

A JORDAN CURVE THAT CANNOT BE CROSSED BY RECTIFIABLE ARCS ON A SET OF ZERO LENGTH

JACK BURKART

ABSTRACT. We construct a Jordan curve $\Gamma \subset \mathbb{C}$ so that for any rectifiable arc σ with endpoints in distinct complementary components of Γ , $H^1(\sigma \cap \Gamma) > 0$.

1. INTRODUCTION AND MOTIVATION

A Jordan curve is an injective image of the unit circle in the plane. The Jordan Curve Theorem asserts that a Jordan curve separates the plane into exactly two complementary components, a bounded interior component and an unbounded exterior component. This gives Jordan curves rather simple topology, but their geometry can be exotic. We say an arc *crosses* a Jordan curve Γ if its endpoints belong in different complementary components of Γ . Let H^1 denote the Hausdorff 1-measure. In this paper, we prove the following theorem, answering a question posed by Sauter in [Sau18].

Theorem 1.1. *There exists a Jordan curve Γ so that if σ is a rectifiable (finite length) arc crossing Γ , $H^1(\sigma \cap \Gamma) > 0$.*

We would like to make some more general remarks about Theorem 1.1. It is a straightforward calculation from the construction that Γ will have positive area, but a simple argument using Fubini's theorem and integrating over a family of line segments shows that any Γ satisfying the conclusion of Theorem 1.1 must have positive area.

By the Riemann mapping theorem and Caratheodory's theorem, any Jordan curve can be crossed by an arc of σ -finite length that intersects the Jordan curve at exactly one point. Such an arc is analytic everywhere except possibly at the point where it crosses the boundary. In this sense, the hypothesis of rectifiability is the weakest hypothesis on σ so that the conclusion of Theorem 1.1 will hold.

Theorem 1.1 has been addressed in other contexts, both in the class of arcs σ considered and the size of the intersection of $\sigma \cap \Gamma$. Examples of Jordan curves Γ that cannot be crossed by line segments at only one point seem well known, see for example [Pet]. In [Bis], Bishop constructs a Jordan curve so that if σ is a line segment crossing Γ , the Hausdorff dimension of $\sigma \cap \Gamma$ is 1, but the case of positive length is not addressed. Motivated by questions about ordinary differential

equations, in [PW12], Pugh and Wu show that Jordan curves that cannot be crossed by rectifiable arcs at exactly one point exist and are actually generic (in a Baire sense) in the space of Jordan curves in the plane (equipped with an appropriate metric).

The curve Γ also has connections to conformal welding. If Γ is a Jordan curve in the plane, let Ω denote its bounded complementary component. Then there are conformal mappings $f : \mathbb{D} \rightarrow \Omega$ and $g : \mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow \mathbb{C} \setminus \overline{\Omega}$ which induce a circle homeomorphism $h : g \circ f^{-1} : \partial\mathbb{D} \rightarrow \partial\mathbb{D}$ which we call a *conformal welding*. Γ is called *log-singular* if there is a Borel set E so that E and $h(S^1 \setminus E)$ both have zero logarithmic capacity (for the definition of logarithmic capacity, see Chapter III of [GM05]). Results of Buerling in [Beu40] show that if $\varphi : \mathbb{D} \rightarrow \Omega$ is a conformal mapping onto a Jordan domain Ω , and $\zeta \in S^1$.

$$\int_0^1 |\varphi'(r\zeta)| dr < \infty$$

except perhaps on a set of zero logarithmic capacity (see also 23(a), p. 127, in [GM05]). It follows that outside of this exceptional set, the image of the hyperbolic geodesic rays $[0, \zeta)$ have finite length. Applying this result to both f and g , it follows from Theorem 1.1 that Γ is log-singular. Otherwise, by using a pair of two finite length geodesics with the same endpoint on Γ , we could construct a finite length arc crossing Γ at exactly one point. Our example gives the first explicit construction of a log singular curve. For more on log-singular curves and conformal welding, see [Bis94] and [Bis07].

We will construct the curve $\Gamma = \bigcap_{n=1}^{\infty} \Gamma_n$ in Theorem 1.1 as a nested intersection of axis aligned tubes which we call *plumbings*. Given a plumbing, we will use a Lakes of Wada type of construction to construct a new plumbing which weaves back and forth inside of itself, see Figure 1. For a rectifiable curve σ that passes from the interior boundary component to the exterior boundary component of Γ , we let $l_n = H^1(\Gamma_n \cap \sigma)$. In order for l_{n+1} to be much smaller than l_n , σ must weave back and forth horizontally so that it can pass through the vertical gaps, as seen in Figure 1. We will tune the parameters defining the plumbings Γ_n so that σ can only pass through an increasingly smaller percentage of the vertical gaps of Γ_n as $n \rightarrow \infty$. It will follow that most of the time, the connected components of $\sigma \cap \Gamma_{n+1}$ must pass through those components approximately as a straight line segment, and we will show directly that a straight line segment intersects Γ with a set of positive length. These observations combined will allow us to prove Theorem 1.1.

The structure of the paper is as follows. In Section 2, we define all the necessary terminology and reduce Theorem 1.1 to the case of rectifiable curves with length 1. In Section 3, we carefully define plumbings and discuss their properties. In

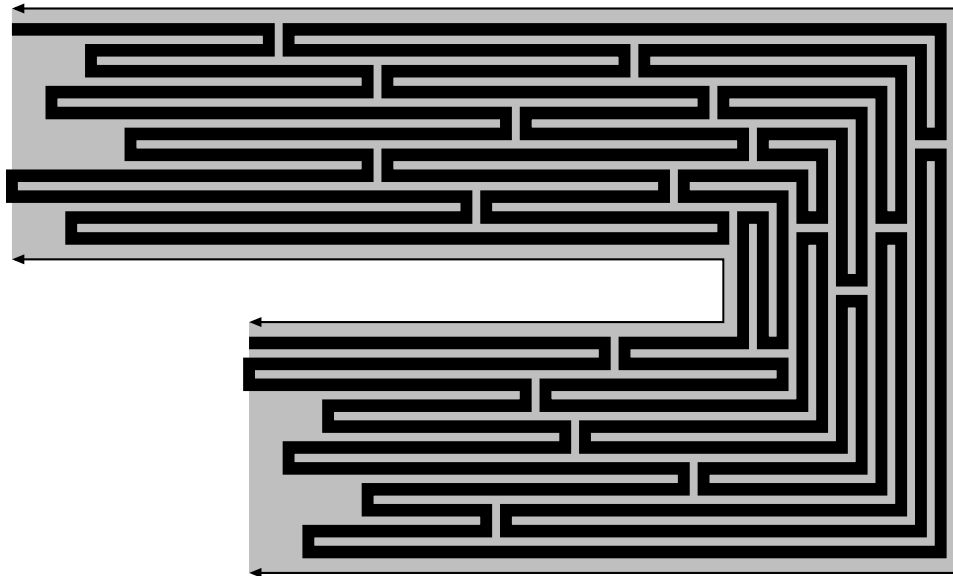


FIGURE 1. An example of the procedure taking a plumbing Γ_n (pictured in gray) to a new plumbing Γ_{n+1} (pictured in black). For a rectifiable curve to intersect Γ on a set of zero length, it must intersect Γ_n on a set of length which tends to 0 as $n \rightarrow \infty$. This requires σ to weave back and forth too much in comparison to the width of the rectangles that define the plumbings, and this will cost a large amount of length if this happens too often.

Sections 4 and 5, we show how to create new plumbings from old to create the limiting Jordan curve Γ . In Section 6, we show that the Jordan curve cannot be crossed at a single point by rectifiable curves, and in Section 7 we refine this to show the intersection of the rectifiable curve with the Jordan curve has positive area.

2. PRELIMINARIES AND GENERALITIES

A *curve* is a continuous function $\sigma : [0, 1] \rightarrow \mathbb{C}$. If σ is a curve with $\sigma(0) = \sigma(1)$ we call σ *closed*. We will sometimes refer to curves that are not closed as *arcs*. A curve σ is called *simple* if for all $t_1 \neq t_2 \in [0, 1]$, we have $\sigma(t_1) \neq \sigma(t_2)$. Similarly, we call a closed curve simple if for all $t_1 \neq t_2 \in [0, 1)$, we have $\sigma(t_1) \neq \sigma(t_2)$. A *Jordan curve* is a simple closed curve. The image $\sigma([0, 1]) \subset \mathbb{C}$ is the *trace* of σ . When it will not cause confusion, we will refer the trace of a curve σ and the curve σ interchangeably. Recall that the *length* of a curve σ is defined to be

$$l(\sigma) = \sup_{0=t_0 \leq t_1 \leq \dots \leq t_n=1} \sum_{k=1}^n |\sigma(t_k) - \sigma(t_{k-1})|.$$

If $l(\sigma) < \infty$, we call the curve or arc σ *rectifiable*.

