algebra to the corresponding one in the complex (i.e. integrable) case, we shall see they are very different strictly speaking but similar in a weak sense (quasi-isomorphic). This is based on joint work with Shamuel Aueyung and Jin-Cheng Guu[AGH22].

1 Decomposition of d on almost complex manifolds

Given a complex manifold M of complex dimension n, locally it admits complex coordinates $z^i = x^i + \sqrt{-1}y^i$, $i = 1, \ldots, n$. Its tangent space locally is spanned by $\partial/\partial x^i, \partial/\partial y^i$ and its complexified tangent space is spanned by $\partial/\partial z^i$ and $\partial/\partial \overline{z}^i$. Let $T^{1,0}M$ denote the span of $\partial/\partial z^i$'s and $T^{0,1}M$ the span of $\partial/\partial \overline{z}^i$'s. Thus we have

$$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$$

The two subspaces $T^{1,0}$ and $T^{0,1}$ of complexified tangent bundle can be characterized without coordinates as follows. Consider the linear map $J : TM \to TM$ defined by

$$J(\partial/\partial x^i) = \partial/\partial y^i, \quad J(\partial/\partial y^i) = -\partial/\partial x^i$$

It turns out J is globally defined and satisfies $J^2 = -1$. Then $T^{1,0}$ and $T^{0,1}$ are eigenspaces of J with eigenvalue $\pm\sqrt{-1}$.

Dually, the complexified cotangent space admits a decomposition

$$T^*M \otimes \mathbb{C} = T^*_{0,1}M \oplus T^*_{1,0}M$$

where locally $T^*_{1,0}$ and $T^*_{0,1}$ are spanned by dz^i's and $d\overline{z}^i$'s respectively. The above decomposition yields a decomposition

$$\Lambda^k T^*M \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^p T^*_{1,0}M \otimes \Lambda^q T^*_{0,1}M$$

and consequently the Hodge decomposition[1]

$$\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M)$$

where Ω^k means group of complex valued differential k-forms and $\Omega^{p,q}$ the group of smooth sections of $\Lambda^p T^*_{1,0}M \otimes \Lambda^q T^*_{0,1}M$, usually referred to as the (p,q)-forms.

Note that the Hodge decomposition of differential forms *does not* rely on the complex coordinates on M, but rather only depends on the linear map $J : TM \to TM$. We call a (real) manifold M equipped with such a linear map an almost complex manifold and such J is called its almost complex structure.

*This is an informal note, citations are not carefully put

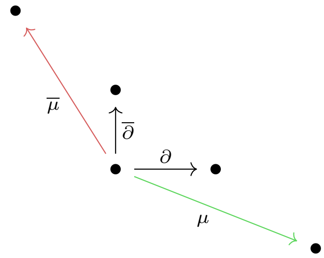
[1] this is not a standard terminology

Now we would like to understand the chain complex $(\Omega^\bullet(M), d)$ with respect to the Hodge decomposition. First of all, $\Omega^\bullet(M)$ is a (graded) algebra when equipped with wedge product, called the de Rham algebra, on which d is a derivation. Since the de Rham algebra is generated by 0-forms (smooth functions) and 1-forms, it suffices to understand the action of d on 0-forms and 1-forms. It follows that d decomposes, with respect to the Hodge decomposition as

$$d = \overline{\mu} + \overline{\partial} + \partial + \mu$$

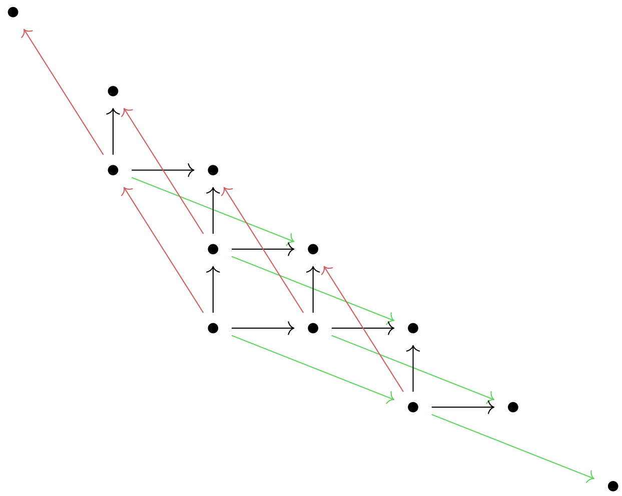
It is better to write this pictorially as

