TRANSCENDENTAL JULIA SETS WITH FRACTIONAL PACKING DIMENSION

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Abstract. We construct transcendental entire functions whose Julia sets have packing dimension in $(1, 2)$. These are the first examples where the computed packing dimension is not 1 or 2. Our analysis will allow us further show that the set of packing dimensions attained is dense in the interval $(1, 2)$, and that the Hausdorff dimension of the Julia sets can be made arbitrarily close to the corresponding packing dimension.

1. Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental (non-polynomial) entire function. We denote the *nth iterate* of f by f^n. We define the *Fatou set*, $\mathcal{F}(f)$, to be the set of all points so that $\{f^n\}_{n=1}^{\infty}$ locally forms a normal family. Thus the Fatou set is the "stable" set for the dynamics of f. We define the *Julia set*, $\mathcal{J}(f)$, to be the complement of the Fatou set. This is the set where the dynamics of f are chaotic. We refer the reader to [CG93] and [Sch10] for an introduction to complex dynamics the rational and transcendental setting, respectively.

One of the goals of complex dynamics is to understand the geometric and topological properties of the Julia set. In this paper we prove the following theorem.

Theorem 1.1. *There exists a transcendental entire function $f : \mathbb{C} \to \mathbb{C}$ such that the packing dimension of $\mathcal{J}(f) \in (1, 2)$.*

Stallard asked in [Sta08] if there exists a transcendental meromorphic (we consider entire functions as a special case of meromorphic functions) function for which the packing and Hausdorff dimensions of the Julia set are non-integer and equal. Our techniques generate a family of entire functions, and we will actually prove the following stronger result which offers positive progress towards the construction of such a function.

Theorem 1.2. *The set of packing dimensions attained by Julia sets of transcendental entire functions is dense in $(1, 2)$. In particular, let $s \in (1, 2)$ and $\epsilon_0 > 0$ be given. Then there exists a transcendental entire f so that*

$$s - \epsilon_0 \leq \dim_{\mathrm{H}}(\mathcal{J}(f)) \leq \dim_{\mathrm{P}}(\mathcal{J}(f)) \leq s + \epsilon_0.$$

In [Bak75], Baker proved that the Julia set of a transcendental entire function must always contain a non-trivial, compact, connected set, and it follows immediately that the Hausdorff dimension of the Julia set must always be greater than or equal to 1. In [Mis81], Misiurewicz showed that the Julia set of e^z was the entire complex plane, and in [McM87] McMullen showed that the Julia sets of the exponential and sine families of entire functions always have Hausdorff dimension 2, but need not be all of $\mathbb{C}$. These examples can also have positive or zero area measure. Reducing the dimension of the Julia set is therefore the difficult task in the transcendental setting, and in [Sta91], Stallard constructed examples in the Eremenko-Lyubich class that had Hausdorff dimension arbitrarily close to 1, and refined this result further in [Sta97] and [Sta00] to include all values in $(1, 2)$. Moreover, in [Sta96], Stallard showed that in the Eremenko-Lyubich class the Hausdorff dimension must be strictly greater than 1. Recently, in [Bis17], Bishop constructed a transcendental entire function with Julia set having Hausdorff dimension

1. This example demonstrates that all values of Hausdorff dimension in $[1, 2]$ can be achieved.

Less is known about the packing dimension in the transcendental setting. In [RS05], Rippon and Stallard show that if f belongs to the Eremenko-Lyubich class, then the packing dimension of the Julia set is 2. Bishop computed the packing dimension of the Julia set of his example above to be 1. Our result is the first of its kind where the computed packing dimension is strictly between 1 and 2. Packing dimension and other various dimensions relevant to the paper are defined in Section 4. Figure 1 below summarizes what has been proven about the possible Hausdorff and packing dimension pairs attained by Julia sets transcendental entire functions.

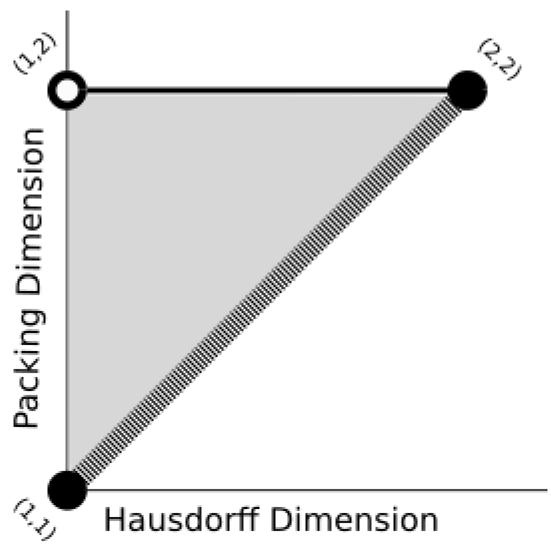

FIGURE 1. A graph showing the possible and attained Hausdorff and packing dimension pairs for transcendental entire functions. All possible pairs are shaded in light grey, and all attained values are colored black. The point $(2, 2)$ is attained by families of the exponential and sine functions. The upper segment is due to the work of Stallard, and the point $(1, 1)$ is due to Bishop. Our contribution uses enlarged, dashed lines, to emphasize that a dense set of dimensions are attained very close to the diagonal.

We would like to point out how our construction differs from the constructions cited above. Since Stallard's examples belong to the Eremenko-Lyubich class, the packing dimension of those Julia sets must be 2, even though the Hausdorff dimension can attain any value in $(1, 2)$. In our examples, the packing and Hausdorff dimensions may be arranged to be arbitrarily close. The dynamical behavior of our examples is also much different; our functions have multiply connected Fatou components which do not occur in the Eremenko-Lyubich class. Stallard uses a family of functions defined via a Cauchy integral, whereas we use an infinite product construction similar to Bishop. Our example looks very similar to Bishop's at first glance but there are many major differences. The most obvious difference is the dynamical behavior near the origin; our examples have an attracting basin with quasicircle boundary near the origin, whereas Bishop's contains a Cantor repeller. This difference introduces what we call 'wiggly' Fatou components, which we describe in Section 2. The Hausdorff and packing dimensions in Bishop's example are supported on the boundaries of Fatou components which escape quickly, and the dynamics are simple on these boundaries. In our examples, we will see that the dimension of the Julia set is supported on buried points that are not on the boundary of any Fatou component. In Bishop's example, the buried points have dimension close to zero. The dynamics on the buried points are more intricate; the buried points contain bounded orbit points, escaping points, and so-called bungee orbits. These dynamically defined sets are defined in Section 13.

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2. Outline of the Proof

We will construct a function $f : \mathbb{C} \to \mathbb{C}$ depending on parameters $N \in \mathbb{N}$, $R \in \mathbb{R}$, $\lambda \in \mathbb{D}$ and c in the main cardioid of the Mandelbrot set. Define $g(z) = z^2 + c$. The function f will be g iterated N times multiplied by an infinite product. As a formula,

$$f(z) = g^N(z) \cdot \prod_{k=1}^{\infty} \left(1 - \frac{1}{2} \left(\frac{\lambda^{2^{Nk}} z}{R_k} \right)^{n_k} \right) = g^N(z)(1 + \epsilon(z)). \tag{2.1}$$

Here, $n_k = 2^{N+k-1}$, and the sequence $\{R_k\}$ grows super-exponentially and is defined inductively starting from a large initial parameter R. The choices are made so that near the origin, the infinite product can be made uniformly close to the constant function 1. We will sometimes write the infinite product as $(1 + \epsilon(z))$ to emphasize this fact, where $\epsilon(z)$ is a holomorphic function uniformly close to the 0 function in a large neighborhood of the origin.

In Section 3 we discuss the facts we will need about conformal, quasiconformal, and polynomial-like mappings. In Section 4, we define and discuss what we call *Whitney-type decompositions*, a generalization of Whitney decompositions composed of dyadic squares and our main tool for calculating the packing dimension. In Section 5, we will carefully define f and show it defines an entire function. In Section 6 we decompose a region of the plane far from the origin into alternating annuli A_k and B_k, where the modulus of A_k is fixed and contains the circle $\{|z| = |R_k|\}$ and the modulus of B_k increases as $k \to \infty$. We will show that $f(B_k) \subset B_{k+1}$, and that if a point ever lands in B_k, it diverges locally uniformly to ∞ under f. The existence of these "absorbing" annuli B_k of increasing modulus is always true for functions with multiply-connected wandering domains; see p.25 in [Zhe06] and also [BRS13] for this and related results. In our example, we will additionally show that on B_k, f is a small perturbation of a constant multiple of the power function z^{2n_k}. Therefore, all the interesting dynamical behavior happens in the annuli A_k. We will show that $A_k \subset f(A_{k-1})$, and that all the zeros and critical points of f and the Julia set are inside the A_k's. To accomplish this, we will show (in a quantitative way) that f is a small perturbation of the kth term of the infinite product on A_k.

Given any $s \in (1, 2)$, we will choose c so that $\dim_{\mathrm{H}}(J(g^N)) = s$. In Section 7, we will show that in a neighborhood of the origin, f is a polynomial-like mapping which is a small perturbation of g^N. In this section, the parameter λ is used to show that the quasicircle Julia set of f viewed as a polynomial-like mapping moves holomorphically with respect to λ. By some standard arguments it will follow that for appropriate choices of λ the Julia set of the entire function f will have Hausdorff dimension bounded below by a value arbitrarily close to s. From here, we will be able to prove that we can sort the Fatou components into two categories depending on if the component remains bounded or if the component escapes to infinity. The first type of Fatou component comes from the connected component containing the critical point 0 of $f(z)$. This component is an attracting basin which we denote by B_f. All the inverse images of B_f are eventually mapped conformally with small distortion onto B_f by some iterate of f.

In Sections 8 and 9 we discuss the second type of Fatou component. These are subsets of the escaping set $I(f)$, where

$$I(f) := \{z \,:\, f^n(z) \to \infty\}.$$

These components will be infinitely connected wandering domains, and their boundary components will be bounded by C^1 closed curves. These boundary curves will accumulate on the outermost boundary of each component. There is a distinguished *central series* $\{\Omega_k\}_{k=-\infty}^{\infty}$ of these Fatou components which surround the origin. We will split these components into two sub-categories. If $k \geq 1$, we will call Ω_k *round* since the inner and outer boundary of Ω_k will be C^1 curves which are approximately circles. See Figure 2. We will call Ω_k for $k \leq 0$ *wiggly.* The inner and outer

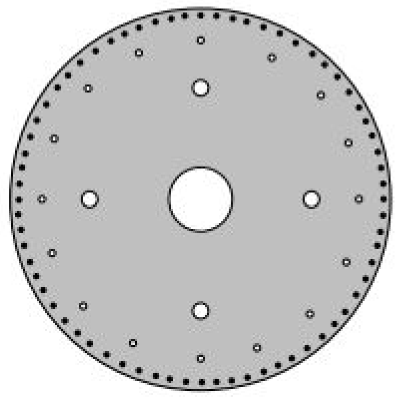

FIGURE 2. A round Fatou component. Such components are infinitely connected, and all the boundary components are C^1 and approximately circles. The circular boundary components accumulate onto the outermost boundary component and are arranged in layers that can be connected by approximately circular Jordan curves. This picture is not to scale; the diameters of the holes are actually much smaller than the diameter of the component.

boundary of wiggly components will be C^1 curves that approximate the fractal boundary of B_f as $k \to -\infty$. If Ω_k is wiggly, then f^{k+1} will map Ω_k to the round component Ω_1 as a covering map. The action of f^{k+1} on Ω_k can be thought of as first mapping Ω_k inside a very thin annulus conformally, then to a thick annulus by a power mapping $z^{n_1(k+1)}$. This is similar to the dynamics on the basin of infinity for a quadratic polynomial with connected Julia set. See Figure 3. We will see that the central series of Fatou components is the main building block for the Fatou and Julia set of f. Indeed, we will show all Fatou components of f map conformally onto an element of the central series with small distortion.

