# ADAMS SPECTRAL SEQUENCE AND APPLICATIONS TO COBORDISM I

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Abstract. In this note, we construct Adams spectral sequence and apply it to compute unoriented ring following Thom.

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## 1. Introduction

In topology, there are two sets of invariants that are of most interest: homology groups and homotopy groups. Homology groups is easier to compute by design, homotopy groups are usually extremely hard to compute but reveal more about the space. One can then ask, can we extract information from homology to get homotopy? Adams spectral sequence is invented to serve this purpose, but instead of homotopy groups we get stable homotopy groups.

In geometry, one wish to classify all manifolds which is very hard. But if one only wants to classify manifolds up to cobordism, then the problem reduces to compute the stable homotopy groups of so-called Thom spaces. Then our strategy is to first understand the homology and cohomology of these Thom spaces and then apply Adams spectral sequence.

## 2. Adams spectral sequence

A continuous map $f : Y \to X$ induces $f^* : \tilde{H}^*(X; \mathbb{Z}_p) \to \tilde{H}^*(Y; \mathbb{Z}_p)$, hence we have a map

$$\phi_Y : [Y, X] \to Hom_{\mathcal{A}}(\tilde{H}^*(X), \tilde{H}^*(Y))$$

where $\mathcal{A}$ is the mod $p$ Steenrod algebra and cohomology are taken to have $\mathbb{Z}_p$-coefficients. Since reduced cohomology and Steenrod powers are stable under suspension, we actually get a map from stable homotopy class of maps from $Y$ to $X$ to $Hom_{\mathcal{A}}(\tilde{H}^*(X), \tilde{H}^*(Y))$. We shall sighltly abuse the notation and use $[Y, X]$ to denote the stable homotopy class of maps.

The benefit of passing to stable maps is that now $[Y, X]$ is an abelian group and $\phi_Y$ is a group homomorphism. But in general $\phi_Y$ is not an isomorphism, so we need more information than just $Hom_{\mathcal{A}}(\tilde{H}^*(X), \tilde{H}^*(Y))$.

Then one realizes that $Hom_{\mathcal{A}}(-,\tilde{H}^*(Y))$ is just one of a sequence of functors $Ext^i_{\mathcal{A}}(-,\tilde{H}^*(Y))$. Now Adams says if we put all those together, then we can get an approximation to $\pi^Y_*(X)$ where $\pi^Y_k(X)=[\Sigma^k Y, X]$.

**Theorem 2.1.** *If $X, Y$ are reasonable spaces, then there exists a spectral sequence*
$$E_2^{s,t}=Ext^{s,t}_{\mathcal{A}}(\tilde{H}^*(X),\tilde{H}^*(Y))\Longrightarrow \pi^Y_{t-s}(X)/\textit{non-p-torsion}$$

The second index $t$ merely comes from the grading of $\tilde{H}^*(X)$, and one can not detect non-$p$-torsion is expected since we are taking $\mathbb{Z}_p$-coefficients. We will take $Y=S^0$ and $X=MO, MSO, MU$ and they are "reasonable".

Let's recall how we compute $Ext^i_R(M,N)$ in homological algebra. First we take a free or projective resolution of $M$
$$\cdots\to F_2\to F_1\to F_0\to M\to 0$$
then we apply $Hom_R(-,N)$ and take cohomology of the sequence
$$0\to Hom_R(F_0,N)\to Hom_R(F_1,N)\to Hom_R(F_2,N)\to\cdots$$
we get $Ext^i_R(M,N)$.

So we wish to construct a "resolution" of the space $X$
$$X\to K_0\to K_1\to K_2\to\cdots$$
so that it induces a free resolution of $\tilde{H}^*(X)$
$$\cdots\to\tilde{H}^*(K_2)\to\tilde{H}^*(K_1)\to\tilde{H}^*(K_0)\to\tilde{H}^*(X)\to 0$$
But this is impossible, because cohomology of a space can never be free over $\mathcal{A}$. The next best thing you have are the Eilenberg-MacLane spaces. $\tilde{H}^*(K(\mathbb{Z}_p,n);\mathbb{Z}_p)$ is a free $\mathcal{A}$-module in the stable range, namely from $n$ to $2n$. This turns out to be enough because we are dealing with a stable problem. We inductively define $K_i$ as follows.