Definition 2.1. Let $\Gamma \subset \mathbb{C}$ be a Jordan curve. We say that an arc σ *crosses* Γ if $\sigma(0)$ and $\sigma(1)$ are in distinct complementary components of Γ .

If σ crosses Γ , we are interested in the subset of points where σ passes from one complementary component of Γ to the other.

Definition 2.2. We say that σ *pierces* Γ at x if for every $\epsilon > 0$, $B(x, \epsilon) \cap \sigma$ contains points in both complementary components of Γ . If σ crosses Γ , the *piercing set* of σ , $\mathcal{P}(\sigma)$, is the nonempty set of all points $x \in \Gamma$ where σ pierces Γ at x . Γ is called *pierceable* by rectifiable curves if there exists a rectifiable curve σ crossing Γ so that $\mathcal{P}(\sigma)$ is exactly one point. Otherwise Γ is *unpierceable* by rectifiable curves.

The definition of pierceability can be easily adjusted to include other families of curves, such as line segments, simple rectifiable curves, and rectifiable curves of some given length. If Γ is unpierceable by rectifiable curves, the piercing set is nontrivial.

Lemma 2.3. *Let Γ be a Jordan curve. The following are equivalent:*

- (1) Γ is unpierceable by rectifiable arcs.
- (2) Γ is unpierceable by simple rectifiable arcs.
- (3) Γ is unpierceable by simple rectifiable arcs with length ≤ 1
- (4) For any simple rectifiable σ crossing Γ , $\mathcal{P}(\sigma)$ is uncountable.
- (5) For any rectifiable σ crossing Γ , $\mathcal{P}(\sigma)$ is uncountable.

Proof. It is easy to see that (1) \Rightarrow (2) \Rightarrow (3) and obviously (5) \Rightarrow (1). (4) \Rightarrow (5) because if σ is a rectifiable arc, there exists a simple rectifiable arc σ' which is a subset of the trace of σ with the same endpoints. So it is sufficient to show that (3) \Rightarrow (4).

First observe that $\mathcal{P}(\sigma)$ cannot have any isolated points, otherwise there exists a sub-curve $\sigma' \subset \sigma$ in a neighborhood of the isolated point that pierces Γ at exactly one point. Since closed countable sets must have isolated points, it will be sufficient to show that $\mathcal{P}(\sigma)$ is closed.

Let Ω_1 be the bounded complementary component of Γ , and Ω_2 be the unbounded complementary component of Γ . Write $\sigma : [0, 1] \rightarrow \mathbb{C}$. Let $A = \sigma^{-1}(\Omega_1)$ and $B = \sigma^{-1}(\Omega_2)$. A and B are disjoint open subsets of $[0, 1]$. We claim that

$$(2.1) \quad \mathcal{P}(\sigma) = \sigma(\overline{A} \cap \overline{B}).$$

If $x \in \mathcal{P}(\sigma)$, there exists sequences $z_n \subset \sigma \cap \Omega_1$ and $w_n \subset \sigma \cap \Omega_2$ so that $z_n \rightarrow x$ and $w_n \rightarrow x$. Since σ is simple, if $\sigma(t) = x$, there are corresponding sequences t_n and t'_n so that $z_n = \sigma(t_n)$, $w_n = \sigma(t'_n)$, and both $t_n, t'_n \rightarrow t$. So $t \in \overline{A} \cap \overline{B}$, and therefore $\sigma(t) = x \in \sigma(\overline{A} \cap \overline{B})$.

If $x \notin \mathcal{P}(\sigma)$, then there exists an $\epsilon > 0$ so that either $B(x, \epsilon) \cap \sigma$ does not contain any points in Ω_1 , or $B(x, \epsilon) \cap \sigma$ does not contain any points in Ω_2 . If $B(x, \epsilon) \cap \sigma$ does not contain any points in Ω_1 , and if $\sigma(t) = x$, then there exists a $\delta > 0$ so that $\sigma(t - \delta, t + \delta) \subset B(x, \epsilon/2)$ which implies that $\sigma(t - \delta, t + \delta) \cap \Omega_1$ is empty. It follows that $t \notin \overline{B}$, and since σ is simple it follows that $x \notin \sigma(\overline{A} \cap \overline{B})$. The argument is exactly the same if $B(x, \epsilon) \cap \sigma$ does not contain any points in Ω_2 . Therefore, (2.1) holds.

Since $\overline{A} \cap \overline{B}$ is compact and σ is continuous, $\mathcal{P}(\sigma)$ is compact and therefore must be closed. This proves the claim. □

In fact, it is not difficult to see additionally that $\mathcal{P}(\sigma)$ is a Cantor set: compact, uncountable, totally disconnected, and no isolated points.

3. PLUMBING

The curve Γ we construct will be the nested intersection of a sequence of topological annuli which we call *plumbings*. We first define the two basic pieces that determine a plumbing. See Figure 2.

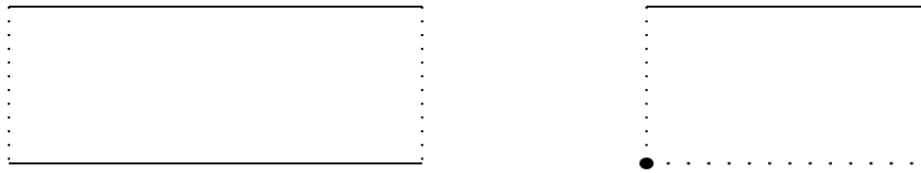


FIGURE 2. Plumbings are constructed by alternating straight pieces and corner pieces, and gluing them together by arc length. The boundary sides are filled in and the openings are denoted by the dotted lines.

To form a *straight piece*, start with a rectangle with sides parallel to the coordinate axes, and remove two of the opposite sides. We call the remaining sides the *boundary sides* of the straight piece and the sides we removed the *openings* of the straight piece. We call the boundary side that is a subset of the interior boundary component of the plumbing the *top* of the straight piece, and the boundary side which is a subset of the exterior boundary component of the plumbing the *bottom* of the straight piece. Whenever we need to analyze a specific straight piece, we will orient our point of view to justify these names.

Corner pieces are formed by taking an axis aligned rectangle and removing two adjacent sides, but keeping the mutual vertex of the two removed sides. This vertex is called an *inner corner* for the corner piece, and the remaining sides are similarly called *boundary sides* and *openings*. Plumbings are formed by alternating straight

and corner pieces and gluing together their openings by arc length. See Figure 3 for an example of a plumbing.

Definition 3.1. A *plumbing* P is a closed topological annulus obtained by gluing together straight pieces and corner pieces so that between every two corner pieces there exists exactly one straight piece.

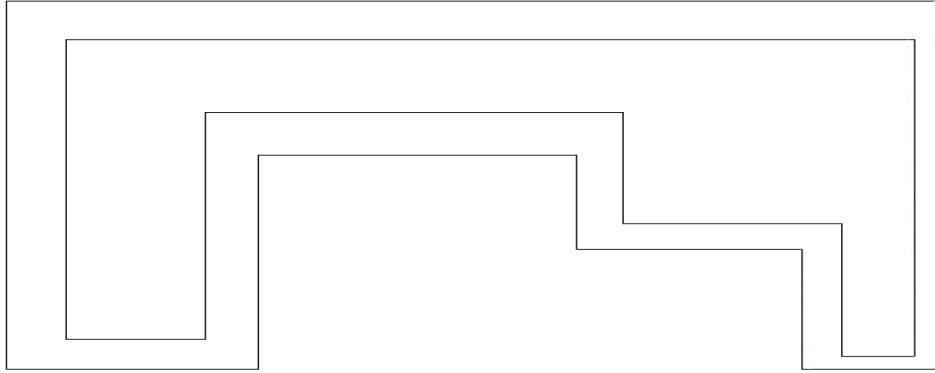


FIGURE 3. An example of plumbing.

We define the following geometric quantities associated to a plumbing. The *width* of a straight piece is defined to be the distance between its boundary sides. The *width* of a plumbing is defined to be the maximum of the widths of the straight pieces in the plumbing. The *length* of a straight piece is the distance between its openings. The *min-length* of a plumbing is the minimum of the lengths of its straight pieces.