$d =$

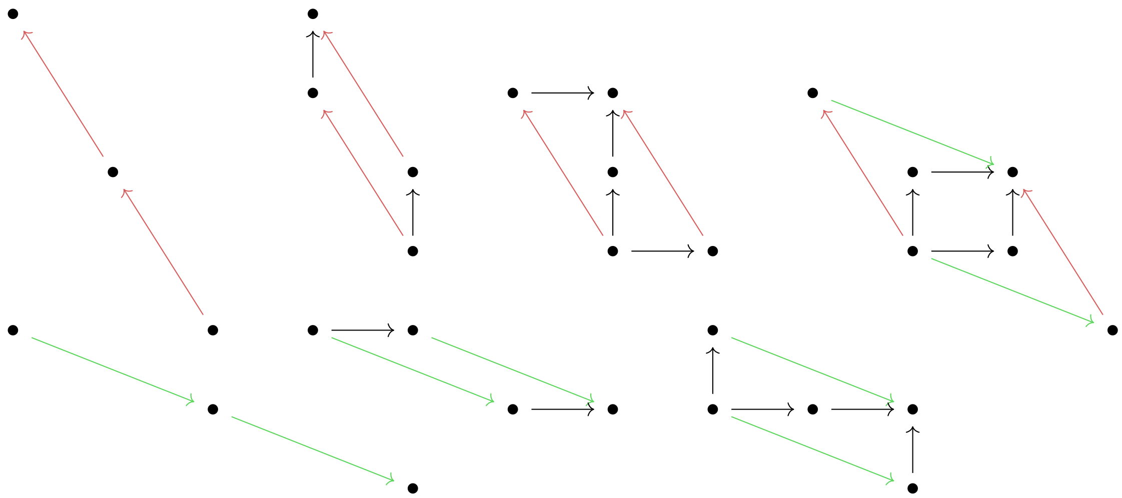


Then d^2 pictorially is

$d^2 =$

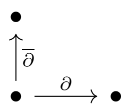


The equation $d^2 = 0$ is equivalent to the vanishing of the following diagrams

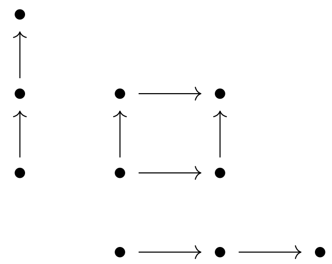


If our almost complex manifold (M, J) actually admits complex coordinates, i.e. complex/holomorphic, then it is well-known that d has only two components $\overline{\partial}, \partial$:

$d =$



subject to relations given by the vanishing of



2 Algebra generated by the components of d

From the above discussion, the relations among the four components of d looks to be rather complicated. We would like to clarify what the relations "mean". More precisely, consider the algebra

$$A = \frac{\text{free associative algebra generated by } \overline{\mu}, \overline{\partial}, \partial, \mu}{\text{relations given above by } d^2 = 0}$$

Question. What is the structure of the algebra A?

This is a somewhat vague question, a more precise one can be: what is the dimension of A in each degree? Our answer to the above question is based on a sequence of observations.

Observation 1. The relations among $\overline{\mu}, \overline{\partial}, \partial, \mu$ can be written using (graded) Lie brackets.

Indeed, $d^2 = 0$ is equivalent to $[d, d] = 2d^2 = 0$. The relations can be written as

$$\begin{aligned} &[\overline{\mu}, \overline{\mu}] = 0, && [\mu, \mu] = 0; \\ &[\overline{\mu}, \overline{\partial}] = 0, && [\mu, \partial] = 0; \\ &[\overline{\mu}, \partial] = -\frac{1}{2}[\overline{\partial}, \overline{\partial}], && [\mu, \overline{\partial}] = -\frac{1}{2}[\partial, \partial]; \\ &[\overline{\mu}, \mu] = -[\overline{\partial}, \partial]. \end{aligned} \tag{1}$$

Theorem 1. $A = U\mathfrak{g}$ where $\mathfrak{g}$ is the graded Lie algebra generated by $\overline{\mu}, \overline{\partial}, \partial, \mu$ and U means universal enveloping algebra.