- Find a set of generators $\alpha_i\in\tilde{H}^*(X)$ as $\mathcal{A}$-module. Each $\alpha_i$ induces a map $X\to K(\mathbb{Z}_p,n_i)$. Let $X_0:=X$, $K_0$ be the product of these $K(\mathbb{Z}_p,n_i)$'s.
- Treat $X\to K_0$ as an inclusion, let $X_1=K_0/X$. And repeat the above to get $X_1\to K_1$, let $X_2=K_1/X_1$.
- Keep doing so.

So we obtain cofibrations $X_s\to K_{s+1}\to X_{s+1}=K_{s+1}/X_s$ thus long exact sequence
$$\cdots\to\pi^Y_*(X_s)\to\pi^Y_*(K_{s+1})\to\pi^Y_*(X_{s+1})\to\pi^Y_{*-1}(X_s)\to\cdots$$
Therefore $\oplus_s\pi^Y_*(X_s)$ and $\oplus_s\pi^Y_*(K_s)$ form an exact couple, hence we get a spectral sequence.

- $E_1^{s,t}=\pi^Y_t(K_s)$ and $d_1: E_1^{s,t}\to E_1^{s+1,t}$ is induced by the map $K_s\to K_{s+1}$.
- Consider the map
$$\pi^Y_t(K_s)\to Hom_{\mathcal{A}}(\tilde{H}^*(K_s),\tilde{H}^*(\Sigma^t Y))=Hom^t_{\mathcal{A}}(\tilde{H}^*(K_s),\tilde{H}^*(Y))$$
Here $Hom^t$ means map of degree $t$. This is an isomorphism if $\tilde{H}^*(K_s)$ is free over $\mathcal{A}$. However $\tilde{H}^*(K_s)$ is only free in the stable range.
- Now we can suspend $X$ so that $X$ is $N$-connected, and thus are $K_s$ from construction so $\tilde{H}^*(K_s)$ is free below $2N$. Finally we take $N\to\infty$ and pretend $\tilde{H}^*(K_s)$ free over $\mathcal{A}$.
- Therefore $E_1^{r,t}\simeq Hom^t_{\mathcal{A}}(\tilde{H}^*(K_s),\tilde{H}^*(Y))$ and $E_2^{s,t}\simeq Ext^{s,t}(\tilde{H}^*(X),\tilde{H}^*(Y))$.

If we take $Y=S^0$, then

**Corollary 2.2.** *There is a spectral sequence*

$$E_2^{s,t} = Ext_{\mathcal{A}}^{s,t}(\tilde{H}^*(X), \mathbb{Z}_p) \Longrightarrow \pi_{t-s}^s(X)$$

**Remark 2.3.** *I was cheating by taking $N \to \infty$. The proper way is to replace spaces by spectra. The cohomology of an Eilenberg-MacLane spectrum is indeed free over $\mathcal{A}$.*

## 3. Determination of unoriented bordism

Let $BO_n = Gr_{\mathbb{R}}(n, \infty)$ be the classfying space of $O_n$, and $EO_n$ the tautological bundle over $BO_n$. Then put a point at infinity for each fiber of $EO_n$ and collapse all the points at infinity to one point, we get the Thom space $MO_n$. Let $MO$ be the limit of all $MO_n$ under map $MO_n \to MO_{n+1}$ induced by the natural map $BO_n \to BO_{n+1}$ classifying $EO_n \oplus \mathbb{R}$.

By Pontryagin-Thom construction, $\pi_*(MO)$ is isomorphic to the unoriented bordism ring $\Omega_*^O$. Since two times of any manifold $M$ is the boundary of $M \times [0,1]$, all elements in $\Omega_*^O$ are 2-torsion. So we only need to run Adams spectral sequence at 2. Thus we take $\mathbb{Z}_2$-coefficient in this section.

**Theorem 3.1** (Thom)**.** *$\tilde{H}^*(MO; \mathbb{Z}_2)$ is a free module over mod 2 Steenrod algebra, hence the Adams spectral sequence collapses on $E_1$-page.*

In order to understand $\tilde{H}^*(MO)$, we first understand $\tilde{H}^*(MO_n)$ and take $n \to \infty$.