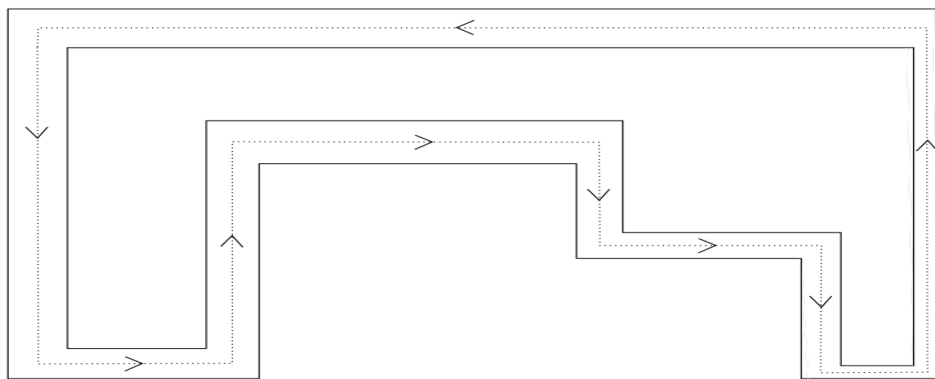


FIGURE 4. A plumbing with its core curve in the middle, oriented counterclockwise.

Plumbings come with a natural foliation of axis aligned polygonal curves that follow the inner and outermost boundary. Start with a plumbing P , and let γ_0

and γ_1 denote the outermost and innermost boundary components, respectively. Decompose P into its straight and corner pieces. Given a straight piece S and a number $t \in (0, 1)$, let w denote its width. Let s_t denote the line segment parallel to the boundary sides of S with distance $t \cdot w$ from γ_0 . Draw this line segment for all straight pieces in P . Any corner piece is adjacent to two straight pieces, and each of its openings contains an endpoint of some s_t and some s'_t . Continue the segments s_t and s'_t until they intersect (this point of intersection will be on the diagonal that separates the openings of the corner piece) and repeat this for all corner pieces to form a Jordan curve γ_t . We define $\gamma_{1/2}$ to be the *core curve* of P . We call $\{\gamma_t\}$ the *plumbing foliation* of P . Whenever we parameterize the elements of a plumbing foliation, we do so counterclockwise, so that the inner boundary component is always to the left of the curve in the foliation. See Figure 4.

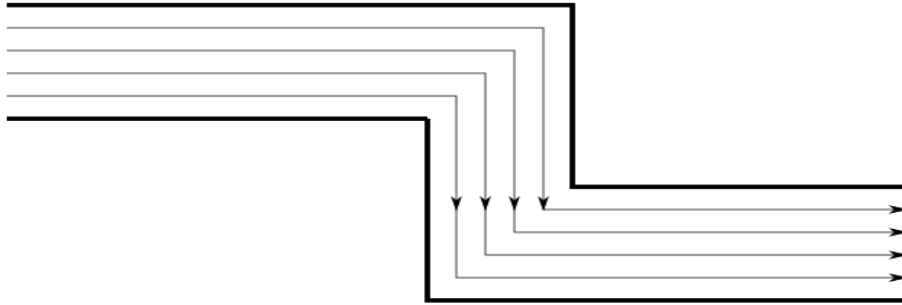


FIGURE 5. An illustration of some elements of the plumbing foliation, with orientation.

A point on the core curve of P where the core curve changes from a vertical segment to a horizontal segment or vice versa is called a *bend point*. Two bend points are called *adjacent* if there is no bend point between them. Every bend point is adjacent to exactly two other bend points. Let z and w be two adjacent bend points of P , so that the core curve passes through z first. The (z, w) -*junction* is the union of the two corner pieces containing z and w , and the straight piece that is between z and w . Each (z, w) junction takes two possible forms, depending on how the openings of the corner pieces are configured. We call the junctions *U-junctions* and *Z-junctions*, respectively. When considered with respect to the orientation of the core curve, there are four junctions to consider. See Figure 6.

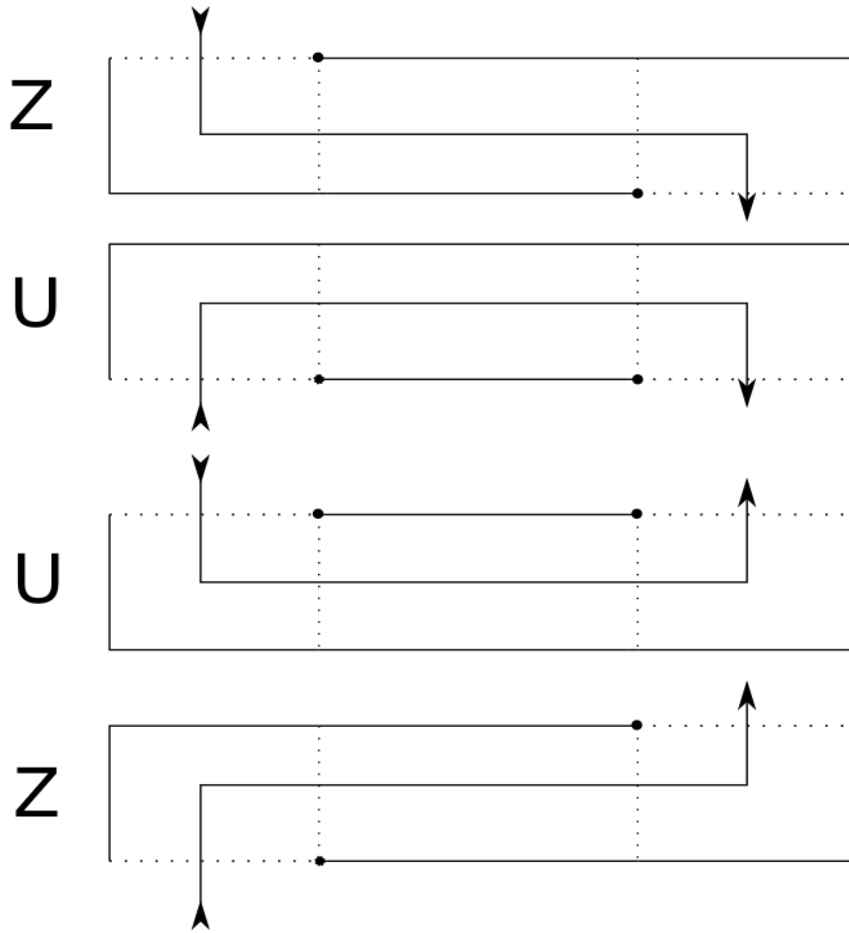


FIGURE 6. Accounting for orientation, there are two Z-junctions and two U-junctions. The pictures have been oriented so that the top boundary side of the straight piece is a subset of the innermost boundary of the plumbing. In fact, since a plumbing is axis aligned, we can always orient our view of a straight piece or junction in this way.

4. CREATING NEW PLUMBING FROM OLD

In this section we define a procedure which takes a plumbing Γ_n and creates a plumbing Γ_{n+1} contained inside of Γ_n taking up most of the area of Γ_n , but having a much smaller width. Roughly put, we will insert very thin vertical and horizontal rectangles segments inside of Γ_n to form a new plumbing. This procedure will form the basis of our construction.

Let Γ_n be a plumbing. Define

$$(4.1) \quad d_n := \min\{l_n, w_n\}$$

where l_n is the min-length of Γ_n and w_n is the width of Γ_n . Subdivide Γ_n into straight pieces and corner pieces, and denote the plumbing foliation of Γ_n by $\{\gamma_t^n\}$. Parameterize $\gamma_{1/2}^n$ as a Jordan curve, say, by arc length. Denote P_0 as the unique plumbing piece of Γ_n so that $\gamma_{1/2}^n(0) \in P_0$ and so that there exists $\delta > 0$ so that $\gamma_{1/2}^n : [0, \delta) \rightarrow P_0$. Label the rest of the plumbing pieces $\{P_i\}_{i=0}^{m-1}$ according to the order $\gamma_{1/2}^n(t)$ passes through them. We will consider this list of pieces modulo m , so that $P_m = P_0$.

Choose some $\delta_n \in (0, 1/100)$. We are going to subdivide the straight pieces P_i further into rectangles by connecting the boundary sides by line segments placed distance approximately $\delta_n d_n$ apart; no line segment will be farther than $2\delta_n d_n$ from the adjacent line segments we place, and no line segment will be closer than $1/2\delta_n d_n$ from the adjacent line segments we place. We will simply refer to these new rectangles as *subdivided rectangles*. If P_i is the straight piece of a U -junction, we must make sure that the number of subdivided rectangles of P_i is even. If P_i is an element of a Z -junction, the number of subdivided rectangles should be odd. See Figure 7.