Proof. Observe A is a primitively generated Hopf algebra in which $\overline{\mu}, \overline{\partial}, \partial, \mu$ are primitive. Then apply Milnor-Moore. □

Therefore, determining the structure of A amounts to determining that of $\mathfrak{g}$. Note that $\mathfrak{g}$ is much smaller than A in the following sense. If $\beta_1, \beta_2, \ldots$ are Betti numbers of $\mathfrak{g}$, that is $\beta_d = \dim \mathfrak{g}_d$ the degree d subspace of $\mathfrak{g}$, then those of A can be read off from its Poincaré series

$$\prod_i \frac{(1 + x^{2i+1})^{\beta_{2i+1}}}{(1 - x^{2i})^{\beta_{2i}}}$$

Observation 2. There are no relations between *only* $\overline{\partial}$ and ∂ in (1).

Theorem 2. The Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ generated by $\overline{\partial}$ and ∂ is free.

The proof of this theorem actually requires an understanding of the role played by $\overline{\mu}, \mu$–they are derivations on $\mathfrak{h}$ and $\mathfrak{g}$ is a "semi-direct product" of $\mathfrak{h}$ and $\overline{\mu}, \mu$.

Observation 3. (The adjoint actions of) $\overline{\mu}$ and μ take $\overline{\partial}, \partial$ into $\mathfrak{h}$.

Theorem 3. $\mathfrak{h}$ is a Lie ideal of $\mathfrak{g}$. Moreover, $\mathfrak{g} = \mathfrak{h}$ in degrees ≥ 2.

Proof. The second assertion follows from an easy induction. □

At this point, we have a rather satisfying understanding of $\mathfrak{g}$ through the exact sequence

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$$

Here $\mathfrak{h}$ is the free Lie algebra generated by $\overline{\partial}, \partial$, and $\mathfrak{g}/\mathfrak{h}$ is an abelian Lie algebra generated by $\overline{\mu}, \mu$. The extension is through the derivation action of $\overline{\mu}, \mu$ on $\mathfrak{h}$.

Now let us compare the situation to the complex case. There is a corresponding algebra A_{hol} and Lie algebra $\mathfrak{g}_{hol}$ where

$$\mathfrak{g}_{hol} = \frac{\text{free graded Lie algebra generated by } \overline{\partial}, \partial}{[\overline{\partial}, \overline{\partial}] = [\overline{\partial}, \partial] = [\partial, \partial] = 0}$$

That is to say, $\mathfrak{g}_{hol}$ is the abelian Lie algebra generated by $\overline{\partial}, \partial$ and thus A_{hol} is the exterior algebra generated by $\overline{\partial}, \partial$.

The natural quotient map

$$\mathfrak{g} \to \mathfrak{g}_{hol}$$

has a gigantic kernel. Indeed, the difference is roughly (or precisely in degrees ≥ 2) the difference between free Lie algebra and abelian Lie algebra.

3 Cohomology

The Lie algebra $\mathfrak{g}$ is equipped with a natural differential $[d, -]$. As we have pointed out, $\mathfrak{g}$ and $\mathfrak{g}_{hol}$ are very different, so the following fact is surprising to us:

Theorem 4. The quotient map $\mathfrak{g} \to \mathfrak{g}_{hol}$ is a quasi-isomorphism.

Let us unwind the statement. Notice that $\mathfrak{g}_{hol}$ is concentrated in degree 1 and $[d, -]$ vanishes on $\mathfrak{g}_{hol}$ as it is abelian, we see the cohomology of $\mathfrak{g}_{hol}$ is itself, concentrated in degree 1. So the theorem actually says $H^k(\mathfrak{g}, [d, -])$ vanishes for $k \geq 2$. Since one can explicitly verify the quotient map yields an isomorphism on H^1, the essence of this theorem is the vanishing of higher cohomologies.