It is well-known that $H^*(BO_n; \mathbb{Z}_2) \simeq \mathbb{Z}_2[w_1, w_2, \ldots, w_n]$ where $w_i$ is the $i$-th Stiefel-Whitney class of $EO_n$. Using splitting principle, we pretend $EO_n$ is a direct sum of rank 1 real bundles with first Stiefel-Whitney classes $t_1, t_2, \ldots, t_n$. Then $w_i = \sigma_i(t_1, \ldots, t_n)$ is the $i$-th symmetric polynomial in $t_1, \ldots, t_n$. So $H^*(BO_n) \simeq Sym(\mathbb{Z}_2[t_1, \ldots, t_n])$.

By Thom isomorphism reduced $H^*(MO_n) \simeq u \cdot \tilde{H}^*(BO_n)/(u \cdot w_n = u^2)$ where $u$ is the Thom class of $EO_n$. Using splitting principle,

$$\tilde{H}^*(MO_n) \simeq t_1 \cdots t_n \cdot Sym(\mathbb{Z}_2[t_1, \ldots, t_n]) \simeq Sym^{\geq n}[t_1, \ldots, t_n]$$

The Steenrod algebra module structure on $\tilde{H}^*(MO_n)$ is determined by

- $Sq(t_i) = (1 + Sq^1 + Sq^2 + Sq^3 + \ldots)(t_i) = t_i + t_i^2$
- $Sq(a \cdot b) = Sq(a) \cdot Sq(b)$ for all $a, b \in \tilde{H}^*(MO_n)$

Now we wish to show $\tilde{H}^*(MO_n)$ is free over $\mathcal{A}_2$ in the stable range.

### 3.1. mod 2 Steenrod algebra.

Before we get into the computation, let's digress and make a remark on the structure of mod 2 Steenrod algebra. Since $\mathcal{A}_2$ is the limit of $\tilde{H}^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$, we shall describe the algebra $H^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$.

**Definition 3.2.** *Let $I = (i_1, i_2, \ldots, i_r)$ be a sequence of natural numbers.*

- *$|I| = i_1 + i_2 + \cdots + i_r$ is called the degree of the sequence.*
- *$I$ is called admissible, if $i_1 \geq 2i_2, i_2 \geq 2i_3, \ldots, i_{r-1} \geq 2i_r$.*
- *Given an admissible sequence $I$, its excess $e(I)$ is defined by*

$$e(I) = (i_1 - 2i_2) + (i_2 - 2i_3) + \cdots + (i_{r-1} - 2i_r) + i_r$$

**Theorem 3.3** (Serre)**.** *The algebra $H^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ is the polynomial algebra generated by the elements $Sq^I(\iota_n)$, where $\iota_n$ is the generator of $H^n(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ and $I$ runs through the set of all admissible sequences with $e(I) < n$.*

*Sketch of proof.* We induct on $n$. $K(\mathbb{Z}_2,1)=\mathbb{R}P^\infty$ hence $H^*(K(\mathbb{Z}_2,1);\mathbb{Z}_2)$ is polynomial algebra with one generator $\iota_1$. For the induction step, we use the fibration $K(\mathbb{Z}_2,n-1)\to\bullet\to K(\mathbb{Z}_2,n)$. Now $H^*(K(\mathbb{Z}_2,n-1);\mathbb{Z}_2)$ is polynomial algebra generated by $z_J=Sq^J(\iota_{n-1})$, where $J$ runs through the set of all admissible sequences, and $e(J)<n-1$. Let us denote by $s_J$ the degree of $z_J$, then $s_J=n-1+|J|$. It is clear that $\iota_{n-1}$ is transgressive element, and that $\tau(\iota_{n-1})=\iota_n$. Since transgression commutes with Steenrod squares, $z_J$ is transgressive and $\tau(z_J)=Sq^J(\iota_n)$. Then by a (not so easy) Serre spectral sequence argument, $H^*(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$ is polynomial algebra generated by the elements $Sq^{L(s_J,r)}\circ Sq^J(\iota_n)$, where $J$ runs through the set of all admissible sequences with $e(J)<n-1$ and $L(s_J,r)$ is the sequence $(2^{r-1}s_J,\dots,2s_J,s_J)$. To complete the proof, we need the following algebraic lemma.