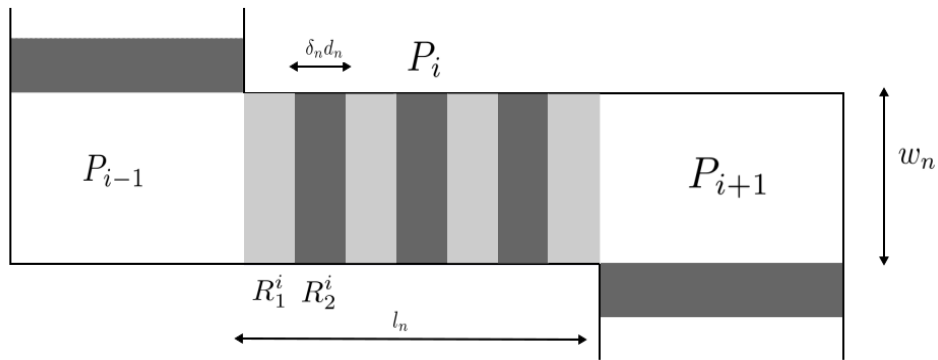


FIGURE 7. Illustrated above is a Z -junction, with pieces P_{i-1}, P_i and P_{i+1} . Since the straight piece is in a Z -junction, we subdivide P_i into an odd number of rectangles. Since the boundary sides of P_{i-1} extend to the bottom boundary side of P_i , the first subdivided rectangle will be labeled as ‘T’. The light gray color coincides with rectangles labeled as ‘T’ and the dark gray color coincides with rectangles labeled as ‘B’. In practice, δ_n is chosen so that P_i is broken into at least 100 rectangles.

Given any $t_n \in \mathbb{N}$, we select the elements of the plumbing foliation $\{\gamma_{j/t_n}^n\}_{j=0}^{t_n}$. We will always assume that t_n is an odd number, so that we are selecting $t_n + 1$ equally

spaced elements of the foliation. If w_n is the width of Γ_n , then adjacent elements of the plumbing foliation inside of a straight piece are no more than distance $\frac{w_n}{t_n}$ apart.

For each straight piece P_i , we want to assign a labels T and B for its subdivided rectangles.

- (1) If P_{i-1} 's boundary side extends the bottom boundary side of P_i , the first subdivided rectangle is marked as T.
- (2) If P_{i-1} 's boundary side extends the upper boundary side of P_i , the first subdivided rectangle is marked as B.

We alternate T and B until we reach the end of the straight piece.

Given a straight piece P , we have $P = \cup R_i$, where R_i are the subdivided rectangles of P described above with base approximately $\delta_n d_n$ and with height no greater than w_n . Choose some positive integer $v_n = k^2$ for some $k \geq 3$ and then choose some subdivided rectangle R_i . Decompose the base of R_i into v_n many equal length segments. We will need the following definition and combinatorial fact.

Definition 4.1. Split the unit square a v_n by v_n square grid. We define a *rook placement* to be a selection of exactly v_n many squares, which we call *rooks*, in the grid so that:

- (1) Each row in the grid contains exactly one rook.
- (2) Each column in the grid contains exactly one rook.

The following lemma is simple, and we leave its proof to the reader, with Figure 8 as a hint. A rook placement satisfying the conditions of Lemma 4.2 is called a *good v_n -rook placement*.

Lemma 4.2. *Let $v_n = k^2$ for some integer $k \geq 3$, and suppose the unit square has been decomposed into a v_n by v_n square grid. Identify the sides of the unit square to create a torus. Then there exists a rook placement of the unit square so that for every rook Q , the 8 adjacent squares, viewed on the torus, do not contain any other rooks in the rook placement.*

Suppose that R_i is a subdivided rectangle of some straight piece, and that R_i is designated as T. The plumbing foliation cuts R_i into t_n many rectangles. Orienting our point of view and counting downwards from the boundary side intersecting the interior complementary component, we call the first $2v_n$ many rectangles a *v_n -slab* inside R_i . A v_n -slab decomposes into a v_n by v_n grid. Rows are determined by every two elements of the plumbing foliation, and the columns are determined by the v_n equally spaced points on the base of R_i . We choose a good v_n -rook placement for this grid. For each rook in this good v_n -rook placement, place a vertical line segment connecting the top and bottom of the rook through the middle. This segment has

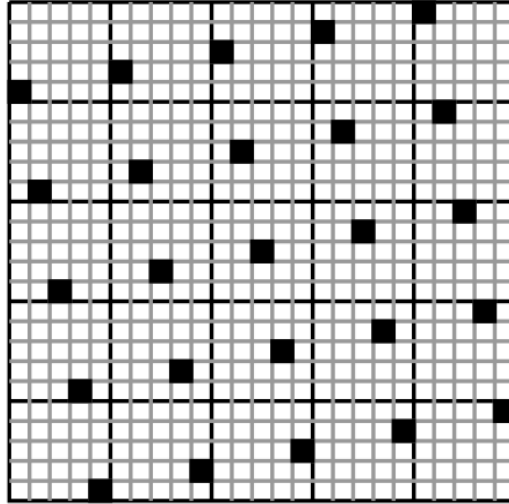


FIGURE 8. A good v_n -rook placement for $v_n = 5^2$.

endpoints on two elements of the plumbing foliation for Γ_n that determine the rows, and passes through exactly one foliation element between the two. See Figure 9.

We can split the rest of R_i into v_n slabs, going from the top down and repeating the rook placement of the top v_n -slab until we reach the element γ_{1/t_n}^n of the plumbing foliation. We can choose t_n to be larger if we need to so that the final v_n -slab has a bottom side determined by γ_{1/t_n}^n ; we will always assume that t_n satisfies this property.

One v_n slab forms a rectangle with base length approximately $\delta_n d_n$ and height $2v_n w_n / t_n$. In our construction, the height $2v_n w_n / t_n$ will be much smaller than $\delta_n d_n$. The amount of v_n slabs we need to stack to form an approximate $\delta_n d_n$ by $\delta_n d_n$ v_n -square is approximately k_n , where,

$$\frac{2v_n w_n}{t_n} \cdot k_n = \delta_n d_n.$$

Or,

$$(4.2) \quad k_n = \frac{\delta_n d_n t_n}{2v_n w_n}.$$

We also can see that R_i decomposes into approximately

$$(4.3) \quad \lambda_n = \frac{w_n}{\delta_n d_n}$$

many v_n -squares.

We do the same thing with the rectangles labeled B , except instead of building v_n -slabs starting from the top down, we build v_n -slabs starting from the bottom

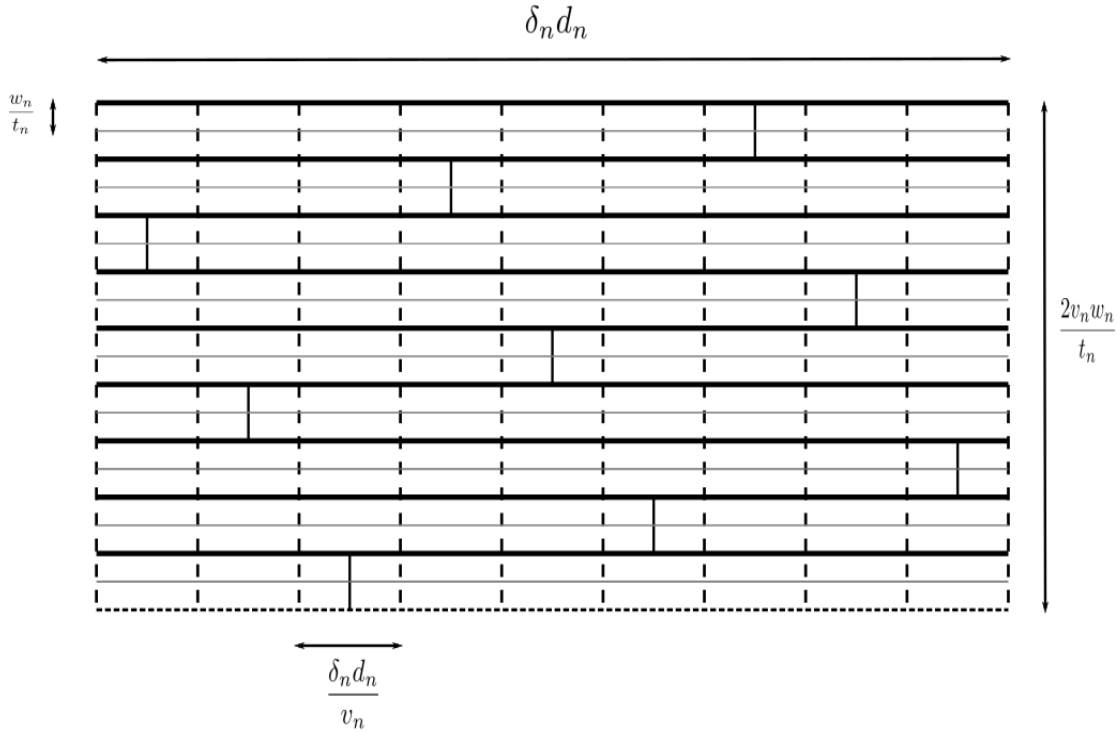


FIGURE 9. A good v_n -rook placement for a v_n -slab. Each combinatorial square has base $\delta_n d_n / v_n$ and height $2w_n / t_n$. The v_n -slab the rook placement is inside has width $\delta_n d_n$ and height $2v_n w_n / t_n$. The foliation elements that determine the rows of the v_n slab are colored darker than the ones that are skipped and that the vertical segments intersect.

up. We do not place any segments connecting elements of the plumbing foliation in corner pieces.