The Julia set of f will contain the boundaries of each of these two types of components. This is not the entire Julia set. Since f has a multiply connected Fatou component, the work of Dominguez ([Dom97]) implies that the Julia set will also contain points that do not lie on the boundaries of either of these two types of components. We call these points in the Julia set *buried points*, and the orbits of buried points either remain bounded, belong to the bungee set or escape slowly (see Section 14 for the definitions of these dynamically defined sets). In Sections 10 through 13, we will perform a detailed analysis of the Hausdorff dimension of the set of buried points. We will show that the Hausdorff dimension of this set is at most ϵ_0 larger than the Hausdorff dimension of the boundary of the fractal basin of attraction. So while the Hausdorff dimension could possibly be strictly larger than

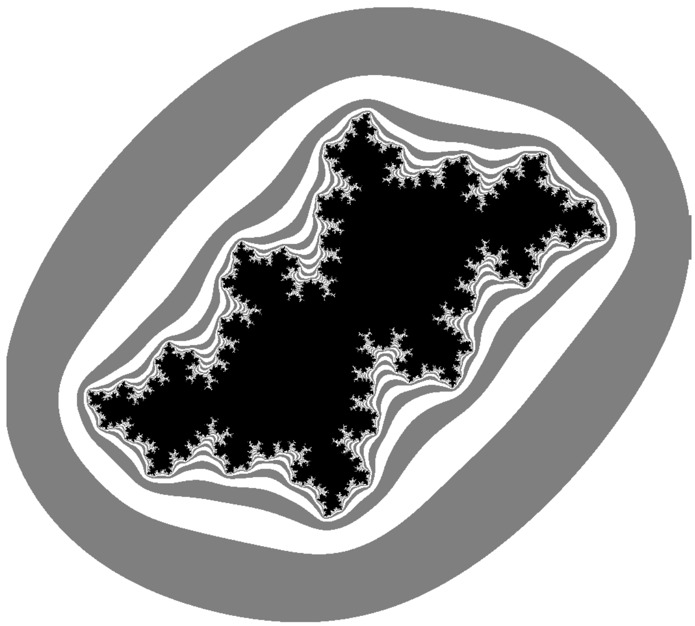

FIGURE 3. A schematic for a sequence of wiggly Fatou components, alternating between gray and white. The holes have been omitted to emphasize the wiggly shape of the inner and outer boundaries. These boundary components approximate level lines for the Green's function of the complement of the basin of attraction, and they surround and accumulate on the basin's fractal boundary.

the Hausdorff dimension of the boundary of the basin of attraction, we show that we can make this difference ϵ_0 arbitrarily small.

To obtain an upper bound the packing dimension, we will follow the strategy in [Bis17] and study the critical exponent of a Whitney-type decomposition of the complement of the Julia set of f in a bounded region. Since the Julia set of f will have zero area, it turns out that this critical exponent coincides with the packing dimension, and we will show that this exponent is at most the Hausdorff dimension of the buried points. The key idea in this part of the proof is to iterate Fatou components, or pieces of Fatou components, conformally onto Fatou components where we can estimate the critical exponent directly. The trade-off is that this conformal rescaling procedure results in various corrective factors that we need to control. We do this with a combination of all the technical work done earlier in the paper.

We conclude with some notation we will use throughout the paper. We denote the complex plane by $\mathbb{C}$, $B(z,r)$ is the open ball in $\mathbb{C}$ with center z and radius r, and $\mathbb{D} = B(0,1)$. Likewise $C(z,r)$ will denote the circle of radius r centered at z. We denote the closure of a set A by $\overline{A}$. If Ω is a multiply connected domain in the plane, we will denote $\widehat{\Omega}$ as the union of Ω with all its bounded complementary components. We will sometimes refer to the simply connected domain $\widehat{\Omega}$ as the *polynomial hull* of Ω. We will say that a Jordan curve $\gamma \subset \mathbb{C}$ *surrounds the origin* if its bounded complementary component contains the origin. We will similarly say that a domain Ω with Jordan boundary components surrounds the origin if at least one of its boundary components surrounds the origin.

We will frequently use big-oh notation. If $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathbb{C}$, then $x_n = O(y_n)$ means that there exists a constant C so that $|x_n| \leq C|y_n|$ for all sufficiently large n. Similarly if $f(z)$ and $g(z)$ are functions, we say $f(z) = O(g(z))$ as $z \to a$ if there exists C and a ball $B(a,r)$ so that $|f(z)| \leq C|g(z)|$ on $B(a,r)$. In certain proofs, a constant C may evolve throughout the proof.

3. Conformal, Quasiconformal, and Polynomial-Like Mappings

In this section, let Ω and Ω' be planar domains. In this paper, we will call a mapping $f : \Omega \to \mathbb{C}$ *conformal* if and only if f is both holomorphic and injective. Equivalently, f is conformal if and only if it is a biholomorphic map onto some domain Ω'. This condition implies that $f'(z) \neq 0$ on Ω, but the converse is not true in general.

If $f : \Omega \to \Omega'$ is conformal, and K is relatively compact in Ω, the *distortion* of f on K is

$$D|_K := \sup_{z,w \in K} \frac{|f'(z)|}{|f'(w)|}.$$

We will often make use of the Koebe growth and distortion theorems for conformal mappings (see [GM05] Theorem 4.5 p. 22) in the following form.

Lemma 3.1. *Fix $r < 1$ and let $B = B(0,r) \subset \mathbb{D}$ be an open ball and let $K \subset \overline{B}$ be a compact set. Suppose that $f : \mathbb{D} \to \mathbb{C}$ is conformal. Then there exists a constant C depending only on r, independent of f, so that*

$$C^{-1}\frac{\operatorname{diam}(f(K))}{\operatorname{diam}(f(B))} \leq \frac{\operatorname{diam}(K)}{\operatorname{diam}(B)} \leq C\frac{\operatorname{diam}(f(K))}{\operatorname{diam}(f(B))}.$$

There exists a constant C' depending only on r, independent of f, so that

$$D|_K \leq C'.$$

We remark that as $r \to 0$, the constants C and C' tend to 1. When the hypotheses of Lemma 3.1 are met we will sometimes say that f has *bounded conformal distortion* on K, or *small conformal distortion* if r is sufficiently close to 0.

We call an orientation preserving homeomorphism $\varphi : \Omega \to \Omega'$ K*-quasiconformal* if φ has locally square integrable distributional derivatives which satisfy

$$|\varphi_{\bar{z}}(z)| \leq k|\varphi_z(z)|$$

for all $z \in \Omega$ for $k = (K-1)/(K+1) < 1$. When the value of K is not important we will refer to f simply as quasiconformal. Given a quasiconformal mapping, we define its *dilatation*

$$\mu(z) = \frac{\varphi_{\bar{z}}(z)}{\varphi_z(z)}.$$

The definition says that the dilatation of a quasiconformal mapping is bounded above by some number strictly less than 1.

Ω is a multiply connected Jordan domain if Ω is not simply connected and all of its boundary components are Jordan curves. A domain $A \subset \mathbb{C}$ is a topological annulus if it has two complementary components, and A is a Jordan annulus if its boundary components are closed Jordan curves. In particular, a Jordan annulus has one bounded and one unbounded complementary component. The boundary of the bounded complementary component is called the inner boundary of A, and the boundary of the unbounded complementary component is called the outer boundary of A. A round annulus is a Jordan annulus of the form $A = A(r_1, r_2) = \{z : r_1 \leq |z| \leq r_2\}$ where $r_1 < r_2$. Given a Jordan annulus A, there exists $1 < r < \infty$ and a conformal mapping

$$\varphi : A \to A(1, r).$$

This allows us to define the modulus of A to be $\mathrm{mod}(A) = \frac{1}{2\pi}\log(r)$. The modulus of an annulus is a quasi-invariant. If $\varphi : A \to A'$ is a quasiconformal homeomorphism between two Jordan annuli then

$$\frac{1}{K}\,\mathrm{mod}(A) \leq \mathrm{mod}(A') \leq K\,\mathrm{mod}(A').$$

Finally, we remark that using our definition of conformal, f is conformal if and only if f is 1-quasiconformal. In particular the modulus of an annulus is invariant under conformal mappings. This allows for the following invariant formulation of Lemma 3.1.

Lemma 3.2. *Let Ω be simply connected, let U be open and compactly contained in Ω, and let K be a compact subset of $\overline{U}$. Suppose $f : \Omega \to \Omega'$ is conformal. Then there is a constant C which depends only on* $\mathrm{mod}(\Omega \setminus \overline{U})$ *so that*

$$C^{-1}\frac{\mathrm{diam}(f(K))}{\mathrm{diam}(f(B))} \leq \frac{\mathrm{diam}(K)}{\mathrm{diam}(B)} \leq C\frac{\mathrm{diam}(f(K))}{\mathrm{diam}(f(B))}.$$

There exists a constant C' which depends only on $\mathrm{mod}(\Omega \setminus \overline{U})$ *so that*

$$D|_{\bar{U}} \leq C'.$$

C *and* $C' \to 1$ *as* $\mathrm{mod}(\Omega \setminus \overline{U}) \to \infty$.

A Jordan curve $\Gamma \subset \mathbb{C}$ is called a κ*-quasicircle* if $\kappa > 1$ and for all points $z, w \in \Gamma$, if γ denotes the subarc of Γ of smallest diameter with endpoints z and w, we have

$$\mathrm{diam}(\gamma) \leq \kappa |z - w|.$$

Γ is a quasicircle if and only if there exists a quasiconformal mapping $\varphi : \mathbb{C} \to \mathbb{C}$ that maps the unit circle onto Γ.

Douady and Hubbard introduced polynomial-like mappings in [DH85]. Recall that a continuous mapping $f : \Omega \to \Omega'$ is called *proper* if the inverse image of every compact set $K \subset \Omega'$ is compact in Ω. A *degree d polynomial-like map* is a triple (f, Ω, Ω'), where $f : \Omega \to \Omega'$ is a proper holomorphic mapping of degree d, and Ω and Ω' are bounded Jordan domains with Ω relatively compact in Ω'. We define the *filled Julia set* of f by

$$K_f := \bigcap_{n \geq 0} f^{-n}(\Omega).$$

The filled Julia set is precisely the set of points that remain in Ω for all iterates of f. The *Julia set* of f is defined to be the boundary ∂K_f, and we denote it by J_f. The straightening lemma of Douady and Hubbard is of great importance, and we will need the following simple formulation.