□

**Lemma 3.4.** *For any $r\geq 0$, and for any admissible sequence $J=(j_1,\dots,j_k)$ with $e(J)<n-1$, consider a sequence*

$$I=(2^{r-1}s_J,\dots,2s_J,s_J,j_1,\dots,j_k),\text{ where } s_J=n-1+|J|.$$

*Then each admissible sequence $I$ with $e(I)<n$ is listed here exactly once.*

**Corollary 3.5.** *For $h\leq n$, the rank $c(h)$ of the group $H^{n+h}(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$ is equal to the number of decompositions of $h$ into summands of type $2^m-1$.*

*Proof.* Let $\theta(\mathbb{Z}_2,n;t)=\sum_k\dim H^k(K(Z_2,n);\mathbb{Z}_2)t^k$, then since $H^*(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$ is a polynomial algebra generated by $Sq^I(\iota_n)$, where $I$ admissible and $e(I)<n$, we see

$$\theta(\mathbb{Z}_2,n;t)=\prod_{e(I)<n}\frac{1}{1-t^{n+|I|}}.$$

In order to simplify this expression, we calculate the number of admissible sequences $I$ with $e(I)<n$ and $|I|=h$ is fixed. Given $I=(i_1,\dots,i_r)$, we set $\alpha_1=i_1-2i_2,\dots,\alpha_{r-1}=i_{r-1}-2i_r,\alpha_r=i_r$. By definition $\alpha_i\geq 0$ and $\sum_{i=1}^r\alpha_i\leq n-1$. Clearly, the numbers $\alpha_i$ determine uniquely the sequence $I$. The condition $|I|=h$ is equivalent to

$$\sum_{i=1}^r\alpha_i(2^i-1)=h.$$

Let $\alpha_0=n-1-\sum_{i=1}^r\alpha_i$, then $\sum_{i=0}^r\alpha_i=n-1$ and

$$n+h=1+\sum_{i=0}^r\alpha_i\cdot 2^i,$$

or equivalently,

$$n+h=1+2^0+\cdots+2^0+2^1+\cdots+2^1+\cdots+2^r+\cdots+2^r,$$

where $2^i$ occurs $\alpha_i$ times. Since $\sum_{i=0}^r\alpha_i=n-1$, there are $n-1$ powers of 2. This means that the number of sequence $I$, satisfying the conditions listed above, is equal to the number of decompositions of integers $h$ in the form

$$h=(2^{m_1}-1)+(2^{m_2}-1)+\cdots+(2^{m_{n-1}}-1),\text{where } m_1\geq\cdots\geq m_{n-1}\geq 0.$$

Hence what we want follows from

$$\begin{aligned}\theta(\mathbb{Z}_2, n; t) &= \prod_{m_1 \geq m_2 \geq \cdots \geq m_{n-1} \geq 0} \frac{1}{t^{n+(2^{m_1}-1)+(2^{m_2}-1)+\cdots+(2^{m_{n-1}}-1)}} \\ &= \prod_{m_1 \geq m_2 \geq \cdots \geq m_{n-1} \geq 0} 1 + t^n \cdot t^{2^{m_1}-1} \dots t^{2^{m_{n-1}}-1} \bmod t^{2n+1} \\ &= 1 + t^n \sum_{m_1 \geq m_2 \geq \cdots \geq m_{n-1} \geq 0} t^{2^{m_1}-1} \dots t^{2^{m_{n-1}}-1} \bmod t^{2n+1}.\end{aligned}$$

□

3.2. **Steenrod algebra module structure of $\tilde{H}^*(MO_n)$.** The following is a sketch of Thom's computation.