The geometric meaning of the above theorem is, any differential operator of order ≥ 2 constructed from $\overline{\mu}, \overline{\partial}, \partial, \mu$, if commutes with d, is of the form $[d, D]$ for some operator D.

Proof sketch. From the exact sequence

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$$

The vanishing of higher (degree> 2) cohomologies of $\mathfrak{g}$ is equivalent to the same vanishing for $\mathfrak{h}$. Then by the generalized Frölicher spectral sequence of Cirici-Wilson [CW21], it suffices to prove $H^k(\mathfrak{h}, [\overline{\mu}, -]) = 0$ for $k > 2$ and $H^2(\mathfrak{g}, [\overline{\mu}, -]) = 0$. Then one passes to universal enveloping algebra of $\mathfrak{h}$, i.e. the free associative algebra generated by $\overline{\partial}, \partial$ and apply induction. $\square$

Proof of vanishing $\overline{\mu}$-cohomology for $\mathfrak{h}$. First of all, we note $\mathrm{ad}_{\overline{\mu}} \overline{\partial} = 0$ and

$$\mathrm{ad}_{\overline{\mu}}[\partial, \overline{\partial}] = [\mathrm{ad}_{\overline{\mu}} \partial, \overline{\partial}] - [\partial, \mathrm{ad}_{\overline{\mu}} \overline{\partial}] = -\frac{1}{2}[[\overline{\partial}, \overline{\partial}], \overline{\partial}] = 0.$$

We leave it to the reader to check when $k = 1, 2$, $H^k(\mathfrak{h}, \mathrm{ad}_{\overline{\mu}})$ is one-dimensional and spanned by (the equivalence class of) $\overline{\partial}$ and $[\partial, \overline{\partial}]$ respectively. Next we observe $H(\mathfrak{h}, \mathrm{ad}_{\overline{\mu}})$ is a Lie algebra and $H^1(\mathfrak{h}) \oplus H^2(\mathfrak{h})$ forms an *abelian* Lie subalgebra since $[\overline{\partial}, \overline{\partial}] = -2[\overline{\mu}, \partial]$ is $\mathrm{ad}_{\overline{\mu}}$-exact, and $[\overline{\partial}, [\partial, \overline{\partial}]] = 0$, $[[\partial, \overline{\partial}], [\partial, \overline{\partial}]] = 0$ by Jacobi identities.

Now consider the universal enveloping algebra $UH(\mathfrak{h}, \mathrm{ad}_{\overline{\mu}})$ of $H(\mathfrak{h}, \mathrm{ad}_{\overline{\mu}})$. It contains the universal enveloping algebra of the abelian Lie subalgebra $H^1(\mathfrak{h}) \oplus H^2(\mathfrak{h})$, which is the free graded commutative algebra $\Lambda(\overline{\partial}, [\partial, \overline{\partial}])$ generated by $\overline{\partial}$ and $[\partial, \overline{\partial}]$. Meanwhile, since the universal enveloping algebra functor commutes with cohomology, we have $UH(\mathfrak{h}, \mathrm{ad}_{\overline{\mu}}) = H(U\mathfrak{h}, \mathrm{ad}_{\overline{\mu}})$ where $\mathrm{ad}_{\overline{\mu}}$ on $U\mathfrak{h}$ is the extended adjoint action. So we get

$$\Lambda(\overline{\partial}, [\partial, \overline{\partial}]) \subset H(U\mathfrak{h}, \mathrm{ad}_{\overline{\mu}}).$$

By Poincaré-Birkhoff-Witt theorem, our proposition is equivalent to $\Lambda(\overline{\partial}, [\partial, \overline{\partial}]) = H(U\mathfrak{h}, \mathrm{ad}_{\overline{\mu}})$. This equality clearly holds in degrees ≤ 2. We plan to prove this by induction on degree, but we need to make some preparations.