Theorem 3.3 (The Straightening Lemma)**.** *Let (f, Ω, Ω') be a degree d polynomial-like mapping. Then there exists a quasiconformal mapping $\varphi : \mathbb{C} \to \mathbb{C}$ and a polynomial p of degree d so that for all $z \in \Omega$ we have*

$$f(z) = \varphi \circ p \circ \varphi^{-1}(z).$$

A polynomial-like mapping (f, Ω, Ω') is called *hyperbolic* if every critical point is attracted to an attracting cycle. Equivalently, (f, Ω, Ω') is hyperbolic if there exists $m \in \mathbb{N}$ so that $|(f^m)'| > 1$ on J_f. In our applications, the polynomial-like mappings will come as the restriction of entire functions, and it will be important that we distinguish between hyperbolicity of polynomial-like mappings, versus hyperbolicity as a transcendental entire function (which our example cannot be, since we will see that it has an unbounded set of critical values. See [RGS17].)

4. Dimension and Whitney Type Decompositions

Given a set $A \subset \mathbb{C}$, we define its α*-Hausdorff measure* to be the quantity

$$H^\alpha(A) := \lim_{\delta \to 0} H^\alpha_\delta(A) := \lim_{\delta \to 0} \left(\inf \left\{ \sum_{i=1}^{\infty} \mathrm{diam}(U_i)^\alpha \; : \; A \subset \bigcup_{i=1}^{\infty} U_i, \, \mathrm{diam}(U_i) < \delta \right\} \right).$$

The infimum is taken over all countable covers by sets $\{U_i\}$ of A. One can check that if $H^t(A) < 0$, then $H^s(A) = 0$ for all $s > t$, and similarly, if $H^t(A) > 0$, then $H^s(A) = \infty$ for all $s < t$. It follows that the *Hausdorff dimension*

$$\dim_{\mathrm{H}}(A) := \sup\{t : H^t(A) = \infty\} = \inf\{t : H^t(A) = 0\}$$

is uniquely defined.

Given a compact set $K \subset \mathbb{C}$, define $N(K, \epsilon)$ to be the minimal number of open balls of radius ϵ needed to cover K. Since K is compact, this number exists and is finite. We define the *upper Minkowski dimension* of K to be

$$\overline{\dim_{\mathrm{M}}}(K) = \limsup_{\epsilon \to 0} \frac{\log(N(K, \epsilon))}{\log(1/\epsilon)} = \sup\left\{s \geq 0 : \limsup_{\epsilon \to 0} N(K, \epsilon)\epsilon^s = 0\right\}.$$

One obtains an equivalent definition using squares of side length ϵ to define $N(K, \epsilon)$. For this reason, upper Minkowski dimension is often referred to as *upper box counting dimension* in the literature.

We define the *packing dimension* of K to be

$$\dim_{\mathrm{P}}(K) = \inf\left\{\sup_i \{\overline{\dim_{\mathrm{M}}}\, K_i : K = \cup K_i\}\right\}.$$

Here, the infimum is taken over all coverings of K by countably many compact subsets K_i. Note that we do not require K itself to be compact.

In this paper, we will investigate the upper Minkowski and packing dimension of *unbounded* Julia sets, so strictly speaking, the definition above does not make sense. We can instead consider the *local upper Minkowski dimension* of the Julia set, which is the upper Minkowski dimension of the Julia set intersected with an open neighborhood of finite diameter. In [RS05], Rippon and Stallard show that the local upper Minkowski dimension of the Julia set of an entire function is constant and coincides with its packing dimension, except perhaps in a neighborhood of 1 point (a point with finite backward orbit; there is at most 1 by the Picard theorem). Our example will not have an exceptional value of this kind, so their result further implies that the packing dimension and local upper Minkowski dimension are the same, no matter where we compute the local upper Minkowski dimension. In light of this, we will abuse notation and refer to the local upper Minkowski dimension of $\mathcal{J}(f)$ by $\overline{\dim_{\mathrm{M}}}(\mathcal{J}(f))$; the neighborhood we are using will always be made clear.

A detailed discussion of these dimensions can be found in [Bis17]. The survey [Sta08] also contains a detailed discussion of the above definitions, along with an overview of many results about the dimension of Julia sets of transcendental entire functions. We focus instead on a detailed discussion of Whitney-type decompositions, which will be our primary tool in estimating the packing dimension.

An interval $I \subset \mathbb{R}$ is called *dyadic* if $I = [j/2^n, (j+1)/2^n]$ for some integers j and n. We denote the set of all dyadic intervals by Δ, and all the dyadic intervals of side length 2^{-n} by Δ_n. Notice that $\Delta = \cup_n \Delta_n$. A *dyadic square* Q in the plane is the product of two dyadic intervals in Δ_n.

Let $F \subset \mathbb{C}$ be a nonempty closed set, and let $\Omega = \mathbb{C} \backslash F$. A *Whitney decomposition* of Ω is a countable collection of dyadic squares $\{Q_j\}$ satisfying the following three properties:

(1) $\Omega = \cup_j \overline{Q_j}$.
(2) For all j, k, Q_j and Q_k have disjoint interior.
(3) There exists a constant $C > 1$ so that for all squares Q_j,
$$\frac{1}{C} \operatorname{dist}(Q_j, \partial\Omega) \leq \operatorname{diam}(Q_j) \leq C \operatorname{dist}(Q_j, \partial\Omega).$$

Whitney decompositions always exist when F is nonempty, (see [Ste70], p. 167). We may always choose the constant $C = 4$.

For our purposes, the key feature of Whitney decompositions will be that the squares are approximately squares with unit area with respect to the hyperbolic metric. It is often advantageous to consider more abstract decompositions with similar properties where the elements will not necessarily be dyadic squares. To distinguish these objects, we define a (C, λ)-*Whitney type decomposition* to be a countable collection of sets $\{S_j\}$ whose boundaries are quasicircles that satisfy the following four properties:

(1) $\Omega = \cup_j \overline{S_j}$
(2) For all j, k, S_j and S_k have disjoint interior.
(3) There exists a constant $C > 1$ so that for all S_j,
$$\frac{1}{C} \operatorname{dist}(S_j, \partial\Omega) \leq \operatorname{diam}(S_j) \leq C \operatorname{dist}(S_j, \partial\Omega).$$
(4) There exists a constant λ so that for all S_j we have
$$\frac{\operatorname{diam}(S_j)^2}{\operatorname{Area}(S_j)} \leq \lambda.$$

For convenience we will often omit the constants and refer to such collections as Whitney type decompositions, and we will still refer to the elements as squares. Note that the Whitney decomposition of dyadic squares described above is a $(4, 2)$-Whitney type decomposition. Whenever we summon a Whitney type decomposition in a proof, unless stated otherwise, we will assume it is the $(4, 2)$-Whitney type decomposition of dyadic squares. A Whitney type decomposition of an open set is defined using $F = \mathbb{C} \setminus \Omega$. A Whitney type decomposition of a neighborhood of a set $\Omega = \mathbb{C} \setminus F$ is a collection of sets $\{S_j\}$ within a bounded distance of F.

Example 4.1. It is often useful to create Whitney-type decompositions well-adapted to the dynamics of polynomial maps. The following example will be important for this paper. Let $f(z) = z^N$, $N \in \mathbb{N}$. We will create a Whitney type decomposition of $\mathbb{C} \setminus \overline{B(0,1)}$ in a neighborhood of $\overline{B(0,1)}$ using the dynamics of f.

First, let C_n denote the circle $C(0, R^{1/2^{Nn}})$. We define $C_0 = C(0,R)$. Then $f(C_n) = C_{n-1}$ and $f^n(C_n) = C_0$ for all $n \geq 1$. Let A_n denote the open round annulus with inner boundary C_n and outer boundary C_{n-1}. Define $S_{j,n} \subset A_n$ via preimages of A_1 under f^n. There are many ways to do this, and we shall make the normalization that $S_{1,n}$ has a radial boundary segment that rests on the real line for each n. See Figure 4.

The collection $\{S_{j,n}\}$ is a (C,λ)-Whitney type decomposition of $B(0,R)\setminus\overline{B(0,1)}$, where C and λ depend on N and R. The Whitney type decomposition is dynamical in the sense that $f(S_{j,n}) = S_{j',n-1}$ for some new j', and if $\mathrm{int}(S_{j,n})$ is the interior of $S_{j,n}$, $f^n(\mathrm{int}(S_{j,n}))$, is A_1 with a radial slit removed.

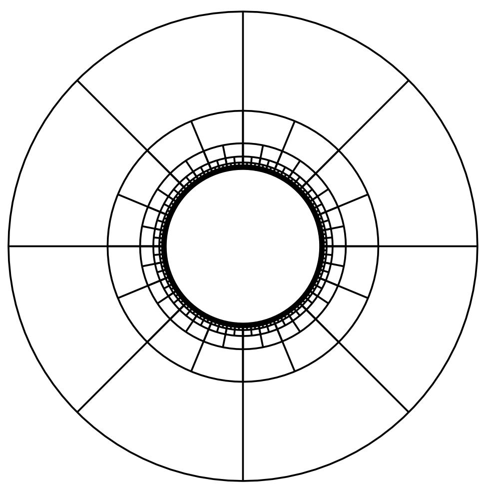

FIGURE 4. The Whitney type decomposition in Example 4.1. In this picture the "squares" form a Whitney type decomposition for the complement of $F = \mathbb{C} \setminus \mathbb{D}$. Each square in the picture has the property z^2 maps a square in one ring of squares to one in the next ring, normalized so that one of the squares in each ring has a radial segment on the real axis.

The following is a simple but very useful geometric lemma.

Lemma 4.2. *Let $W_1(\Omega)$ and $W_2(\Omega)$ be (C_j, λ_j)-Whitney type decompositions, $j = 1, 2$, of Ω. Then there exists a constant $L = L(C_1, C_2, \lambda_1, \lambda_2)$ such that if $Q \in W_1(\Omega)$, then Q is covered by at most L elements $S \in W_2(\Omega)$, and vice versa.*

The key to the proof of the lemma is comparing the area of a square Q in one collection with the area of all the squares in the other collection that intersect Q. We omit the details.

Let $K \subset \mathbb{C}$ be compact, and let $\Omega = \mathbb{C} \setminus K$. Let $W(\Omega)$ be a Whitney type decomposition of Ω. We define the *critical exponent* of K to be

$$\alpha(K) = \inf \left\{ \alpha \geq 0 \,:\, \sum_{Q \in W(\Omega)} \operatorname{diam}(Q)^\alpha < \infty \,,\, \operatorname{diam}(Q) < 1 \right\}.$$

The critical exponent does not depend on the Whitney type decomposition.

Lemma 4.3. *Suppose that $W(\Omega)$ and $W'(\Omega)$ are Whitney type decompositions. Then*

$$\sum_{Q \in W(\Omega)} \operatorname{diam}(Q)^\alpha < \infty \quad \textit{if and only if} \quad \sum_{Q \in W'(\Omega)} \operatorname{diam}(Q)^\alpha < \infty.$$

Proof. This follows immediately from Lemma 4.2. □

Given a Whitney type decomposition, we will sometimes call the sum of diameters of the squares as in Lemma 4.3 the α-*Whitney sum.* Thus $\alpha(K) = \alpha$ if and only if the t-Whitney sums for some Whitney type decomposition of the complement of K converge for all $t > \alpha$.

The critical exponent will is the main tool we will use to estimate the packing dimension. The following is Lemma 2.6.1 in [BP17].