- Let $u \in \tilde{H}^n(MO_n)$ be the Thom class, we can show $Sq^I(u)$ are linearly independent, where $I$ runs through the set of all admissible sequences with $e(I) < n$ and $|I| < n$. Hence the map corresponding to the Thom class $MO_n \to K(\mathbb{Z}_2, n)$ induces injection on cohomology $H^*(K(\mathbb{Z}_2, n)) \to H^*(MO_n)$ in the stable range $n \leq * \leq 2n$.
- Then we peel off the image of $H^*(K(\mathbb{Z}_2, n))$ in $H^*(MO_n)$, and in the remaining we look for the set of linear generators of the smallest degree, say $n_1$. These then induce a map from $MO_n$ to a product of copies of $K(\mathbb{Z}_2, n_1)$'s. Similarly we can show $H^*(\text{product of } K(\mathbb{Z}_2, n_1)\text{'s}) \to H^*(MO_n)$ is again injective in the stable range $n \leq * \leq 2n$.
- You know how the story goes now. Again we peel off those cohomology classes coming from product of Eilenberg-MacLane spaces and seek for the smallest degree ones in the remaining, and show that they induce injection from the cohomology of a product of Eilenberg-MacLane spaces in the stable range.
- Repeat doing so, we eventually get a map from $MO_n$ to a product of Eilenberg-MacLane spaces which induces isomorphism on cohomology in the stable range. Pass $n \to \infty$, we see that $H^*(MO)$ is free over mod 2 Steenrod algebra.

Let's carry out the computation in details.

Note that any class of the type $Sq^I(w_k)$, where the sequence $I$ is not necessarily admissible, looks like $w_k \cdot Q_I$, where $Q_I \in H^h(BO_n)$ is a polynomial of total weight $h$ with respect to $w_i$. One can see this using splitting principle $w_n = t_1 \cdots t_n$. For example $Sq^i(w_k) = Sq^i(t_1 \cdots t_n) = t_1 \cdots t_n \cdot Q_I = w_n \cdot Q_I$, and $Q_I$ is symmetric in $t_1, \dots, t_n$ hence a polynomial in $w_1, \dots, w_n$. More explicitly, $Sq^i(w_n) = w_n \cdot w_i$ for $i \leq n$.

Let us introduce on the set of monomials $w_i$ the lexicographic ordering $(R)$ by setting $w_k \prec w_l$ if $k < l$. For instance, $w_4 \prec w_4(w_1)^2 \prec w_4w_2w_1 \prec w_4w_3$.

Now let $Sq^I = Sq^{i_1}Sq^{i_2} \dots Sq^{i_r}$, where $I = (i_1, \dots, i_r)$ is admissible, and let $Sq^I(w_k) = w_k \cdot Q_I$. We examine the highest term of $Q_I$ with respect to $(R)$.

We claim that $Q_I = w_{i_1}w_{i_2} \dots w_{i_r}$+lower terms. For $r = 1$, $Q_{i_1} = w_{i_1}$. Assume this is true for $r - 1$, then

$$Sq^I(w_n) = Sq^{i_1}(Sq^{i_2} \dots Sq^{i_r}(w_n)) = Sq^{i_1}(w_n \cdot P),$$

where, by assumption $P$ looks like $w_{i_2}w_{i_3}\dots w_{i_r}$+lower terms. Hence

$$Sq^{i_1}(w_n \cdot P) = \sum_{0\le j\le i_1} Sq^j(w_n)Sq^{i_1-j}(P) = w_n \sum_j w_j Sq^{i_1-j}(P).$$

Consequently, $Q_I = \sum_{0\le j\le i_1} w_j Sq^{i_1-j}(P)$. To analyze the highest term, we need Wu formulae, which can be shown using splitting principle:

$$Sq^r(w_i) = \sum_t \binom{r-i+t-1}{t} w_{r-t}w_{i+t}.$$

In the sum $\sum_{0\le j\le i_1} w_j Sq^{i_1-j}(P)$, the term with $j = i_1$ looks like $w_{i_1}\cdot P = w_{i_1}w_{i_2}\dots w_{i_r}$+monomials of lower order in $(R)$. On the other hand, by Wu formulae, $Sq^{i_1-j}(P), j < i_1$ only contains those classes $w_i$ for which $i < 2i_2 \le i_1$. So $Q_I = w_{i_1}\dots w_{i_r}$+lower terms with respect to $(R)$.