For simplicity of notation, denote $B = U\mathfrak{h}$, which is the free tensor algebra on $\overline{\partial}, \partial$. Under the isomorphism

$$\phi : B_{k-1} \oplus B_{k-1} \cong B_k, (x, y) \mapsto \partial x + \overline{\partial} y,$$

the differential $\mathrm{ad}_{\overline{\mu}} |_{B_k}$ can be written as the matrix

$$\mathrm{ad}_{\overline{\mu}} |_{B_k} \cong \begin{pmatrix} -\mathrm{ad}_{\overline{\mu}} |_{B_{k-1}} & 0 \\ -\overline{\partial}|_{B_{k-1}} & -\mathrm{ad}_{\overline{\mu}} |_{B_{k-1}} \end{pmatrix} \tag{2}$$

by using the relations (1). To see this, we compute for $x, y \in B_{k-1}$

$$\begin{aligned}
[\overline{\mu}, \partial x + \overline{\partial} y] &= \overline{\mu}\partial x - (-1)^k \partial x \overline{\mu} + \overline{\mu}\overline{\partial} y - (-1)^k \overline{\partial} y \overline{\mu} \\
&= -\overline{\partial}^2 x - \partial\overline{\mu} x - (-1)^k \partial x \overline{\mu} - \overline{\partial}\overline{\mu} y - (-1)^k \overline{\partial} y \overline{\mu} \\
&= -\partial \left(\overline{\mu} x - (-1)^{k-1} x \overline{\mu}\right) - \overline{\partial} \left(\overline{\partial} x + \overline{\mu} y - (-1)^{k-1} y \overline{\mu}\right) \\
&= -\partial[\overline{\mu}, x] - \overline{\partial}(\overline{\partial} x + [\overline{\mu}, y]).
\end{aligned}$$

In particular, by setting $x = 0$, we see $\mathrm{ad}_{\overline{\mu}}$ skew commutes with $\overline{\partial}$. This means both $\pm\overline{\partial}$ are morphisms of cochain complexes $\pm\overline{\partial} : B_\bullet \to B_\bullet[1]$, where $B_\bullet = (B, \mathrm{ad}_{\overline{\mu}})$. Moreover, (2) shows the mapping cone of $-\overline{\partial}$ is isomorphic to $B_\bullet[2]$ by ϕ. Then the inclusion of $B_\bullet[1]$ into the mapping cone of $-\overline{\partial}$ is identified with $\overline{\partial} : B_\bullet[1] \to B_\bullet[2]$, and the projection from the mapping cone of $-\overline{\partial}$ onto $B_\bullet[1]$ is identified with $\delta : B_\bullet[2] \to B_\bullet[1]$ which takes $\partial x + \overline{\partial} y$ to x. It follows we have an exact triangle

$$B_\bullet \xrightarrow{-\overline{\partial}} B_\bullet[1] \xrightarrow{\overline{\partial}} B_\bullet[2] \xrightarrow{\delta} B_\bullet[1].$$

This exact triangle induces a long exact sequence in cohomology

$$\cdots \to H^{k-2}(B_\bullet) \xrightarrow{-\overline{\partial}} H^{k-1}(B_\bullet) \xrightarrow{\overline{\partial}} H^k(B_\bullet) \xrightarrow{\delta} H^{k-1}(B_\bullet) \xrightarrow{-\overline{\partial}} H^k(B_\bullet) \to \cdots$$