Lemma 4.4. *Let $K \subset \mathbb{C}$ be compact. Then $\alpha(K) \leq \overline{\dim}_{\mathrm{M}}(K)$, and if K has zero area, then $\alpha(K) = \overline{\dim}_{\mathrm{M}}(K)$.*

In this paper, the fundamental Whitney type decompositions we consider are decompositions with dyadic squares, and the decomposition in Example 4.1. It will also be important that these Whitney type decompositions behave well under conformal and quasiconformal mappings.

Lemma 4.5. *Let $f : \Omega \to \Omega'$ be a K-quasiconformal homeomorphism of two domains in $\mathbb{C}$. Let $W(\Omega)$ be a Whitney type decomposition for Ω. Define*

$$f(W(\Omega)) := \{f(Q_j) \,:\, Q_j \in W(\Omega)\}.$$

Then $f(W(\Omega))$ is a (C', λ')-Whitney type decomposition of Ω', and the constants C and λ only depend on the constant K and on the constants C and λ.

It follows that conformal and quasiconformal mappings take Whitney type decompositions to new Whitney type decompositions. When f is conformal, the lemma above follows from the Lemma 3.2. When f is K-quasiconformal, this follows from Theorem 11.14 in [Hei01]. One can also argue using standard modulus of path family arguments to the topological annuli $\Omega \setminus \overline{Q}_j$ for each square $Q_j \in W(\Omega)$.

Theorem 4.6. *Let $f : \Omega \to \Omega'$ be a K-quasiconformal homeomorphism. Let $W(\Omega)$ and $W(\Omega')$ be Whitney type decompositions. Then there exists a constant L that depends only on K and the constants defining the Whitney type decompositions so that each $S \in f(W(\Omega))$ is covered by at most L elements $Q \in W(\Omega')$.*

Proof. This follows immediately from Lemma 4.2 and Lemma 4.5. □

We will use the following corollary often.

Corollary 4.7. *Let $f : \Omega \to \Omega'$ be K-quasiconformal. Then the α-Whitney sums of $W(\Omega')$ and $f(W(\Omega))$ are comparable with constant depending only on K and on the constants defining the Whitney type decompositions.*

5. The Definition of f

In this section, we specify the parameters defining f and show that it is an entire function.

Recall that the main cardioid of the Mandelbrot set is the region consisting of all parameters $c = \mu/2(1 - \mu/2)$, where $\mu \in \mathbb{D}$. If c is a parameter in the main cardioid, the Julia set of $z^2 + c$ is a quasicircle with an attracting fixed point in its interior. For each $s \in (1, 2)$, we may choose c in the main cardioid so that $\dim_{\mathrm{H}}(\mathcal{J}(z^2+c)) = \dim_{\mathrm{P}}(\mathcal{J}(z^2+c)) = \overline{\dim_{\mathrm{M}}}(\mathcal{J}(z^2+c)) = s$ (see [Shi98] p.232 and [Sul83] p.742, along with Theorem 7.6.7 in [PU10]).

Having chosen such a c, recall that we defined $g(z) = z^2 + c$, and $g^N(z)$ denotes the Nth iterate of g. Since g^N is a degree 2^N monic polynomial there exists some $R > 0$ so that if $|z| \geq R$ we have

$$\frac{1}{2} \leq \left| \frac{g^N(z)}{z^{2^N}} \right| \leq 2. \tag{5.1}$$

In fact, given any $\epsilon > 0$, there exists $R > 0$ so that if $|z| \geq R$ we have

$$\frac{1}{(1+\epsilon)} \leq \left| \frac{g^N(z)}{z^{2^N}} \right| \leq (1+\epsilon). \tag{5.2}$$

We will always assume R is big enough so that (5.1) holds.

Next given some integer $N > 0$ define a sequence of integers for $k = 0, 1, 2 \ldots$

$$n_k := 2^{N+k-1}.$$

Note that when $k \neq 0$, $n_k \geq 2^N$, $n_1 = 2^N$, and for all k we have $2n_k = n_{k+1}$. Given the R above, define

$$R_1 = 2R.$$

We will construct our infinite product as a sequence of partial products inductively as follows. Given R as above we can define

$$F_1(z) := \left(1 - \frac{1}{2}\left(\frac{z}{R_1}\right)^{n_1}\right),$$

$$f_1(z) := g^N(z) \cdot F_1(z),$$

$$R_2 := M(f_1, 2R_1) := \max\{|f_1(z)| \, : \, |z| = 2R_1\}.$$

Next, assume that f_{k-1}, F_{k-1} and R_k have all been defined. From there, we define

$$F_k(z) := \left(1 - \frac{1}{2}\left(\frac{z}{R_k}\right)^{n_k}\right),$$

$$f_k(z) := g^N(z) \prod_{j=1}^{k} F_j(z),$$

$$R_{k+1} := M(f_k, 2R_k) = \max\{|f_k(z)| \, : \, |z| = 2R_k\}.$$

With these starting parameters, we want to investigate the convergence of infinite products of the form

$$f(z) = \lim_{k\to\infty} f_k(z) = g^N(z) \cdot \lim_{k\to\infty} \prod_{j=1}^{k} F_j(z). \tag{5.3}$$

Next, we will introduce the parameter $\lambda \in \mathbb{D}$. Define $m_k = 2^{Nk}$, and then define

$$f_\lambda(z) = g^N(z) \cdot \lim_{k\to\infty} \prod_{j=1}^{k} \left(1 - \frac{1}{2}\left(\frac{\lambda^{m_k} z}{R_k}\right)^{n_k}\right),$$

$$f_{k,\lambda} = g^N(z) \cdot \prod_{j=1}^{k} \left(1 - \frac{1}{2}\left(\frac{\lambda^{m_k} z}{R_k}\right)^{n_k}\right)$$

We define

$$\widehat{R}_k := \frac{R_k}{\lambda^{m_k}}.$$

When $\lambda = 0$, f_λ is simply the polynomial $g^N(z)$.

To prove that f_λ defines an entire function, we first must record some basic facts about the growth rate of $\{\widehat{R}_k\}$ and n_k. The following Lemma is completely elementary but used often. We only remark that (3) below is just a restatement of (2).

Lemma 5.1 (The Growth Rate of n_k)**.** *For all $k = 1, 2, \ldots,$ we have*

(1) $n_k = 2n_{k-1}$*, and* $n_k \geq 2^N$*.*
(2) $2^N + \sum_{j=1}^{k} n_j = n_{k+1}$*.*

(3) $\deg(f_k) = 2\deg(F_k)$

Before introducing λ, the original sequence $\{R_k\}$ was defined in terms of the maximum modulus of the partial products defining f. The next Lemma says that $\{|\widehat{R}_k|\}$, despite not being defined in terms of the partial products, approximately behaves this way.

Lemma 5.2 ($\hat{R}_k$ grows similarly to R_k)**.** *Given any $\epsilon > 0$, the starting parameter R may be chosen large enough so that*

$$\frac{|\widehat{R}_{k+1}|}{(1+\epsilon)} \le \max_{|z|=2|\hat{R}_k|} |f_{k,\lambda}(z)| \le (1+\epsilon)|\widehat{R}_{k+1}|.$$

Proof. By (5.2), there exists $R > 0$ so that for all $|z| > R$ we have

$$\frac{1}{(1+\epsilon)^{1/2}} \left|\frac{g^N(z)}{z^{2^N}}\right| \le (1+\epsilon)^{1/2},$$

Given such a choice, we estimate

$$\begin{aligned}
\max_{|z|=2|\widehat{R}_k|} |f_{k,\lambda}(z)| &= \max_{|z|=2|\widehat{R}_k|} \left(|g^N(z)| \prod_{j=1}^{k} |F_j(\lambda^{m_k} z)| \right) \\
&\le (1+\epsilon)^{1/2} \max_{|z|=2|\widehat{R}_k|} \left(|z|^{2^N} \prod_{j=1}^{k} |F_j(\lambda^{m_k} z)| \right) \\
&= (1+\epsilon)^{1/2} \max_{|z|=2|\widehat{R}_k|} \left(|z|^{2^N} \prod_{j=1}^{k} |F_j(|\lambda^{m_k}|z)| \right).
\end{aligned}$$

The last equality from the fact the the maximum of $|F_j(\lambda^{n_k} z)|$ only depends on $|\lambda^{m_k}|$. Then using the definition of $\widehat{R}_k$,

$$\begin{aligned}
\max_{|z|=2|\widehat{R}_k|} |f_{k,\lambda}(z)| &\le \frac{(1+\epsilon)^{1/2}}{|\lambda|^{2^N \cdot 2^{kN}}} \max_{|z|=2R_k} \left(|z|^{2^N} \prod_{j=1}^{k} |F_j(z)| \right) \\
&\le \frac{(1+\epsilon)}{|\lambda|^{m_{k+1}}} \max_{|z|=2R_k} \left(|g^N(z)| \prod_{j=1}^{k} |F_j(z)| \right) \\
&= \frac{(1+\epsilon)}{|\lambda|^{2^{(k+1)N}}} R_{k+1} \\
&= (1+\epsilon)|\widehat{R}_{k+1}|
\end{aligned}$$

A similar argument shows that

$$\max_{|z|=2\widehat{R}_k} |f_\lambda(z)| \ge \frac{1}{(1+\epsilon)} |\widehat{R}_{k+1}|.$$

$\square$

Lemma 5.3 (The Growth Rate of $\widehat{R}_k$)**.** *If $k \geq 1$, R satisfies* (5.2) *for some $\epsilon < 10^{-1}$, and if $N \geq 10$, we have*

$$|\widehat{R}_{k+1}| \geq 2^{n_k} |\widehat{R}_k|^{2^{N-1}+n_{k-1}} \geq 2^N |\widehat{R}_k|^{2^N}.$$

Proof. By the assumptions on the initial parameters, we have

$$\begin{aligned}
|\widehat{R}_2| &\geq \frac{1}{1+\epsilon} \max_{|z|=2|\widehat{R}_1|} |f_{1,\lambda}(z)| = \frac{1}{1+\epsilon} \max_{|z|=2|\widehat{R}_1|} |g^N(z)| \cdot \left| \left(1 - \frac{1}{2}\frac{z^{n_1}}{\widehat{R}_1^{n_1}}\right) \right| \\
&\geq \frac{1}{(1+\epsilon)^2} \cdot |2\widehat{R}_1|^{2^N} \cdot \max_{|z|=2|\widehat{R}_1|} \left| \left(1 - \frac{1}{2}\frac{z^{n_1}}{\widehat{R}_1^{n_1}}\right) \right| \\
&\geq \frac{1}{(1+\epsilon)^2} \cdot 2^{2^N} \cdot |\widehat{R}_1|^{2^N} \cdot (2^{n_1-1} - 1) \\
&\geq 2^{2^N} |\widehat{R}_1|^{2^N} = 2^{n_1} |\widehat{R}_1|^{2^{N-1}+n_0}.
\end{aligned}$$

This is the base case for an induction. Suppose that for some $k \geq 3$, and for all $2 \leq j \leq k$, we have

$$|\widehat{R}_j| \geq 2^{n_{j-1}} |\widehat{R}_{j-1}|^{2^{N-1}+n_{j-2}} \geq 2^{2^N} |\widehat{R}_{j-1}|^{2^N} \geq 4|\widehat{R}_{j-1}|^2$$