Thus, we see that all classes $Sq^I(w_n)$, where $I$ is any admissible sequence of total degree $h$, are linearly independent in the group $H^{n+h}(BO_n)$. Indeed, if there were a non-trivial relation between these classes, then, taking the highest term with respect to $(R)$, which looks like $w_{i_1}w_{i_2}\dots w_{i_r}$ by our computation. We would then see that this term can be linearly expressed as a combination of strictly lower terms, which is impossible. This proves

**Lemma 3.6.** *The classes $Sq^I(t_1\dots t_n)$, where $I$ runs through the set of all admissible sequences of total degree $h\le n$, are linearly independent symmetric functions in $t_i$. Therefore, the map $MO_n \to K(\mathbb{Z}_2,n)$ corresponding to the Thom class induces an injection $H^*(K(Z_2,n);\mathbb{Z}_2)\to H^*(MO_n;\mathbb{Z}_2)$ in the stable range $n\le * \le 2n$.*

So we have found a copy of Steenrod algebra in the stable range of $H^*(MO_n)$ generated by the Thom class. Next I'm going to write down a set of generators for $H^*(MO_n)$ as Steenrod algebra module, and prove $H^*(MO_n)$ is freely generated by them.

For any number $h\le k$ consider the class

$$X^h_\omega = \sum_{S_\omega}(t_1)^{a_1+1}(t_2)^{a_2+1}\dots(t_r)^{a_r+1}t_{r+1}\dots t_n,$$

where $S_\omega$ is the set of essential permutations and $\omega = (a_1,\dots,a_r)$ is an arbitrary decomposition of $h$ into summands, with no summand of type $2^m-1$ (non-dyadic decomposition of $h$). Denote the number of such decomposition by $d(h)$.

**Lemma 3.7** (Key lemma)**.** *For any dimension $m\le n$, the following classes form a linear basis of $H^{n+m}(MO_n;\mathbb{Z}_2)$.*

$$X^m_{\omega_m}, Sq^1X^{m-1}_{\omega_{m-1}}, Sq^2X^{m-2}_{\omega_{m-2}},\dots, Sq^{I_h}X^h_{\omega_h},\dots,Sq^I w_n,$$

*where $Sq^{I_h}$ is an admissible of total degree $(m-h)$, and $\omega_h$ is a non-dyadic decomposition of $h$.*

Using the Lemma, we can prove

**Theorem 3.8** (Thom)**.** *$MO_n$ has the same homotopy $2n$-type as*

$$K(\mathbb{Z}_2,n)\times K(\mathbb{Z}_2,n+2)\times\cdots\times(K(\mathbb{Z}_2,n+h))^{d(h)}\times\cdots\times(K(\mathbb{Z}_2,2n))^{d(n)},$$

*where $d(h)$ is the number of non-dyadic decompositions of $h$.*

*Proof.* Each class $X^h_\omega$ corresponds to a mapping $F_\omega : MO_n \to K(\mathbb{Z}_2, n+h)$. Take the product of these we get $F : MO_n \to Y = K(\mathbb{Z}_2, n) \times K(\mathbb{Z}_2, n+2) \times \cdots \times (K(\mathbb{Z}_2, n+h))^{d(h)} \times \cdots \times (K(\mathbb{Z}_2, 2n))^{d(n)}$ induces isomorphism $F^* : H^*(Y) \to H^*(MO_n)$ in the stable range. By a standard obstruction theoretical argument, □

**Corollary 3.9.** *The stable homotopy group $\pi_h(MO) \simeq \Omega^O_h$ is isomorphic to the direct sum of $d(h)$ groups $\mathbb{Z}_2$.*

*Proof.* There's only 2-torsion in $\pi_*(MO)$ and Adams spectral sequence at 2 collapses on $E_1$-page. □

Now we must prove the key lemma. As before, to show the classes

$$X^m_{\omega_m}, Sq^1 X^{m-1}_{\omega_{m-1}}, Sq^2 X^{m-2}_{\omega_{m-2}}, \ldots, Sq^{I_h} X^h_{\omega_h}, \ldots, Sq^I w_n,$$

are linearly independent, we need to define a quasi-order relation.