We now draw line segments along the plumbing foliation. In corner pieces, we add the intersection of all elements in the plumbing foliation with the corner piece. See Figure 10. In straight pieces, we do the same thing, except we remove the portion of the plumbing foliation within distance w_n / t_n of the center of vertical line segments we placed in the previous step. See Figure 14.

With the procedure above of adding in vertical and horizontal line segments into straight and corner pieces, we have constructed a topological subannulus contained inside of Γ_n . The only issue remaining is that its boundary components are not Jordan curves. To do this, we will thicken all of the segments in the horizontal and vertical directions by a small amount so that the segments we added become

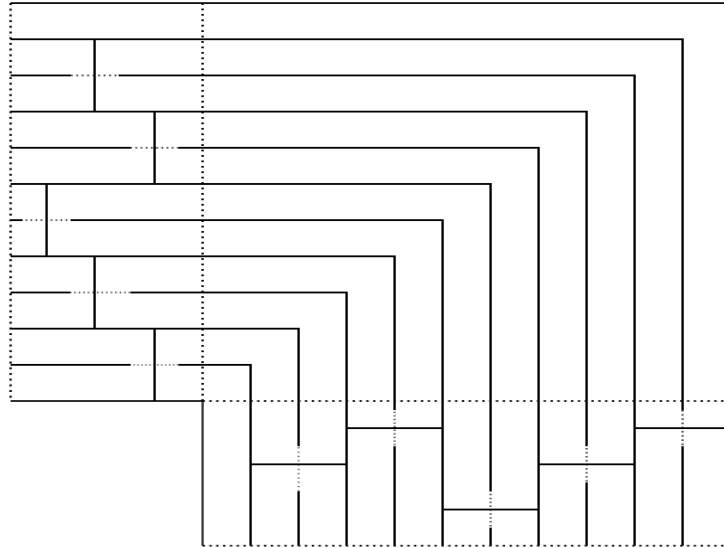


FIGURE 10. In the corner pieces of the plumbing, we just fill in the elements of the plumbing foliation.

rectangles. The amount we thicken is given by

$$(4.4) \quad \eta_n := \frac{1}{100^n} \frac{w_n}{\prod_{j=1}^n t_j}.$$

The following theorem is now clear:

Theorem 4.3. Γ_{n+1} is a plumbing compactly contained in Γ_n .

The core curve of Γ_{n+1} can be visualized in Figure 11.

5. THE LIMITING OBJECT IS A JORDAN CURVE

We now construct the Jordan curve Γ . Fix $\epsilon_0 > 0$ and let Γ_0 be the topological annulus formed by taking the open square of side length $2(1 + \epsilon_0)$ centered at the origin and removing the closed square of side length $2(1 - \epsilon_0)$ centered at the origin. Γ_0 is a square plumbing, and parameterize its core curve using a constant speed parameterization. Using the procedure in the previous section, we construct a plumbing with the following parameters. At each stage n , we must choose a parameter v_n . We assume that v_n is strictly increasing, and we always choose v_n to be a perfect square. Moreover, we will demand that

$$(5.1) \quad \sum_{n=1}^{\infty} \frac{1}{v_n} < \frac{1}{100}.$$

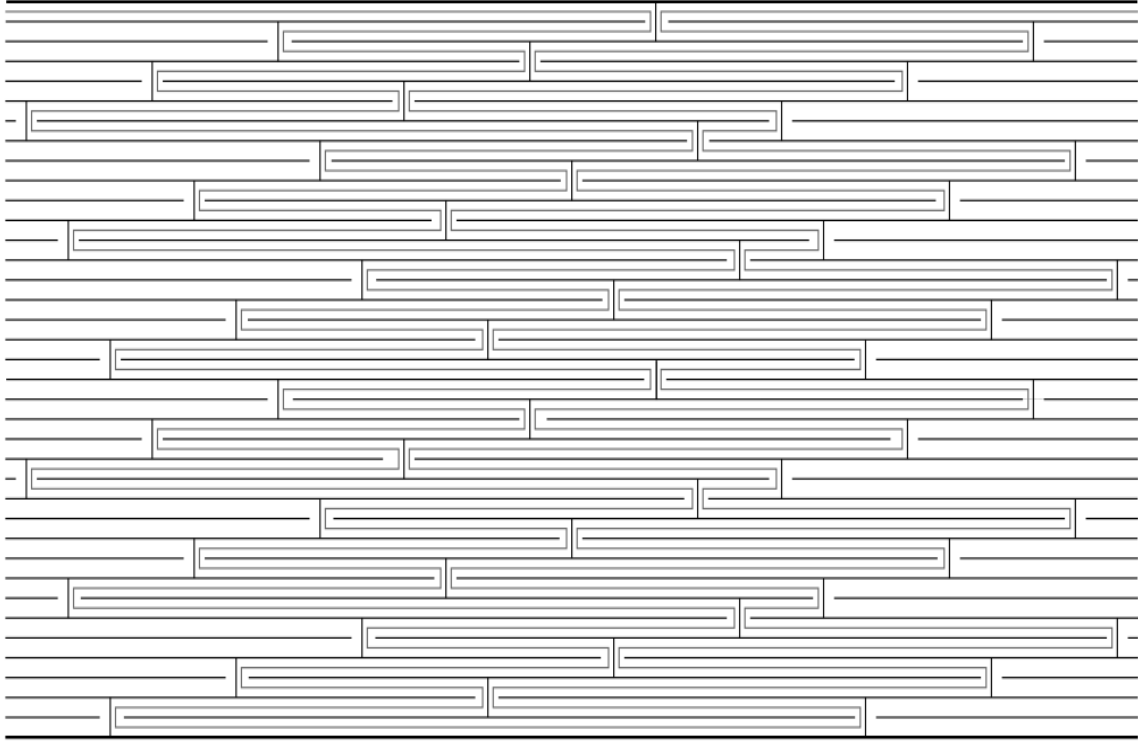


FIGURE 11. A visualization of Γ_{n+1} inside of a straight piece, with its core curve in gray. With this point of view, the core curve enters from the top left hand corner, weaves back and forth between vertical segments we place down to the bottom, and then weaves back up. This picture was constructed with one v_n -square made out of 2 v_n -slabs. In practice, the number of v_n -slabs and v_n -squares tends to infinity.

We will always assume that the parameters δ_n satisfy

$$(5.2) \quad \sum_{n=1}^{\infty} \delta_n < \frac{1}{100}.$$

Notice that for any valid choice of t_n , the widths w_n will always satisfy the inequality

$$(5.3) \quad w_{n+1} \leq \frac{w_n}{t_n}.$$

Given these above quantities, we will always choose t_n large enough so that it satisfies

$$(5.4) \quad \frac{t_n}{v_n} \cdot \frac{(\delta_n d_n)^2}{2w_n} \geq 1$$

By (5.1) and (5.4) we know that

$$(5.5) \quad \sum_{n=1}^{\infty} \frac{1}{t_n} < \frac{1}{100}.$$

From this it easily follows that

$$(5.6) \quad \sum_{n=1}^{\infty} w_n < \infty.$$

In this section, we will prove the following:

Theorem 5.1. *With all the parameters defined as above,*

$$\Gamma = \bigcap_{n=0}^{\infty} \Gamma_n$$

is a Jordan curve.

To prove this, the first step is to construct appropriate parameterizations of $\gamma_{1/2}^{n+1}$ given some parameterization of the core curves $\gamma_{1/2}^n$ of Γ_n . Since we will always work with the core curves in this section, we will call $\gamma_{1/2}^n := \gamma^n$.

Recall that each straight piece P of Γ_n is decomposed into subdivided rectangles with base approximately $\delta_n d_n$ and height w_n . We do not decompose the corner pieces and leave them as is. For some small choice of δ , $\gamma^n(0, \delta)$ is contained in one of these subdivided rectangles or corners, so we denote it as R_0 . We label the rest of the subdivided rectangles and corner pieces in the order that γ^n passes through them. This gives a list of subdivided rectangles and corners $\{R_i\}_{i=0}^{m-1}$. We will consider this list modulo m , so that $R_0 = R_m$. Let $t_i \in [0, 1]$ for $i = 1, \dots, m$ be the first entry time of γ^n in R_i . t_m will be the exit time for R_{m-1} , which coincides with entering back inside R_0 . $\gamma^n : (t_m, 1] \cup [0, t_1)$ goes from the end of R_{m-1} to the end of R_0 .