This induction hypothesis implies that $|\widehat{R}_k|^{1/2} \geq |\widehat{R}_j|$ for all $j \leq k-1$. Therefore,

$$\begin{aligned}
|\widehat{R}_{k+1}| &\geq \frac{1}{(1+\epsilon)} \max_{|z|=2|\widehat{R}_k|} |f_{k,\lambda}(z)| = \frac{1}{(1+\epsilon)} \max_{|z|=2|\widehat{R}_k|} |g^N(z)| \cdot \prod_{j=1}^{k} \left| \left(1 - \frac{1}{2}\frac{z^{n_j}}{\widehat{R}_j^{n_j}}\right) \right| \\
&\geq \frac{1}{(1+\epsilon)^2} \cdot 2^{2^N} \cdot |\widehat{R}_k|^{2^N} \cdot \prod_{j=1}^{k} \left| 2^{n_j-1} \frac{\widehat{R}_k^{n_j}}{\widehat{R}_j^{n_j}} - 1 \right| \\
&\geq \frac{1}{(1+\epsilon)^2} \cdot 2^{2^N} \cdot |\widehat{R}_k|^{2^N} \cdot \prod_{j=1}^{k} \left| 2^{n_j-2} \frac{\widehat{R}_k^{n_j}}{\widehat{R}_j^{n_j}} \right| \\
&\geq \frac{1}{(1+\epsilon)^2} \cdot 2^{2^N} \cdot |\widehat{R}_k|^{2^N} \cdot 2^{n_k-2} \prod_{j=1}^{k-1} \left| 2^{n_j-2} \widehat{R}_k^{n_{j-1}} \right| \\
&\geq 2^{2^N - 2k + \sum_{j=1}^k n_j} \cdot |\widehat{R}_k|^{2^N + \sum_{j=1}^{k-1} n_{j-1}}.
\end{aligned}$$

We used the fact that $N > 10$ to move from the second line to the third line above. To wrap up we use Lemma 5.1 again to see that

$$\begin{aligned}
|\widehat{R}_{k+1}| &\geq 2^{2^N - 2k + n_{k+1}} \cdot |\widehat{R}_k|^{2^N + n_{k-1}} \\
&\geq 2^{2^N + n_k} \cdot |\widehat{R}_k|^{2^N + n_{k-1}}.
\end{aligned}$$

This completes the proof. $\square$

For the rest of the paper, we will always assume that $N \geq 10$, so that the conclusion of Lemma 5.3 is always valid. The lemma above also contains the following simpler inequalities that will often be sufficient for our purposes.

Corollary 5.4 (Other Useful Inequalities). *For $k \geq 1$ we have*

(1) $|\widehat{R}_{k+1}| \geq 4|\widehat{R}_k|^2$.
(2) $|\widehat{R}_{k+1}| \geq (2R)^{2^{kN}}$.

The proof of (1) above is obvious, and the proof of (2) is a simple induction. See Corollary 8.3 of [Bis17]. The above inequalities allow us to apply the same argument as Lemma 5.2 in [Bis17] and conclude that $f_\lambda(z)$ is a transcendental entire function for all $\lambda \in \mathbb{D} \setminus \{0\}$.

Corollary 5.5. *Let $\lambda \in \mathbb{D} \setminus \{0\}$. The function*

$$f_\lambda(z) = g^N(z) \cdot \prod_{k=1}^{\infty} F_k(z)$$

converges uniformly on compact subsets of $\mathbb{C}$. In particular, $f_\lambda(z)$ is a transcendental entire function.

We conclude this section by recording some useful estimates regarding the relative growth rates of $\{|\widehat{R}_k|\}$ that will be useful in Section 7. The proof follows from Theorem 5.3 and a use of Taylor series approximations, and we refer the reader to Sections 6 and 8 of [Bis17] for the details.

Lemma 5.6. *Suppose that $\{\widehat{R}_k\}$ has been defined as in this section, and $m \geq 1$. Then*

$$\prod_{j=1}^{k-1} \left(1 + \left(\frac{|\widehat{R}_j|}{|\widehat{R}_k|}\right)^m\right) = 1 + O\left(|\widehat{R}_k^{-m/2}|\right), \tag{5.4}$$

$$\prod_{j=k+1}^{\infty} \left(1 + \frac{|\widehat{R}_k|}{|\widehat{R}_j|}\right) = 1 + O\left(|\widehat{R}_k^{-1}|\right). \tag{5.5}$$

Finally, if $|z| \leq 4|\widehat{R}_k|$, we have

$$\prod_{j=k+1}^{\infty} F_j(\lambda^{m_j} z) = 1 + O\left(|\widehat{R}_k^{-1}|\right). \tag{5.6}$$

6. The Mapping Behavior of f away from the origin

We now move on to analyzing the function f_λ far away from the origin when $\lambda \in \mathbb{D} \setminus \{0\}$. The purpose of this section is to show that f_λ behaves like simpler functions on suitably defined regions of $\mathbb{C}$. To be more specific, recall that

$$f_\lambda(z) := g^N(z) \cdot \prod_{j=1}^{\infty} F_j(\lambda^{m_j} z).$$

We will show that we can decompose $\mathbb{C} \setminus B(0, |\widehat{R}_1|/4)$ into regions where f_λ looks approximately like the jth term of the infinite product. The observations and estimates here are vital for understanding to precise dynamical behavior of f_λ.

We define

$$H_m(z) = z^m(2 - z^m).$$

A detailed description of the conformal mapping behavior of H_m can be found in Section 9 of [Bis17]. For our purposes, we will need to consider the connected components of $\mathbb{C} \setminus \{|H_m(z)| = 1\}$. This set has $m+2$ connected components, one unbounded, one containing the origin, and m petals. We denote a single petal by Ω_m^p. Then $H_m : \Omega_m^p \to \mathbb{D}$ is a conformal mapping, and $\operatorname{diam}(\Omega_m^p) = O(1/m)$. See Figure 5.

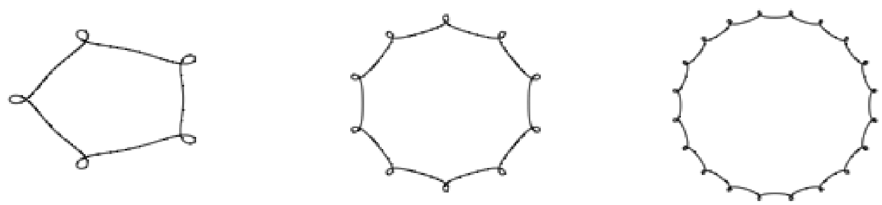

FIGURE 5. An illustration of the level sets of $\{|H_m(z)| = 1\}$ for $m = 5, 10$ and 20. There are m petals where H_m is a conformal mapping to the disk, and as m grows, the diameter of the petals shrinks. All the points on $\{|H_m(z)| = 1\}$ are distance $O(1/m)$ from the unit circle $|z| = 1$.

Next, we decompose $\mathbb{C} \setminus B(0, |\widehat{R}_1|/4)$ into annuli as follows.

$$A_k := \left\{ z : \frac{1}{4}|\widehat{R}_k| \le |z| \le 4|\widehat{R}_k| \right\}, \qquad B_k := \left\{ z : 4|\widehat{R}_k| \le |z| \le \frac{1}{4}|\widehat{R}_{k+1}| \right\},$$

$$V_k := \left\{ z : \frac{3}{2}|\widehat{R}_k| \le |z| \le \frac{5}{2}|\widehat{R}_k| \right\}, \qquad U_k := \left\{ z : \frac{5}{4}|\widehat{R}_k| \le |z| \le 3|\widehat{R}_k| \right\}.$$

Note that V_k is compactly contained inside of U_k. See Figure 6.

The following is Lemma 10.1 in [Bis17]. We include its simple proof.

Lemma 6.1. *With H_m defined above, for all integers $k \ge 1$ we have*

$$F_k(z) = \frac{1}{2}\left(\frac{R_k}{z}\right)^{n_k} H_{n_k}\left(\frac{z}{R_k}\right).$$

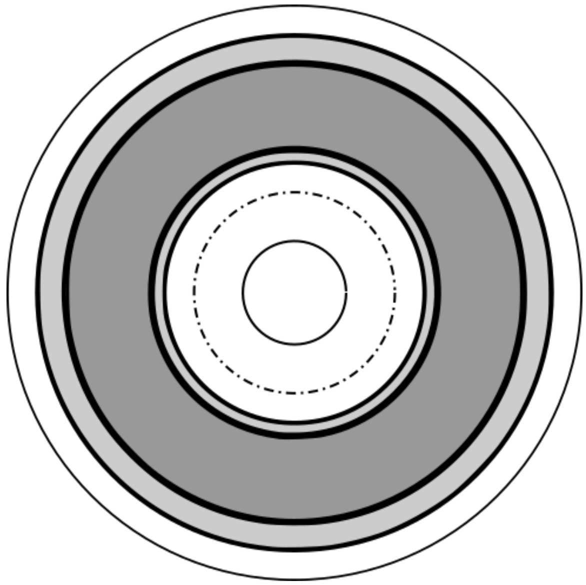

FIGURE 6. A schematic for A_k, $k \geq 1$. The innermost circle and outermost circle form the boundary of A_k. The dashed line is the circle $|z| = R_k$. The lightly shaded region is U_k, and the darker region is V_k, which is compactly contained in U_k. In the upcoming sections, we will see that the Julia set of f is contained near the circle $|z| = R_k$ and in V_k.

Proof. We compute this directly by factoring as follows,

$$\begin{aligned} \frac{1}{2}\left(\frac{R_k}{z}\right)^{n_k} H_{n_k}\left(\frac{z}{R_k}\right) &= \frac{1}{2}\left(\frac{R_k}{z}\right)^{n_k}\left(\frac{z}{R_k}\right)^{n_k}\left(2-\left(\frac{z}{R_k}\right)^{n_k}\right) \\ &= \left(\frac{R_k}{z}\right)^{n_k}\left(\frac{z}{R_k}\right)^{n_k}\left(1-\frac{1}{2}\left(\frac{z}{R_k}\right)^{n_k}\right) \\ &= F_k(z). \end{aligned}$$

This is exactly what we wanted. $\square$

In particular, we have

$$F_k(\lambda^{m_k} z) = \frac{1}{2}\left(\frac{\widehat{R}_k}{z}\right)^{n_k} H_{n_k}\left(\frac{z}{\widehat{R}_k}\right).$$

The next lemma says that f_λ looks like a slightly perturbed multiple of H_{n_k} on the annuli A_k.

Lemma 6.2. *Let $k \geq 1$. If $z \in A_k$, there is a constant C_k so that*

$$f_\lambda(z) = C_k H_{n_k}\left(\frac{z}{\widehat{R}_k}\right)(1 + O(|\widehat{R}_k|^{-1})).$$

For $k \geq 2$, the constant C_k is given by the formula

$$C_k = (-1)^{k-1} 2^{-k} \widehat{R}_k^{n_k} \prod_{j=1}^{k-1} \widehat{R}_j^{-n_j}.$$

For $k = 1$ the constant is given by

$$C_1 = \frac{1}{2} \widehat{R}_1^{n_1}.$$

The proof is almost exactly the same as Lemma 10.2 in [Bis17]. The idea is very simple. We break f_λ into the product of three pieces:

$$f_\lambda(z) = \left(g^N(z) \cdot \prod_{j=1}^{k-1} F_j(\lambda^{m_j} z) \right) \cdot F_k(\lambda^{m_k} z) \cdot \left(\prod_{j=k+1}^{\infty} F_j(\lambda^{m_j} z) \right). \tag{6.1}$$

The first piece is g^N followed by the the first $k-1$ terms of the infinite product. That is estimated by some factoring and applying (5.4) from Lemma 5.6. The second piece is just F_k, which we rewrite using Lemma 6.1. The third part is the tail of the infinite product, which we estimate using (5.6) from Lemma 5.6.