**Definition 3.10.** *Let $P$ be an arbitrary polynomial in variables $t_i$. A variable $t_i$ is a dyadic variable for $P$, if the exponent of the variable in $P$ is either zero or a power of 2.*

**Definition 3.11.** *By a non-dyadic factor of the monomial $(t_1)^{a_1}(t_2)^{a_2}\ldots(t_r)^{a_r}$ we mean the monomial consisting of all non-dyadic variables; denote the number of the non-dyadic factors by $\nu$. For the set of monomials in $(t_i)$ variables we define a quasi-order relation $(Q)$ as follows: a monomial $X$ is greater than the monomial $Y$ with respect to $(Q)$ if $u(X) > u(Y)$ or if $u(X) = u(Y)$ and $\nu(X) < \nu(Y)$.*

The motivation for the above definition is the following observation: any variable $t_k$, which is dyadic for $P$, is dyadic for $Sq^i(P)$ as well. Indeed, since $Sq^a(t_k)^m = \binom{m}{a}(t_k)^{m+a}$ (one can see this by induction on $a+m$), if $m$ is a power of two, then $\binom{m}{a}$ is congruent to zero except for $a = 0, m$.

*Proof of Key lemma.* For any number $h \le n$ let's consider the classes

$$X^h_\omega = \sum (t_1)^{a_1+1}(t_2)^{a_2+1}\ldots(t_r)^{a_r+1} t_{r+1}\ldots t_n,$$

where $\omega = a_1, a_2, \ldots, a_r$ is a non-dyadic decomposition of $h$. Note that $u(X^h_\omega) = r$ and $\nu(X^h_\omega) = h + r$. After applying $Sq^I$ to $X^h_\omega$, the dyadic variables stays dyadic, so the index $u$ of the monomials in $Sq^I(X^h_\omega)$ is essentially less or equal to $u(X^h_\omega)$. Thus the leading term in $Sq^I(X^h_\omega)$ looks like

$$\sum (t_1)^{a_1+1}(t_2)^{a_2+1}\ldots(t_r)^{a_r+1} \cdot Sq^I(t_{r+1}\ldots t_n).$$

Hence any non-trivial linear dependency between

$$X^m_{\omega_m}, Sq^1 X^{m-1}_{\omega_{m-1}}, Sq^2 X^{m-2}_{\omega_{m-2}}, \ldots, Sq^{I_h} X^h_{\omega_h}, \ldots, Sq^I w_n,$$

arises from non-trivial linear dependency between those classes $Sq^I X^h_\omega$, whose leading terms are of the same index $u = r$ and of the same index $\nu = r+h$. Furthermore, the decompositions $\omega$ of $h$ of these $X^h_\omega$'s, for which this linear dependence hold, should be the same to make sure the $Sq^I(X^h_\omega)$ have the same non-dyadic factors. Therefore, any linear dependence is of the type $\sum_\lambda c_\lambda Sq^{I_\lambda} X^h_\omega = 0$, containing only one class $X^h_\omega$.

Let's write down the $(Q)$-leading terms of this relation:

$$\sum_\lambda c_\lambda (t_1)^{a_1+1}(t_2)^{a_2+1}\ldots(t_r)^{a_r+1} Sq^{I_\lambda}(t_{r+1}\ldots t_n) = 0.$$

All terms of this relation containing a fixed factor $(t_1)^{a_1+1}(t_2)^{a_2+1}\dots(t_r)^{a_r+1}$ should sum to zero. Thus,

$$\sum_{\lambda} c_\lambda Sq^{I_\lambda}(t_{r+1}\dots t_n) = 0.$$

But $|I_\lambda| = m - h \leq n - r$ for $m \leq n$ and $h \geq 2r$, and $I_\lambda$ is admissible, by Lemma 3.5 the coefficients $c_\lambda$ are equal to zero.

Finally, by a direct dimension counting, the linearly independent classes

$$X^m_{\omega_m}, Sq^1 X^{m-1}_{\omega_{m-1}}, Sq^2 X^{m-2}_{\omega_{m-2}}, \dots, Sq^{I_h} X^h_{\omega_h}, \dots, Sq^I w_n,$$

do form a base for $H^{n+m}(MO_n;\mathbb{Z}_2)$. The rank of $H^{n+m}(MO_n)$ is equal to the total number $p(m)$ of decompositions of $m$ into summands. On the orther hand, the number of the above classes is equal to $\sum_{h\leq m} c(m-h)d(h)$. It is easy to see that

$$p(m) = \sum_{h\leq m} c(m-h)d(h).$$

Indeed, to each decomposition of $m$ there correspond two decompositions: the decomposition of $h$, consisting of summands of the type $2^k-1$ and the decomposition of $h$, consisting of the remaining summands. $\square$