Now we can inductively prove $H(B_\bullet) = \Lambda(\overline{\partial}, [\partial, \overline{\partial}])$. We note $\Lambda(\overline{\partial}, [\partial, \overline{\partial}])$ is one-dimensional in each degree and spanned by powers of $[\partial, \overline{\partial}]$ and $\overline{\partial}$ times powers of $[\partial, \overline{\partial}]$. Assume the desired equality is proved in degrees $< k$. Observe that $\overline{\partial}$ vanishes on $\left(\Lambda(\overline{\partial}, [\partial, \overline{\partial}])\right)^{\mathrm{odd}}$, so if k is even then from the above long exact sequence we have $\delta : H^k(B_\bullet) \to H^{k-1}(B_\bullet)$ is an isomorphism. On the other hand, if k is odd, then $\overline{\partial} : H^{k-1}(B_\bullet) \to H^k(B_\bullet)$ is monic since $\overline{\partial}$ takes $\left(\Lambda(\overline{\partial}, [\partial, \overline{\partial}])\right)^{\mathrm{even}}$ injectively into $\left(\Lambda(\overline{\partial}, [\partial, \overline{\partial}])\right)^{\mathrm{odd}} \subset H^{\mathrm{odd}}(B_\bullet)$ and thus the above long exact sequence implies $\overline{\partial} : H^{k-1}(B_\bullet) \to H^k(B_\bullet)$ is an isomorphism. So in either case we have $H^k(B_\bullet) \cong H^{k-1}(B_\bullet)$, and in particular $H^k(B_\bullet)$ is one-dimensional. But $H^k(B_\bullet) \supset \left(\Lambda(\overline{\partial}, [\partial, \overline{\partial}])\right)^k$, therefore $H^k(B_\bullet)$ must be equal to $\left(\Lambda(\overline{\partial}, [\partial, \overline{\partial}])\right)^k$. This finishes the inductive step and thus completes the proof. □

Proof of vanishing of H^2 for $\mathfrak{g}$. One can compute $H^2(\mathfrak{g}, [\overline{\mu}, -]) = 0$ and apply the spectral sequence of Cirici-Wilson. □

Observation 4. There are many inner differentials on $\mathfrak{g}$.

For example $[\overline{\mu}, -]$ and $[\mu, -]$ are inner differentials on $\mathfrak{g}$.

Question. [2] Find all inner differentials on $\mathfrak{g}$.

This amounts to solving the equation $[a, a] = 0$ for $a \in \mathfrak{g}_1$. Writing $a = x\overline{\mu} + y\overline{\partial} + z\partial + w\mu$ we have

$$\begin{aligned}
xz - y^2 &= 0, \\
yw - z^2 &= 0, \\
xw - yz &= 0.
\end{aligned}$$

Denote the set of solutions $MC(\mathfrak{g}) = \{a \in \mathfrak{g}_1 : [a, a] = 0\}$.

[2]This question is also addressed in [TT20].

Theorem 5. $\mathbb{P}MC(\mathfrak{g})$ is the twisted cubic in $\mathbb{P}^3$, parametrized by

$$\mathbb{P}^1 \to \mathbb{P}MC(\mathfrak{g}), [s{:}t] \mapsto d_{s,t} = s^3\overline{\mu} + s^2t\overline{\partial} + st^2\partial + t^3\mu$$

Proposition 6. For $(s,t) \neq (0,0)$, $H^1(\mathfrak{g}, [d_{s,t}, -])$ is spanned by $\frac{\partial}{\partial s}d_{s,t}$ and $\frac{\partial}{\partial t}d_{s,t}$.

Proof. $[d_{s,t}, d_{s,t}] = 0$ implies $0 = \frac{\partial}{\partial s}[d_{s,t}, d_{s,t}] = 2[d_{s,t}, \frac{\partial}{\partial s}d_{s,t}]$. Similar for t. The fact that they span can be proved by explicit computation or deduced from deformation theory (cf. Goldman-Milson). □

Corollary 7. For $st \neq 0$, $H^1(\mathfrak{g}, [d_{s,t}, -])$ is spanned by $d_{s,t}$ and

$$d^J_{s,t} = \sqrt{-1}(3s^3\overline{\mu} + s^2t\overline{\partial} - st^2\partial - 3t^3\mu).$$

The operator $d^J = \sqrt{-1}(3\overline{\mu}+\overline{\partial}-\partial-3\mu)$ is in fact the commutator $[d, J]$ where J is extended as a derivation on the de Rham algebra.

Corollary 8. The quotient map $\mathfrak{g} \to \mathfrak{g}_{hol}$ is a quasi-isomorphism with respect to $[d_{s,t}, -]$ for $st \neq 0$.