The next lemma says that f_λ looks like a power function on B_k. This fact is used but not proved directly in [Bis17], so we include it for completeness.

Lemma 6.3. *For $z \in B_k$, we have*

$$f_\lambda(z) = -C_k \left(\frac{z}{\widehat{R}_k} \right)^{2n_k} (1 + O(|\widehat{R}_{k+1}|^{-1})) \cdot (1 + O(4^{-n_{k+1}})) \cdot (1 + O(4^{-n_k})).$$

Proof. The reasoning above and a similar decomposition to (6.1) allows us to conclude that on B_k,

$$f_\lambda(z) = C_k H_{n_k} \left(\frac{z}{\widehat{R}_k} \right) F_{k+1}(\lambda^{m_{k+1}} z)(1 + O(|R_{k+1}|^{-1})).$$

The only change is that we keep the F_{k+1} term. However, when $z \in B_k$, $4|\widehat{R}_k| \leq |z| \leq |\widehat{R}_{k+1}|/4$, so that,

$$\begin{aligned} H_{n_k}\left(\frac{z}{\widehat{R}_k} \right) &= \left(\frac{z}{\widehat{R}_k} \right)^{n_k} \left(2 - \left(\frac{z}{\widehat{R}_k} \right)^{n_k} \right) \\ &= \left(\frac{z}{\widehat{R}_k} \right)^{2n_k} \left(2 \left(\frac{\widehat{R}_k}{z} \right)^{n_k} - 1 \right) \\ &= -\left(\frac{z}{\widehat{R}_k} \right)^{2n_k} \left(1 + O(4^{-n_k}) \right). \end{aligned}$$

A similar computation yields

$$F_{k+1}(\lambda^{m_{k+1}}z) = \left(1 - \frac{1}{2}\left(\frac{z}{\widehat{R}_{k+1}}\right)^{n_{k+1}}\right) = (1 + O(4^{-n_{k+1}})).$$

□

Corollary 6.4. *f_λ is never zero on B_k.*

Next, we have the following estimates on the size of the coefficients C_k.

Lemma 6.5. *Let $R > 8$, $N > 3$.*

(1) *When $k \geq 2$, we have $|C_k| \geq |\widehat{R}_k|^{n_{k-1}}/2^k \geq 8|\widehat{R}_k|$.*
(2) *When $k = 1$, we have $|C_1| = |\widehat{R}_1|^{n_1}/2 \geq 8|\widehat{R}_1|$.*
(3) *For all $k \geq 1$, we have $|C_{k+1}| \geq |C_k| > 1$.*

Proof. The $k = 1$ case can be checked directly. For $k \geq 2$, we can compute, using the fact that $|\widehat{R}_j| \leq |\widehat{R}_k|^{1/2}$, that

$$\begin{aligned} |C_k| &= \frac{1}{2^k}|\widehat{R}_k|^{n_k}\prod_{j=1}^{k-1}|\widehat{R}_j|^{-n_j} \geq \frac{1}{2^k}|\widehat{R}_k|^{n_k}\prod_{j=1}^{k-1}|\widehat{R}_k|^{-n_j/2} \\ &= \frac{1}{2^k}|\widehat{R}_k|^{n_k-n_{k-1}} = \frac{1}{2^k}|\widehat{R}_k|^{n_{k-1}}. \end{aligned}$$

So in this case we see that

$$|C_k| \geq \frac{1}{2^k}|\widehat{R}_k|^{n_{k-1}} > 8|\widehat{R}_k|.$$

Part (3) is easily checked by computing and estimating $|C_{k+1}|/|C_k|$, see Lemma 10.4 in [Bis17]. □

The next two proofs are Lemmas 10.5 and 10.6 of [Bis17]. They are proved using Lemma 6.2 and factoring techniques similar to the proof of Lemma 6.3. They are quantitative statements that say that far enough away from the set of points where $|H_m(z)| = 1$, H_m looks like z^{2m} and near the origin, H_m looks like z^m. Lemma 10.5 in [Bis17] actually has an error in the statement with a missing factor of $|\widehat{R}_k|^{-n_k}$, although the proof is correct. The correct statement is below.

Lemma 6.6. *For all $k \geq 1$, and for z satisfying $5R_k/4 \leq |z| \leq 4R_k$, we have*

$$f_\lambda(z) = C_k\left(\frac{z}{\widehat{R}_k}\right)^{2n_k}\left(1 + O\left(\left(\frac{4}{5}\right)^{n_k}\right)\right)(1 + O(|\widehat{R}_k|^{-1})).$$

Lemma 6.7. *For $k \geq 1$, and $R_k/4 \leq |z| \leq 4R_k/5$, we have*

$$f_\lambda(z) = 2C_k\left(\frac{z}{\widehat{R}_k}\right)^{n_k}\cdot\left(1 + O\left(\left(\frac{4}{5}\right)^{n_k}\right)\right)(1 + O(|\widehat{R}_k^{-1}|)).$$

We will see that most of the interesting mapping behavior for f_λ happens near $|z| = |\widehat{R}_k|$, where f_λ "interpolates" from being a perturbed degree n_k power mapping to a perturbed degree n_{k+1} power mapping.

The conformal mapping behavior of f_λ and its iterates will be very important later on, so having control of the critical points and critical values of f_λ will be very important. The statement below is Corollary 10.7 of [Bis17].

Theorem 6.8. *$f'_\lambda(z)$ is non-zero on V_k.*

The proof of Theorem 6.8 requires the following lemma.

Lemma 6.9. *On U_k, we have*

$$f_\lambda(z) = C_k \left(\frac{z}{\widehat{R}_k}\right)^{2n_k} (1 + h_k(z)).$$

The function $h_k(z)$ is holomorphic on U_k with

$$|h_k(z)| = O\left(\left(\frac{4}{5}\right)^{n_k} + |\widehat{R}_k|^{-1}\right).$$

Lemma 6.9 follows immediately from Lemma 6.6. To see how this implies Theorem 6.8, define

$$\epsilon_k = C\left(\left(\frac{3}{4}\right)^{n_k} + |\widehat{R}_k^{-1}|\right).$$

The constant $C > 0$ is chosen so that $|h_k(z)| \leq \epsilon_k$ on the annuli U_k. It follows that $\sum \epsilon_k$ can be made arbitrarily small, given that N and R are sufficiently large. This will come up again so we state this carefully below.

Lemma 6.10. *Let C, $h_k(z)$, and ϵ_k be defined as above. Let $\delta > 0$ be given. Then for N and R sufficiently large,*

$$\sum_{k=1}^{\infty} \epsilon_k < \delta.$$

Theorem 6.8 now follows from the Cauchy estimates applied to $h'_k(z)$ and a direct estimate of $f'_\lambda(z)$ on U_k using Lemma 6.6.

The following lemma is equation (10.5) on p. 435 of [Bis17], but contains a typo (due to the typo in the statement of Lemma 6.6) and the proof is omitted. We include a corrected version along with a proof.

Lemma 6.11. *For all $k \geq 1$ we have*

$$\frac{1}{4} \leq \frac{|\widehat{R}_{k+1}|}{|C_k| \cdot 2^{n_{k+1}}} \leq 4.$$

Proof. By (5.6), if $z \in A_k$ we have

$$f_\lambda(z) = f_{k,\lambda}(z)(1 + O(|\widehat{R}_k|^{-1})).$$

Recall that $f_{k,\lambda}$ was the kth partial product of the infinite product defining f_λ. By (5.1) it follows that if R is sufficiently large, then

$$\frac{1}{2} \leq \frac{M(f_\lambda, 2|\widehat{R}_k|)}{|\widehat{R}_{k+1}|} \leq 2.$$

Next note that Lemma 6.6 applies to f_λ on $|z| = 2|\widehat{R}_k|$ as well, and we also have

$$\frac{1}{2} \leq \frac{M(f_\lambda, 2|\widehat{R}_k|)}{|C_k| \cdot 2^{n_{k+1}}} \leq 2.$$

The conclusion follows immediately. □

With Corollary 6.11 and the estimates of the previous section, we can prove the following theorem.

Theorem 6.12. *For $N > 10$ and sufficiently large R, we have that $A_{k+1} \subset f_\lambda(V_k) \subset f_\lambda(A_k)$ and $f_\lambda(B_k) \subset B_{k+1}$. Moreover, f_λ maps the outermost boundary component of V_k into B_{k+1} and the innermost boundary component of V_k into B_k.*

For example, if $|z| = 4|\widehat{R}_k|$, we have

$$|f_\lambda(z)| \geq \frac{1}{4}|C_k|4^{n_k+1} \geq \frac{1}{16}2^{n_{k+1}}|\widehat{R}_{k+1}|. \tag{6.2}$$

We also have

$$|f_\lambda(z)| \leq 4|C_k|4^{n_{k+1}} \leq 16 \cdot 2^{n_{k+1}}|\widehat{R}_{k+1}| \leq \frac{16}{|\widehat{R}_k|^{2^N-1}}|\widehat{R}_{k+2}|. \tag{6.3}$$

Using Lemma 5.3, we can conclude that the outermost boundary component of B_k is mapped well inside B_{k+1} by f_λ. Observations like this allow us to deduce Theorem 6.12, whose straightforward but somewhat tedious proof can be found in Sections 11 and 12 of [Bis17].

As a result we can obtain our first dynamical corollaries.

Corollary 6.13. *Each set B_k is in the Fatou set of f_λ.*

Proof. Since B_k maps into B_{k+1} we know that the iterates tend to infinity locally uniformly. □

An *asymptotic value* of an entire function g is a point $w \in \mathbb{C}$ such that there exists a curve $\gamma : [0, \infty) \to \mathbb{C}$ such that $\gamma(t) \to \infty$ and $g(\gamma(t)) \to w$ as $t \to \infty$.

Corollary 6.14. *f_λ has no finite asymptotic values.*

Proof. If $\gamma : [0, \infty) \to \mathbb{C}$ satisfies $\gamma(t) \to \infty$ as $t \to \infty$, then there exists t_k with $t_k < t_{k+1}$, $t_k \to \infty$ and $\gamma(t_k) \in B_k$. Therefore by Theorem 6.12 $f_\lambda(\gamma(t_k)) \in B_{k+1}$, so that $f_\lambda(\gamma(t_k)) \to \infty$. Therefore f_λ has no finite asymptotic values. □

When we prove that f_λ has multiply connected Fatou components, that will also imply f has no asymptotic values (see [Sch10] Corollary 2.7).

7. Behavior of f near the Origin

Having analyzed the behavior of f_λ away from the origin, we now analyze f_λ near the origin. The primary goal of this section is to show that the Fatou set of f_λ contains an attracting basin containing the origin, and that the boundary of this fractal basin of attraction moves holomorphically with respect to the parameter λ. This will allow us to control the Hausdorff dimension of the boundary of the basin of attraction.