We'll show how to parameterize γ^{n+1} inside of the intervals (t_i, t_{i+1}) . Here we take the convention that $(t_m, t_{m+1}) = (t_m, 1] \cup [0, t_1)$. To do this, we will show where to place the points $\gamma^{n+1}(t_i)$ and $\gamma^{n+1}(t_{i+1})$. Then for the subarc of γ^{n+1} in between these points, we will use a constant speed parameterization. We will handle this with three cases.

For the first case, suppose that R_i is a subdivided rectangle of a straight piece so that R_{i+1} is not a corner piece. Orient R_i so that the top boundary side is a subset of the innermost boundary component of Γ_n . If R_i was labeled as B, then $\gamma^{n+1}(t_i)$ will be the point on the intersection of γ^{n+1} and the left side of R_i between $\gamma_{\frac{t_n-1}{t_n}}^n$ and γ_1^n . We let $\gamma_{1/2}^{n+1}(t_{i+1})$ be the point on the intersection of γ^{n+1} and the right side of R_i between $\gamma_{\frac{1}{t_n}}^n$ and γ_0^n . If R_i was labeled as T, we do the opposite. $\gamma^{n+1}(t_i)$ will be the point on the intersection of γ^{n+1} and the left side of R_i between and

$\gamma_{\frac{1}{t_n}}^n$ and γ_0^n , and $\gamma^{n+1}(t_{i+1})$ will be the point on the intersection of γ^{n+1} and the left side of R_i between $\gamma_{\frac{1}{t_n}}^n$ and γ_1^n . See Figure 12.

Suppose that R_i is a subdivided rectangle so that R_{i+1} is a corner boundary piece. Then we label $\gamma^{n+1}(t_i)$ using the procedure above. We let $\gamma^{n+1}(t_{i+1})$ be the point in the same straight piece as $\gamma^{n+1}(t_i)$ that lies on the right side of R_i . See Figure 13.

Finally, suppose that R_i is a corner piece. Then R_{i-1} and R_{i+1} fall into the above cases, which means that $\gamma^{n+1}(t_i)$ and $\gamma^{n+1}(t_{i+1})$ have already been defined. The former corresponds to the first time that γ^{n+1} enters the corner piece, and the latter corresponds to the last time γ^{n+1} exits the corner piece. We again use the constant speed parameterization to parameterize the subarc between these two points. See Figure 13.

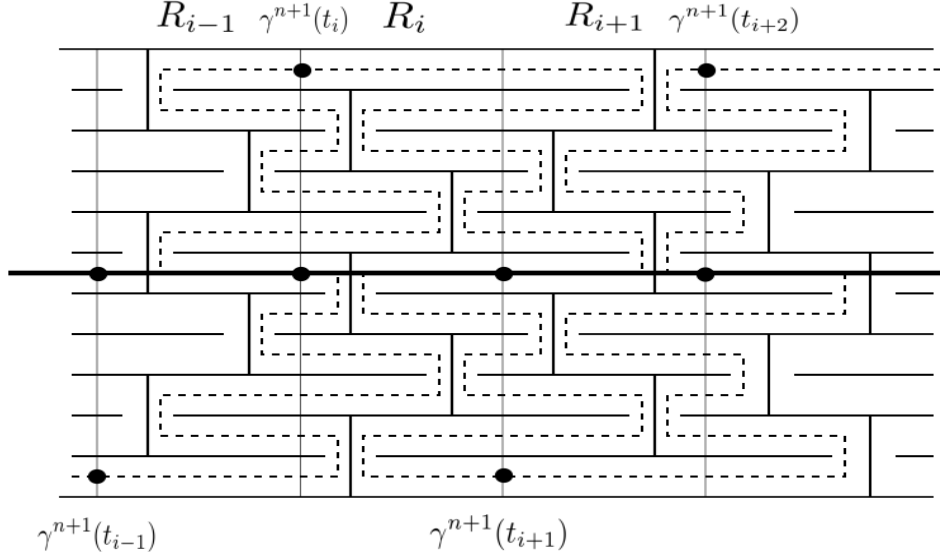


FIGURE 12. Parameterizing in straight pieces. Denote the subdivided rectangles by R_{i-1}, R_i, R_{i+1} . The bold curve is γ^n , and $\gamma^n(t_j)$ is marked for $j = i-1, i, i+1, i+2$. Since R_{i-1} is a T subdivided rectangle, $\gamma^{n+1}(t_{i-1})$ is the point on γ^{n+1} with farthest distance below $\gamma^n(t_{i-1})$. Since R_i is a B subdivided rectangle, $\gamma^{n+1}(t_i)$ is the point on γ^{n+1} with farthest distance above $\gamma^n(t_i)$. We parameterize the portion of the curve between those points with a constant speed parameterization.

Lemma 5.2. *The sequence of functions $\{\gamma^n\}_{n=1}^\infty$ is uniformly Cauchy. In fact, for sufficiently large n , we have the estimate for all $t \in [0, 1]$*

$$|\gamma^{n+1}(t) - \gamma^n(t)| \leq 5w_n.$$

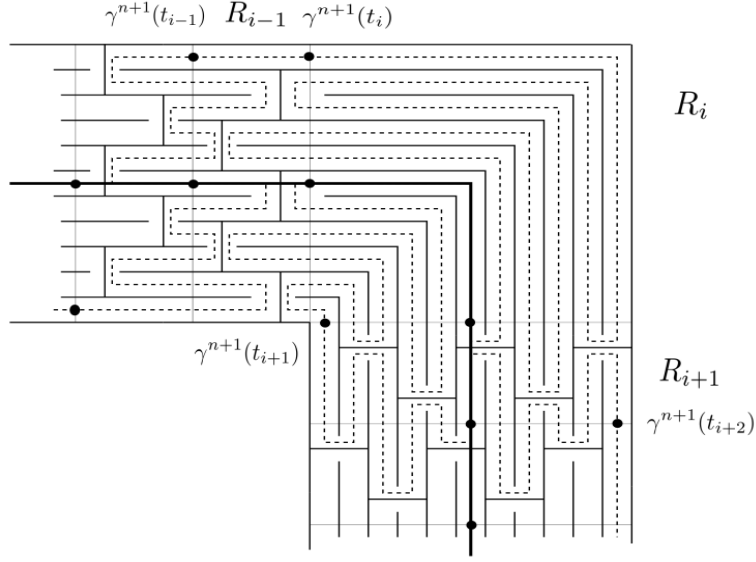


FIGURE 13. Parameterizing near a corner piece. Let R_{i-1} be the subdivided rectangle to the left of the corner piece, R_i . $\gamma^{n+1} : (t_{i-1}, t_i)$ is simply the portion of $\gamma^n(t_{i-1}, t_i)$ translated up to be between the final two elements of the plumbing foliation. $\gamma^{n+1}(t_i, t_{i+1})$ is the constant speed parameterization between the already placed $\gamma^{n+1}(t_i)$ and $\gamma^{n+1}(t_{i+1})$.

Proof. If $\gamma^n(t) \in R_i$, and $t \in [t_i, t_{i+1}]$, observe that the curve $\gamma_{1/2}^{n+1} : [t_i, t_{i+1}]$ can only intersect the rectangles R_{i-1}, R_i and R_{i+1} . Since $\text{diam}(R_i) \leq \sqrt{2}w_n$ for all corner pieces and subdivided rectangles, this means that $\text{diam}(\gamma_{1/2}^{n+1}(t_i, t_{i+1})) \leq 3\sqrt{2}w_n \leq 5w_n$. Since $\gamma^n(t) \in R_i$, it follows that

$$|\gamma^{n+1}(t) - \gamma^n(t)| \leq 5w_n.$$

By (5.6), this is a summable estimate independent of $t \in [0, 1]$, so it follows that γ^n is uniformly Cauchy. \square

It follows that γ^n converges uniformly to γ for some continuous function $\gamma : [0, 1] \rightarrow \mathbb{C}$. Since γ is continuous and the widths of Γ_n are strictly decreasing, we must have $\gamma([0, 1]) = \Gamma$. So to show Γ is a Jordan curve it is sufficient to show that γ is injective.