Proof. $d_{s,t}$ is conjugate to d by multiplication by $s^{p+2q}t^{2p+q}$ on bidegree (p,q) subspace. □

4 Interpretations and comments

4.1 Deformation theory

It is expected from deformation theory that the first cohomology $H^1(\mathfrak{g}, [d_{s,t}, -])$ is the Zariski tangent space of $MC(\mathfrak{g})$ at $d_{s,t}$. Also the quasi-isomorphism should yield a bijection between $MC(\mathfrak{g}, [d, -])$ and $MC(\mathfrak{g}_{hol}, [d, -])$. But the former is isomorphic to $MC(\mathfrak{g})$ and the latter is isomorphic to $\mathbb{A}^2$.

4.2 Rational homotopy theory

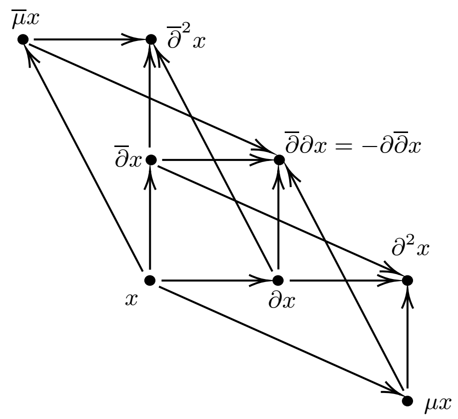
The subalgebra $\mathfrak{h}$ is isomorphic to the homotopy Lie algebra of $\mathbb{CP}^1 \vee \mathbb{CP}^1$ and the subalgebra of $\mathfrak{g}_{hol}$ generated by $\overline{\partial}, \partial$, which is $\mathfrak{g}_{hol}$ itself, is isomorphic to the homotopy Lie algebra of $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$. So the difference between $\mathfrak{g}$ and $\mathfrak{g}_{hol}$ is captured by higher dimensional cells of $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$. For example, the 4-dimensional cell corresponds to $[\partial, \overline{\partial}]$. The quotient

$$\mathfrak{g}/([\partial, \overline{\partial}] = 0)$$

is 6-dimensional whose subalgebra generated by $\partial, \overline{\partial}$ is isomorphic to the homotopy Lie algebra of $\mathbb{CP}^1 \times \mathbb{CP}^1$.

4.3 Representation theory

The ultimate goal is through the understanding of A and its representations, we can understand almost complex manifolds better. But the representation theory for A or equivalently for $\mathfrak{g}$ looks to be very complicated.



$$\overline{\mu}(\partial x) = \left(-\tfrac{1}{2} - \alpha\right)\overline{\partial}^2 x$$
$$\partial(\overline{\mu}x) = \left(-\tfrac{1}{2} + \alpha\right)\overline{\partial}^2 x$$
$$\overline{\mu}(\mu x) = \beta\overline{\partial}\partial x$$
$$\mu(\overline{\mu}x) = -\beta\overline{\partial}\partial x$$
$$\mu(\overline{\partial}x) = \left(-\tfrac{1}{2} - \gamma\right)\partial^2 x$$
$$\overline{\partial}(\mu x) = \left(-\tfrac{1}{2} + \gamma\right)\partial^2 x$$

This is a faithful representation of $\mathfrak{g}/[\partial, \overline{\partial}]$. We remark that $\mathfrak{g}/[\partial, \overline{\partial}]$ contains two (or three) copies of graded Heisenberg Lie algebra.

One might wish to design certain cohomology groups that are able to distinguish different representations of $\mathfrak{g}$. My suggestion is to consider the role of d^J and define

$$H^\bullet_{BC} = \frac{\ker d}{\operatorname{im} dd^J}$$

and its dual notion

$$H^\bullet_A = \frac{\ker dd^J}{\operatorname{im} d}$$

The catch of this definition is that H_{BC} is graded, real, but not multiplicative because $(d^J)^2 \neq 0$ in general. One might say the failure of H_{BC} to be multiplicative measures the failure of J to be integrable.

We note there are other definitions of Bott-Chern and Aeppli cohomology for almost complex manifolds. See [CPS21] for example.

One probably should justify either of the definition by relating those groups to geometry like Bott and Chern did, or to quantities from representations of $\mathfrak{g}$.

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