A non-spin^c open 5-manifold whose compact submanifolds are all spin^c

Jiahao Hu

Let E be a rank 3 oriented real vector bundle over $\mathbb{CP}^1$ with $w_2(E) \neq 0$. For example, one can take E to be $\mathcal{O}(1) \oplus \mathbb{R}$ where $\mathcal{O}(1)$ is the dual to the tautological complex line bundle over $\mathbb{CP}^1$ and $\mathbb{R}$ denotes the trivial real line bundle. Let $\mathcal{E}$ be the total space of E and let i denote the embedding of $\mathbb{CP}^1$ into $\mathcal{E}$ as the zero-section. Next choose a degree 3 map $f : \mathbb{CP}^1 \to \mathbb{CP}^1$. Then since $\mathcal{E}$ is 5-dimensional, the composition

$$\mathbb{CP}^1 \xrightarrow{f} \mathbb{CP}^1 \xrightarrow{i} \mathcal{E}$$

can be approximated by a smooth embedding $j : \mathbb{CP}^1 \hookrightarrow \mathcal{E}$.

Lemma 1. The normal bundle to j is isomorphic to E.

Proof. Note that rank 3 orientable real vector bundles over $\mathbb{CP}^1$ are classified by their second Stiefel-Whitney classes since $\pi_2(BSO_3) = \mathbb{Z}_2$. So to show the normal bundle to j is isomorphic to E, it suffices to show its second Stiefel-Whitney class is non-zero. Since $w_2(\mathbb{CP}^1) = 0$, we have that the second Stiefel-Whitney class of the normal bundle to j coincides with

$$j^* w_2(\mathcal{E}) = f^* i^* w_2(\mathcal{E}) = i^* w_2(\mathcal{E}) = w_2(E).$$

Here the second equality follows from that $f^* : H^2(\mathbb{CP}^1; \mathbb{Z}_2) \to H^2(\mathbb{CP}^1; \mathbb{Z}_2)$ is an isomorphism since f is of degree 3. □

Now by choosing a tubular neighborhood of $j(\mathbb{CP}^1)$ in $\mathcal{E}$, we obtain a smooth embedding

$$g : \mathcal{E} \hookrightarrow \mathcal{E}$$

that satisfies:

(i) $g_* : H_2(E; \mathbb{Z}) \to H_2(E; \mathbb{Z})$ is isomorphic to $\mathbb{Z} \xrightarrow{\times 3} \mathbb{Z}$.

(ii) $g_* : H_2(E; \mathbb{Z}_2) \to H_2(E; \mathbb{Z}_2)$ is an isomorphism.

Define $M_i := \mathcal{E}$ for $i = 1, 2, 3, \dots$ and regard M_i as a submanifold of M_{i+1} through the embedding g. Let M be the union of the M_i's.

Lemma 2. $H^2(M; \mathbb{Z}) = 0$.

Proof. Since every M_i is simply-connected, so is M and in particular $H_1(M; \mathbb{Z}) = 0$. The second homology of M is $\varinjlim_i H_2(M_i; \mathbb{Z}) = \mathbb{Z}[1/3]$. Then by universal coefficient theorem, we have a short exact sequence

$$0 \to \mathrm{Ext}(H_1(M; \mathbb{Z}), \mathbb{Z}) \to H^2(M; \mathbb{Z}) \to \mathrm{Hom}(H_2(M; \mathbb{Z}), \mathbb{Z}) \to 0.$$

Since both $H_1(M; \mathbb{Z})$ and $\mathrm{Hom}(H_2(M; \mathbb{Z}), \mathbb{Z})$ vanish, we have $H^2(M; \mathbb{Z}) = 0$. □

Corollary 3. M is not spin^c.

Proof. Recall a manifold is spin^c if and only if its second Stiefel-Whitney class admits an integral lift. Now $w_2(M) \neq 0$ because the restriction of $w_2(M)$ to M_i is $w_2(M_i) \neq 0$. But $H^2(M; \mathbb{Z}) = 0$, therefore $w_2(M)$ admits no integral lift. □

Proposition 4. Every compact submanifold of M admits a spin^c structure.

Proof. Every compact submanifold of M is contained in M_i for some i. Since each M_i is spin^c, so are its submanifolds. □

Remark 5. The construction of this example fails in dimension 4 because every orientable 4-manifold, compact or not, is spin^c. This fact plays a fundamental role in Seiberg-Witten theory. From an alternative point of view, if one takes $\mathcal{E}$ to be the total space of $\mathcal{O}(1)$, then the composition

$$\mathbb{CP}^1 \xrightarrow{f} \mathbb{CP}^1 \xrightarrow{i} \mathcal{E}$$

can never be approximated by an embedding because now $\mathcal{E} = \mathbb{CP}^2 - \mathrm{pt}$ and the existence of such an embedding will violate Thom conjecture (which is proved using Seiberg-Witten theory).