Lemma 7.1. *On $B(0, 1/4\widehat{R}_1)$, we have*

$$f_\lambda(z) = g^N(z)(1 + O(4^{-n_1}))(1 + O(|\widehat{R}_{k+1}|^{-1})).$$

Assume that R is large enough so that $|\widehat{R}_1|^{1/2} < 1/4|\widehat{R}_1|$. Then by perhaps choosing R larger, on $B(0, 2|\widehat{R}_1|^{1/n_2})$ we also have

$$f_\lambda(z) = g^N(z) \cdot (1 + O(|\widehat{R}_1|^{\frac{1}{2}-n_0})).$$

Proof. The techniques are similar to the previous section. Indeed, when $|z| \leq 1/4|\widehat{R}_1|$ we have

$$f_\lambda(z) = g^N(z)\left(1 - \frac{1}{2}\left(\frac{z}{\widehat{R}_1}\right)^{n_1}\right)\prod_{k=2}^{\infty} F_k(\lambda^{m_k}z) = g^N(z)(1+O(4^{-n_1}))(1+O(|\widehat{R}_2|^{-1})).$$

The proof on $B(0, 2|\widehat{R}_1|^{1/n_2})$ is similar. □

When $z \in B(0, 2|\widehat{R}_1|^{1/n_2})$ we write $f_\lambda(z) = g^N(z) \cdot (1 + \epsilon(z))$, where $|\epsilon(z)| = O(|\widehat{R}_1|^{\frac{1}{2}-n_1})$ as $R \to \infty$. We define $r := |\widehat{R}_1|^{1/n_2}$.

Lemma 7.2. *Let $\eta > 0$ be given. Then there exists sufficiently large R so that on $B(0, r)$ we have*

$$\sup_{z \in B(0,r)} |f'_\lambda(z) - (g^N)'(z)| < \eta.$$

In particular, the estimate holds for all $\lambda \in \mathbb{D}$.

Proof. Let $K = \sup_{z\in B(0,2r)} |g^N(z)|$ and $K' = \sup_{z\in B(0,r)} |(g^N)'(z)|$. Then we choose R so that r is large enough so that (5.1) applies, and we have

$$K \leq 2^{n_1+1} r^{2^N} = 2^{n_1+1}|\widehat{R}_1|^{\frac{1}{2}}.$$

By the Cauchy estimate we have

$$K' \leq 2^{n_1+1} r^{2^N-1} = 2^{n_1+1}|\widehat{R}_1|^{\frac{1}{2}-\frac{1}{n_2}}.$$

Next note that $|\epsilon(z)| = O(|\widehat{R}_1|^{\frac{1}{2}-n_0})$ on $B(0, 2r)$. Then the Cauchy estimate says

$$\max_{z\in B(0,r)} |\epsilon'(z)| = O(|\widehat{R}_1|^{-\frac{1}{2}-n_0})$$

Therefore, given $\eta > 0$, there exists R sufficiently large so that

$$\begin{aligned}\sup_{z\in B(0,r)} |f'(z) - (g^N)'(z)| &= \sup_{z\in B(0,r)} |(g^N)'(z)\epsilon(z) + g^N(z)\epsilon'(z)| \\ &\leq O(|\widehat{R}_1|^{-1}) < \eta.\end{aligned}$$

This concludes the proof. □

For the rest of this section we define $B = B(0,r)$. Since $g^N(z)$ is a polynomial, for sufficiently large values of R, $(g^N, (g^N)^{-1}(B), B)$ is a degree 2^N polynomial-like mapping. The Julia set and filled Julia set of the polynomial-like mapping $(g^N, (g^N)^{-1}(B), B)$ are the same as the Julia set and filled Julia set of the polynomial g^N. g^N is also hyperbolic on its quasicircle Julia set. In fact, there exists a topological annulus A that contains $\mathcal{J}(g^N)$ so that $|(g^N)'(z)| \geq \mu > 1$ on A. By Lemma 7.2, if R is sufficiently large then $|f_\lambda'(z)| \geq \mu_0 > 1$ on A as well for all $\lambda \in \mathbb{D}$. Let D_λ denote the bounded complementary component of $f_\lambda^{-1}(A)$. All $2^N - 1$ finite critical points (counted with multiplicity) and values of f_λ contained in B are contained in the interior of D_λ, and all $2^N - 1$ finite critical values are also contained in the interior of D_λ.

Lemma 7.3. *Let B and r be defined as above. Define $U_\lambda = (f_\lambda)^{-1}(B)$. Then for any $\lambda \in \mathbb{D}$ and all R sufficiently large, $(f_\lambda, U_\lambda, B)$ is degree 2^N polynomial-like mapping.*

Note here again that $f_0 = g^N$.

Proof. By the discussion above and Lemma 7.1 and Lemma 7.2, if R is sufficiently large, all the critical points and critical values of f_λ in B are contained in D_λ. Therefore U_λ is a topological disk compactly contained in B, and the result follows. □

We now want to show that as we vary λ, the Julia sets J_{f_λ} of the polynomial-like mappings move holomorphically. First, we recall the definition.

Definition 7.4. A *holomorphic motion* of a set $E \subset \mathbb{C}$ is a family of injective maps

$$\varphi_\lambda : E \to \mathbb{C},$$

one for each $\lambda \in \mathbb{D}$, so that $\varphi_\lambda(\cdot)$ is holomorphic for fixed $z \in E$, and φ_0 is the identity.

We also want our holomorphic motion to respect the dynamics of f_λ when viewed as polynomial-like mappings. That is, the holomorphic motion needs to satisfy

$$\varphi_\lambda : J_{f_0} \to J_{f_\lambda},$$

and

$$f_\lambda \circ \varphi_\lambda(z) = \varphi_\lambda \circ f_0(z).$$

If these conditions hold, we will call the holomorphic motion equivariant.

We need the following to fundamental facts about holomorphic motions. First, we have the λ-lemma. See Theorem 4.1 of [McM94].

Theorem 7.5 (The λ-lemma)**.** *A holomorphic motion of a set E has a unique extension to a holomorphic motion of $\overline{E}$. The extended motion is continuous $h_\lambda : \mathbb{D} \times \overline{E} \to \hat{\mathbb{C}}$. For each λ, h_λ extends to a quasiconformal map of $\mathbb{C}$ to itself.*

The λ-lemma implies that to show the Julia set moves holomorphically, it suffices to construct the holomorphic motion on the repelling periodic points, since these points are dense in the Julia set. Bers and Royden showed in [BR86] that the dilatation of holomorphic motions can be controlled in the following precise way:

Theorem 7.6. *If $h_\lambda : \mathbb{D} \times E \to \mathbb{C}$ is a holomorphic motion, then every h_λ is the restriction to E of a K-quasiconformal self map H_λ of $\mathbb{C}$, with dilatation not exceeding*

$$K = \frac{1 + |\lambda|}{1 - |\lambda|}.$$

Finally, we will use the following version of the holomorphic implicit function theorem (see, for example, [Var11], Theorem 2.3.10).

Theorem 7.7. *Let $U \subset \mathbb{C} \times \mathbb{C}$ be open with $(z_0, w_0) \in U$. Let $F : U \to \mathbb{C}$ be continuous and holomorphic in each variable separately. Suppose that $F(z_0, w_0) = 0$ and $F_z(z_0, w_0) \neq 0$. Let $\Pi_2 : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ denote projection onto the second coordinate. Then there exists an open set V containing (z_0, w_0) so that for all $w \in \Pi_2(V)$, the equation $F(z, w) = 0$ has a unique solution $\varphi(w)$ so that $(\varphi(w), w) \in V$. Moreover, the function $w \mapsto \varphi(w)$ is holomorphic.*

Now let's move onto the family of functions f_λ. We have shown that independent of $\lambda \in \mathbb{D}$, there exists a ball B so that (f, U_λ, B) is a degree 2^N polynomial-like mapping, where $U_\lambda = f_\lambda^{-1}(B)$.

Lemma 7.8. *The filled Julia set of each $(f_\lambda, U_\lambda, B)$ is the closure of exactly one basin of attraction.*

Proof. By Lemma 7.2 there exists a Jordan annulus A so that $|f_\lambda'| > \mu_0 > 1$ on A for all $\lambda \in \mathbb{D}$. Let D_λ be the bounded complementary component of $f_\lambda^{-1}(A)$. Then $f_\lambda : D_\lambda \to D_\lambda$, and there exists an attracting fixed point c_λ for each f_λ contained in D_λ. This fixed point is the only possible fixed point or periodic cycle contained in D_λ.

Any other periodic cycles for $(f_\lambda, U_\lambda, B)$ must be contained in A. Since $|f'_\lambda| > 1$ on A, such a cycle must be repelling. Therefore, f_λ has exactly one attracting fixed point, and all other periodic cycles are repelling. It follows that K_{f_λ}, the filled Julia set of $(f_\lambda, U_\lambda, B)$, is the closure of exactly one basin of attraction. □

We denote this basin of attraction by B_{f_λ}.

Corollary 7.9. *The Julia set of $(f_\lambda, U_\lambda, B)$ is a quasicircle.*

Proof. This follows since the filled Julia sets are closures of a basin of attraction, combined with the straightening lemma. See Theorem 2.1, p.102 of [CG93]. □

Lemma 7.10. *The repelling periodic cycles of f_λ depend holomorphically on λ.*

Proof. We study the equation

$$Q(z, \lambda) = f_\lambda^m(z) - z = 0.$$

The solutions to this equation are exactly the periodic cycles of f_λ of period dividing m. Let z_0 be a repelling periodic cycle of period m for f_0. Let μ be the multiplier of f_0 at z_0. Then $z_0 \in J_{f_0}$, and $\mu \geq \mu_0 > 1$ and

$$|Q_z(z_0, 0)| = |\mu - 1| > 0.$$

By the holomorphic implicit function theorem, z_0 moves holomorphically.

Such a cycle remains repelling as we vary λ. The cycle cannot become attracting since the filled Julia set is the closure of exactly one basin of attraction. If the cycle became neutral, that is, $|(f_\lambda^m)'(z_0)| = 1$, there would exist a nearby λ' so that the cycle became attracting. □

For a repelling periodic point z we define $h_\lambda(z)$ to be the corresponding holomorphic motion of that point. If E is the set of all repelling periodic cycles, this defines a mapping

$$h_\lambda : E \times \mathbb{D} \to E_\lambda,$$

where E_λ is the set of repelling periodic cycles for $(f_\lambda, U_\lambda, B)$.

Lemma 7.11. *$h_\lambda : E \times \mathbb{D} \to E_\lambda$ is a holomorphic motion.*

Proof. It only remains to check injectivity. Again, for some given λ, a periodic cycle z_λ of period diving m is a solution of

$$Q(z, \lambda) = f_\lambda^m(z) - z = 0.$$

By the implicit function theorem, there exists a neighborhood of (z_λ, λ) such that for all choices of λ' in this neighborhood there is a unique $z_{\lambda'}$ so that $Q(z_{\lambda'}, \lambda') = 0$. So by the uniqueness statement of the implicit function theorem, we must have injectivity. □

Corollary 7.12. *The Julia sets of* $(f_\lambda, U_\lambda, B)$ *move holomorphically.*

Proof. By the λ-Lemma, the holomorphic motion h_λ extends to the closures of the repelling periodic points. We just need to check the equivariance

$$f_\lambda \circ h_\lambda(z) = h_\lambda \circ f_0(z),$$

for all $z \in J_{f_0}$. Since h_λ maps periodic points of period m onto distinct periodic points of period m, if z is periodic with period m, we have

$$h_\lambda(f_0^m(z)) = h_\lambda(z) = f_\lambda^m(h_\lambda(z)).$$

It follows that $f_\lambda(h_\lambda)(z)$ and $h_\lambda(f_0(z))$ must belong to the same periodic cycle, so they must be equal. Therefore, we have equivariance on the repelling periodic points, and by density, this extends to equivariance on the Julia sets of the polynomial like mappings. □

Corollary 7.13. $\lambda \in \mathbb{D}$ *may be chosen so that the Hausdorff dimension of the Julia set of* (f_0, U_0, B) *is arbitrarily close to the Hausdorff dimension of the Julia set of* $(f_\lambda, U_\lambda, B)$.