Lemma 5.3. *For all sufficiently large n , if $\gamma^n(t) \in R_i$ for some i , then $\gamma(t) \in R_{i-2}, \dots, R_{i+2}$.*

Proof. We know that $\gamma^{n+1}(t) \in R_{i-1}, R_i$, or, R_{i+1} . Then we can estimate using (5.3) that

$$\begin{aligned} |\gamma(t) - \gamma^{n+1}(t)| &\leq \sum_{j=n+1}^{\infty} |\gamma^j(t) - \gamma^{j+1}(t)| \\ &\leq \sum_{j=n+1}^{\infty} 5w_j \\ &\leq 5w_{n+1} \left(1 + \sum_{j=n+1}^{\infty} \frac{1}{\prod_{k=n+1}^j t_k} \right) \\ &\leq 6w_{n+1} \end{aligned}$$

Next observe that by (5.3) and (5.4),

$$w_{n+1} \leq \frac{w_n}{t_n} \leq \frac{(\delta_n d_n)^2}{2v_n} \leq \frac{1}{100} \delta_n d_n.$$

From this and the construction of Γ we deduce that $\gamma(t) \in R_{i-2}, \dots, R_{i+2}$. \square

Lemma 5.4. *There exists n large enough so that if $s \neq t$, $\gamma_{1/2}^n(s)$ and $\gamma_{1/2}^n(t)$ are separated by at least 1 straight piece.*

Proof. Suppose that $\gamma^n(s)$ and $\gamma^n(t)$ belong to the same straight piece of Γ_n . What happens if $\gamma^{n+1}(s)$ and $\gamma^{n+1}(t)$ belong to the same straight piece of Γ_{n+1} ? Then we must have

$$(5.7) \quad |\gamma^n(s) - \gamma^n(t)| \leq |\gamma^{n+1}(s) - \gamma^{n+1}(t)|$$

This is because of how we parameterize γ^{n+1} with respect to γ^n . Indeed, the procedure we used to parameterize the γ^n 's implies that the length of γ^{n+1} restricted to $[t_i, t_{i+1}]$ is always bounded below by the length of γ^n restricted to $[t_i, t_{i+1}]$. Since we use the constant speed parameterization to define γ^{n+1} on $[t_i, t_{i+1}]$, we must have inequality (5.7).

Note that in Γ_{m+1} for any m , the longest a straight piece can be is bounded above by $10w_m$. This follows from the construction and Lemma 5.3. Combined with (5.7), it follows that $\gamma^m(s)$ and $\gamma^m(t)$ cannot belong to the same straight piece for all $m \geq n$. \square

Lemma 5.5. *The limit function γ above is injective.*

Proof. Choose some $s < t$. By Lemma 5.4, there exists a value n so that $\gamma^n(s)$ and $\gamma^n(t)$ are separated by at least 1 straight piece. Since $\delta_n < \frac{1}{100}$, there are at least 50 subdivided rectangles between $\gamma^n(s)$ and $\gamma^n(t)$. Therefore, Lemma 5.3 says that $\gamma(s) \neq \gamma(t)$. \square

Corollary 5.6. *Γ is a Jordan curve.*

6. THE JORDAN CURVE IS NON-PIERCE-ABLE BY RECTIFIABLE ARCS

By Lemma 2.3, to show that Γ is non-pierceable by rectifiable arcs, we only must prove that Γ is not pierceable by unit length simple rectifiable curves which cross Γ .

Lemma 6.1. *Suppose that σ is a rectifiable arc which crosses Γ . Then there exists n so that the endpoints of σ are in distinct complementary components of Γ_n .*

Proof. This follows from the fact that (4.1) tends to 0 as $n \rightarrow \infty$. □

We will always parameterize σ crossing Γ so that $\sigma(0)$ is in the bounded complementary component of Γ . The following lemma is easy to visualize, but its proof is cumbersome. The idea is that if σ has endpoints in distinct complementary components of Γ_n , to cross Γ at exactly one point, σ is forced to remain in the part of Γ_n we removed to construct Γ_{n+1} , otherwise it will cross more than once. Therefore, such a σ must remain inside of one subdivided rectangle of Γ_n , so that it must completely cross a v_n -slab. See Figure 12 and Figure 14.

Lemma 6.2. *If σ is a rectifiable arc which crosses Γ and $\mathcal{P}(\sigma)$ has only one point, then σ must enter every rook contained in some v_n -slab of some subdivided rectangle of a straight piece of Γ_n . Moreover, such a v_n -slab may be chosen so that it does not intersect the boundary of Γ_n .*

Proof. By equation (5.1), combined with equation (5.4), we see that the number of v_n -slabs contained in a subdivided rectangle in Γ_n tends to ∞ as $n \rightarrow \infty$. Therefore, we may always assume that every subdivided rectangle in Γ_n has at least one v_n -slab in each complementary component of the core curve of Γ_n that does not intersect the boundary of Γ_n .

Suppose that the lemma is false: there is no subdivided rectangle so that σ passes through every single rook of some v_n -slab contained in that subdivided rectangle. Then there exists a subdivided rectangle R and a v_n -slab S contained in R so that σ passes through at least one, but not all, of the rooks in S . We may assume that S is in the bounded complementary component of the core curve of Γ_n . Indeed, if no such v_n -slab existed, then either $\mathcal{P}(\sigma)$ has more than one element, or the conclusion of the lemma holds.

In this case, two of the opposite sides of a rook of S that σ does not enter are determined by elements of the plumbing foliation γ_{j/t_n}^n and $\gamma_{(j+2)/t_n}^n$. Then by the construction of Γ_{n+1} from Γ_n , σ must pierce Γ at least once between γ_{j/t_n}^n and $\gamma_{(j+2)/t_n}^n$. This must happen either in the subdivided rectangle R , or one of the subdivided rectangles or corner pieces adjacent to it.

For the exact same reasons as above, there must exist a subdivided rectangle R' and a v_n -slab S' contained in R' so that σ passes through at least one, but not all

of the rooks in the v_n -slab, and this v_n -slab is in the unbounded complementary component of the core curve of Γ_n . But then by the same reasoning σ must pierce Γ again, contradicting the fact that $\mathcal{P}(\sigma)$ has only one point. \square

The statement and proof of Lemma 6.2 apply in exactly the same way if v_n -slabs are replaced by v_n -squares, since equation (5.2) implies that λ_n , the number of v_n -squares contained inside of a subdivided rectangle, also tends to ∞ as $n \rightarrow \infty$. This means we may assume there are many v_n -squares in the bounded and unbounded complementary components of the core curve of Γ_n , and the reasoning of the proof still applies.

In fact, since the number of v_n -slabs and v_n -squares tends to ∞ , for large enough n , the reasoning of Lemma 6.2 allows us to conclude that there are at least three adjacent v_n -slabs or v_n -squares so that σ enters every single rook of all three v_n -slabs or v_n -squares. We will use this observation in the basic estimates below.

Lemma 6.3. *Suppose that σ crosses Γ and $\mathcal{P}(\sigma)$ only contains one point. Then there exists a v_n -slab R in some Γ_n so that*

- (1) σ passes through every rook in the v_n -slab R
- (2) σ passes through every rook in the adjacent v_n -slabs above and below R

Moreover, we have

$$l(\sigma \cap R) > \delta_n d_n.$$

Proof. The proof is a picture; see Figure 14. The distance σ must travel to go between two rooks in consecutive rows in the rook placement is at least $\delta_n d_n / v_n$ distance apart by Lemma 4.2. σ must also cross from the top of R to the bottom of R , contributing a length of at least $v_n \cdot \frac{2w_n}{t_n}$. So we calculate

$$l(\sigma \cap R) \geq v_n \cdot \frac{2w_n}{t_n} + v_n \cdot \frac{\delta_n d_n}{v_n} > \delta_n d_n.$$

This proves the claim. \square

Corollary 6.4. *Suppose that σ crosses Γ and $\mathcal{P}(\sigma)$ only contains one point. Then there exists a v_n -square R in some Γ_n so that*

- (1) σ passes through every rook in the v_n -square R .
- (2) σ passes through and every rook of the v_n -square above and below R .

Moreover, we have

$$l(\sigma \cap R) > 1.$$

Proof. To cross R in such a way, one must pass through all the rooks in k_n many v_n -slabs without piercing Γ . Therefore by repeated applications of Lemma 6.3, and

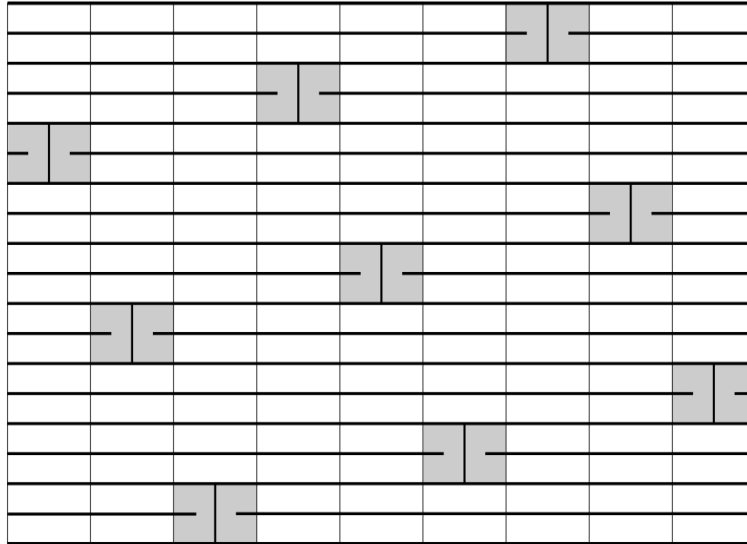


FIGURE 14. Illustrating Lemma 6.3. We have a v_n -slab that is a subset of Γ_n , along with the thin boundary rectangles that form part of the boundary of Γ_{n+1} . If σ crosses Γ_n without crossing Γ , it must enter all the rook squares (shaded in gray) of every v_n -slab it passes through.

then using (5.4), we have

$$l(\sigma \cap R) \geq k_n \delta_n v_n = \frac{t_n}{v_n} \cdot \frac{(\delta_n d_n)^2}{2w_n} > 1.$$

□

Corollary 6.5. Γ is non-pierceable by rectifiable arcs.

Proof. This follows immediately from Lemma 2.3, since Lemma 6.4 shows that Γ is non-pierceable by simple rectifiable arcs with length bounded by 1. □

7. ANY CROSSING RECTIFIABLE CURVE INTERSECTS ON POSITIVE LENGTH

We would like to focus on a convenient subarc of σ , which exists by the following lemma.

Lemma 7.1. *Suppose σ is simple and rectifiable with length ≤ 1 and crosses Γ . Then there exists an integer $n \geq 0$ and a straight piece P of Γ_n so that σ enters the top of P and exits out the bottom of P .*

Proof. By Lemma 6.1, σ must cross Γ_n for some n . If σ did not enter the top and exit out of the bottom of the same straight piece P for Γ_{n+1} , then similar

reasoning to Lemma 6.2 and Lemma 6.3 would imply that σ must enter too many rook squares and must have length greater than 1. \square

We will show that the subarc that begins at the top of a straight piece and exits at the bottom of a straight piece intersects with positive length. We first consider the special case that this subarc is a line segment.

Lemma 7.2. *If σ is a axis aligned line segment that crosses Γ , then*

$$H^1(\sigma \cap \Gamma) > 0.$$

Recall that we always may orient our view so that that this axis aligned line segment is vertical.

Proof. Let P denote the straight piece that σ crosses contained in some Γ_n . We define

$$l_m := l(\sigma \cap \Gamma_m).$$

Then l_m is a decreasing sequence and

$$\lim_{m \rightarrow \infty} l_m = H^1(\sigma \cap \Gamma).$$

We will estimate the length lost between each stage, $l_m - l_{m+1}$. By our assumptions we have $l_n \geq w_n$.

σ passes through t_n many rows in P determined by the plumbing foliation for Γ_n . The length l_{n+1} can decrease for two reasons. First, if σ passes through a rook square, then it may pass through the thickened vertical segment placed in that square, which does not belong to Γ_{n+1} . The number of rooks a single line segment can pass through is no more than $k_n \lambda_n$ (recall that k_n and λ_n are 4.2 and 4.3, respectively). Second, l_n can decrease by η_n for every element of the plumbing foliation that σ passes through. This means that the amount of length lost can be bounded above by

$$l_n - l_{n+1} \leq t_n \cdot \eta_n + k_n \cdot \lambda_n \frac{2w_n}{t_n} \leq \frac{w_n}{100^n \prod_{j=1}^{n-1} t_j} + \frac{w_n}{v_n}.$$

Here we used (4.4), (4.3), and (4.2).

Estimating $l_{n+1} - l_{n+2}$ is similar; we just have to count how many straight pieces σ crosses in Γ_{n+1} . This number is no more than t_n . Therefore, we can apply the same estimates above and see that

$$\begin{aligned} l_{n+1} - l_{n+2} &\leq t_n \cdot (t_{n+1} \cdot \eta_{n+1}) + t_n \left(k_{n+1} \cdot \lambda_{n+1} \cdot \frac{2w_{n+1}}{t_{n+1}} \right) \\ &\leq \frac{w_{n+1}}{100^{n+1} \prod_{j=1}^{n-1} t_j} + \frac{t_n w_{n+1}}{v_{n+1}} \\ &\leq \frac{w_n}{100^{n+1}} + \frac{w_n}{v_{n+1}} \end{aligned}$$

In general, we can use this argument to deduce that

$$(7.1) \quad l_m - l_{m+1} \leq w_n \left(\frac{1}{100^m} + \frac{1}{v_m} \right).$$

Putting it all together,

$$\begin{aligned} l_n - l(\sigma \cap \Gamma) &= \lim_{m \rightarrow \infty} \sum_{j=n}^m l_j - l_{j+1} \\ &\leq w_n \sum_{j=n}^{\infty} \left(\frac{1}{100^{j+1}} + \frac{1}{v_j} \right) \\ &\leq \frac{w_n}{10}. \end{aligned}$$

Therefore,

$$l(\sigma \cap \Gamma) \geq l_n - w_n/10 \geq \frac{9}{10} w_n.$$

□

With a little more care, we can upgrade these observations above to the proof of Theorem 1.1.

Lemma 7.3. *Suppose σ is a simple rectifiable arc with length ≤ 1 and crosses a straight piece P of Γ_n . Then σ cannot intersect more than $k_n v_n + k_n \lambda_n$ many thickened vertical segments placed in rooks in Γ_n .*

Proof. Since σ must pass through at least $k_n \cdot \lambda_n$ many v_n -slabs, σ can intersect exactly one thickened vertical segment in one rook of each of the $k_n \cdot \lambda_n$ -slabs of Γ_n without costing any length, just like a vertical segment. Any additional thickened vertical segments that σ can intersect come from σ going to additional v_n -slabs or from σ entering multiple thickened vertical segments in other rooks in the same v_n -slab.

Suppose that σ passes through thickened vertical segments in more than $k_n \cdot \lambda_n$ many v_n slabs. By Lemma 8, each additional slab of this type that σ passes through must take at least a length of

$$\frac{\delta_n d_n}{2v_n}.$$

Any additional thickened vertical segments passed through in any of the v_n slabs that σ intersects also costs a length of $\frac{\delta_n d_n}{2v_n}$, again by Lemma 8.

This means the amount of additional thickened vertical segments that σ can intersect is no more than $2k_n v_n$, since (5.4) implies that

$$k_n \delta_n d_n > 1$$

This is not possible if we assume that σ has length ≤ 1 .

□

If σ intersects a thickened line segment in some rook, we will just assume that σ passed through the thickened line segment. This causes l_{n+1} to decrease, but fortunately this cannot happen often. Indeed, note that $2k_nv_n$, is a very small percentage of t_n , since by (4.2),

$$\frac{2k_nv_n}{t_n} = \frac{\delta_n d_n}{w_n}.$$

This quantity tends to 0 very rapidly by (5.2).

This observation allows us to prove Theorem 1.1.

Proof of the Main Theorem. Again, let P be the straight piece of Γ_n that σ crosses. Again we denote $l_n = l(\sigma \cap \Gamma_n)$.

σ must travel between each adjacent element of the plumbing foliation for Γ_n . Therefore, to estimate $l_m - l_{m+1}$, we can use all of the same estimates from Lemma 7.2; we just have to additionally discard the length of the excess vertical segments that σ can visit. But by Lemma 7.3, this number is $2k_nv_n$, so the amount of excess length discarded is no more than

$$k_nv_n \cdot \frac{2w_n}{t_n} = 2\delta_n d_n.$$

At the next stage, Γ_{n+1} , the amount of excess length discarded is no more than

$$t_n k_{n+1} v_{n+1} \cdot \frac{2w_{n+1}}{t_{n+1}} \leq t_n \delta_{n+1} d_{n+1} \leq \frac{w_n}{w_{n+1}} \delta_{n+1} d_{n+1} \leq w_n \delta_{n+1}.$$

We used (5.3) in the second inequality. Similarly, at the m th stage, the amount of excess length discarded compared to a vertical line segment is no more than

$$w_n \delta_m.$$

So the amount of excess length discarded compared to a vertical segment is no more than

$$w_n \sum_{j=n}^{\infty} \delta_j < \frac{w_n}{100}.$$

This combined with the estimates in Lemma 7.2 show that

$$l(\sigma \cap \Gamma) > w_n \left(\frac{9}{10} - \frac{1}{100} \right) > 0$$

□

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DEPARTMENT OF MATHEMATICS, STONY BROOK UNIVERSITY, STONY BROOK, NY 11794
Email address: `jack.burkart@stonybrook.edu`