Proof. Theorem 7.6 shows that the h_λ extends to a quasiconformal self map of $\mathbb{C}$ and the dilatation of h_λ may not exceed $K = 1 + |\lambda|/(1 - |\lambda|)$. Since K-quasiconformal mappings are locally $1/K$-Holder continuous, it follows that as $\lambda \to 1$, $\dim_H(J_\lambda) \to \dim_H(J_0)$, which proves the claim. □

Lemma 7.14. $J_{f_\lambda} \subset \mathcal{J}(f_\lambda)$. *In other words, the Julia set of the polynomial like mapping* $(f_\lambda, U_\lambda, B)$ *is a subset of the Julia set of the entire function* f.

Proof. As a polynomial-like map, f_λ has exactly one attracting basin, B_{f_λ}. B_{f_λ} and J_{f_λ} are invariant for $(f_\lambda, U_\lambda, B)$, and it follows that they are forward invariant sets for the entire function f_λ. If $z \in J_{f_\lambda}$, then there always exists a nearby point w so that $f_\lambda^n(w)$ converges to the attracting fixed point in B_{f_λ}. But $f_\lambda^n(z) \in J_{f_\lambda}$ for all n, so f_λ and its iterates cannot form a normal family in any neighborhood of z. Therefore $z \in \mathcal{J}(f_\lambda)$. □

It follows from the reasoning above that if $s = \dim_{\mathrm{H}}(\mathcal{J}(g^N))$ and $\epsilon_0 > 0$ is given that we may choose λ so that

$$\dim_{\mathrm{H}}(\mathcal{J}(f_\lambda)) \geq s - \epsilon_0. \tag{7.1}$$

Important Remark on Notation: The primary purpose of the parameter λ is to control the change in Hausdorff dimension from J_λ to J_0 with arbitrary precision. We will always assume the λ chosen in the corollary above is real and so that (7.1) holds. Therefore, it will no longer be necessary to write $|\widehat{R}_k|$. Furthermore, for readability, we will assume that λ has been chosen so that Corollary 7.13 is true, and for the rest of the paper we will suppress the notation λ and just refer to the function as f. We will also refer to the values of $\widehat{R}_k$ simply as R_k.

8. Conformal Mapping Properties of the Fatou and Julia Set

Let $k \geq 0$. Define,

$$A_0 = \left\{ |z| \leq \frac{R_1}{4} \;:\; f(z) \in A_1 \right\},$$

$$A_{-k} = \left\{ |z| \leq \frac{R_1}{4} \;:\; f^j(z) \in |z| \leq \frac{R_1}{4},\ j = 1, 2, \ldots, k, \text{ and } f^{k+1}(z) \in A_1 \right\}.$$

So after $k+1$ iterates, f maps A_{-k} into A_1. We make similar definitions for V_k and B_k, $k \leq 0$.

Let C denote the set of all points who eventually get mapped into the filled Julia set K_f of f viewed as a polynomial-like mapping. Then

$$C = \bigcup_{k=1}^{\infty} f^{-k}(K_f).$$

Now define

$$A := \bigcup_{k \in \mathbb{Z}} (A_k \setminus C).$$

It follows from Lemma 6.12 that $f^{-1}(A) \subset A$.

The first lemma of this section tells us where the Julia set is located in each A_k, $k \geq 1$. Recall that $\Omega^p_{n_k}$ is a "petal" region where $H_{n_k}(z) = z^{n_k}(2 - z^{n_k})$ restricts to a conformal mapping onto the disk. Let $\mathcal{P}_{n_k}$ denote the union of all the petals of H_{n_k}. Then $R_k \cdot \Omega^p_{n_k}$ is a petal of $H_{n_k}(z/R_k)$, so we will let $R_k \cdot \mathcal{P}_{n_k}$ denote the petals of $H_{n_k}(z/R_k)$.

Lemma 8.1. *Let $k \geq 1$. Then $(\mathcal{J}(f) \cap A_k) \subset V_k \cup (R_k \cdot \mathcal{P}_{n_k})$. The diameter of the portion of the Julia set contained in each petal $R_k \cdot \Omega^p_{n_k}$ is at most $R_k^{-2} n_k^{-2}$.*

To prove this lemma, one uses Lemma 6.6 to conclude that the region, $\{z : 2R_k \leq |z| \leq 4R_k\}$ gets mapped into B_{k+1}, and then uses Lemma 6.7 to show that $\{z : 1/4R_k \leq |z| \leq 4/5R_k\}$ gets mapped into B_k, so these regions belong in the Fatou set. To deal with the points in $\{z : 4/5R_k \leq |z| \leq 3/2R_k\}$, we first observe that the zeros of f in A_k belong to this sub-annulus, and define

$$T^\eta_k = \{z : \ 4/5R_k \leq |z| \leq 3/2R_k \text{ and } H_{n_k}(z/R_k) > \eta\}.$$

The points where $4/5R_k \leq |z| \leq 3/2R_k$ and $H_{n_k}(z/R_k) \leq |\eta|$ for $\eta < 1/2$ are contained in $R_{n_k} \cdot \mathcal{P}_{n_k}$. One can show that there exists a small value η which depends only on R such that $f(T^\eta_k) \subset B_k$, and is therefore in the Fatou set. The diameter argument follows from this observation; H_{n_k} must be very small on the Julia set contained inside $R_k \cdot \Omega^p_{n_k}$, therefore the Julia set has small diameter inside of the petal. The details for all of these arguments follow similarly as in Sections 11-13 of [Bis17].

The next lemma characterizes the dynamics of the critical points of f. Recall that B_f is the basin of attraction for the polynomial-like mapping f. Since f has no asymptotic values, it's postcritical set is $P(f) = \{f^n(z) : n \geq 1, z \text{ a critical point}\}$.

Lemma 8.2. *For sufficiently large R, the critical points of f are either contained in B_f or A_k for some $k \geq 1$. If z is a critical point contained in A_k, then $f(z) \in B_k$. In both cases, z is in the Fatou set.*

This lemma is proved by studying the size of H_{n_k} at a critical point of f. Since f is close to H_{n_k}, its critical points are very close to the critical points of H_{n_k}. However, $|H_{n_k}| = 1$ at any of its critical points, so we expect that H_{n_k} should have modulus close to 1 at a critical point for f. This calculation is done explicitly in Lemma 14.2 in [Bis17], where at a critical point it is shown that

$$|1 - H_{n_k}(z)| \leq n_k^{-2}.$$

It follows from Lemma 6.2 and Lemma 6.5 that the distance between a critical value and the circle $C(0, 4R_k)$ is approximately

$$|C_k| - 4R_k \geq R_k^{n_{k-1}}/2^k - 4R_k$$

when $k \geq 2$ and approximately

$$R_1^{n_1}/2 - 4R_1,$$

when $k = 1$. So choosing R large enough, this distance can be made arbitrarily large. Finally, it follows that the distance between $P(f)$ and $\mathcal{J}(f)$ is strictly larger than 0.

Recall that for a set A, $\widehat{A}$ denotes its polynomial hull; the union of A and all its bounded complementary components. Lemma 8.2 implies the following.

Lemma 8.3. *Let $k \geq 1$ and let D_{k-1} denote the component of $f^{-1}(\widehat{A}_k)$ containing the origin. Then*

$$f : D_{k-1} \to \widehat{A}_k$$

is a degree n_{k+1} branched covering map when $k \geq 1$. When $k \leq 0$, then

$$f : D_{k-1} \to \widehat{A}_k$$

is a degree 2^N branched covering map.

The basic covering map lemma above can be refined as follows:

Lemma 8.4. *Let W be a connected component of $f^{-1}(A_k)$ for $k \in \mathbb{Z}$. Then we have the following possibilities.*

(1) *If* $k \geq 1$, $W \subset V_{k-1}$, *and* $f : W \to A_k$ *is a degree* n_k *covering map. If* $k \leq 0$, $W \subset A_{k-1}$ *and* $f : W \to A_k$ *is a degree* 2^N *covering map.*

(2) $W \subset A_j$ *for* $j \geq k$ *and* $j \geq 1$, *and* W *is contained in some petal in* $R_j \cdot \mathcal{P}_{n_j}$. *Conversely, for all such* j *and every petal* $R_{n_j} \cdot \Omega_{n_j} \in R_{n_j} \cdot \mathcal{P}_{n_j}$, *there exists exactly one component* $W = f^{-1}(A_k) \subset R_{n_j} \cdot \Omega_{n_j}$. *In both cases,* $f : W \to A_k$ *is a conformal mapping.*

Moreover, these are the only possibilities.

Proof. (1). By Theorem 6.12, a component W certainly exists in both cases. By Lemma 8.2, each point $z \in A_k$ is *evenly covered*; there is a ball $B(z, r)$ so that $f^{-1}(B(z, r))$ is the disjoint union of n_k simply connected topological disks in W, and f is conformal on each of these disks.

(2). The fact that all possible $j \geq k$ occurs follows from the fact that all the zeros of f are in the annuli A_j for $j \geq 1$, and the fact that f is continuous. W must also lie completely in some petal region. Indeed, by Lemma 8.1, W must contain elements of the Julia set, but if $W \subset V_j$, then $f(W)$ could not be a subset of A_k. W must be strictly inside the petal, or else the proof of Lemma 8.1 would show that there are points in W that map to B_k. The fact that there is at least one component per petal again follows from the continuity of f.

It remains to show the desired conformal mapping behavior occurs. Let $z \in A_k$, and view f restricted as $f : W \to A_k$. Again Lemma 8.2 shows that each z in A_k is evenly covered. We want to further show that f is actually one-to-one on W. To do this, notice that for the globally defined f, z has n_k many preimages in V_{k-1}, and z has n_l many preimages in each A_l (one for each petal), for $l = k, k+1, \ldots, j$. This gives

$$n_k + (n_k + \cdots + n_j) = n_{j+1}$$

total preimages in $\widehat{A}_j$. But $f : D_j \to \widehat{A}_{j+1}$ is a degree n_{j+1} branched covering map, so going back to the original W, we see that f must be one-to-one on W. □

We would like to remark on the following important consequence. If $W \subset A_j$ is a component of $f^{-1}(A_k)$, then $f : \widehat{W} \to \widehat{A}_k$ is conformal. Since f is injective on the Jordan annulus W, it follows from the argument principle that it is injective on $\widehat{W}$.

Let $W \subset A_j$ be a component of $f^{-1}(A_k)$ for $j \geq k$. $f : W \to A_k$ is conformal, and the distortion of this conformal mapping can be controlled to be as small as we would like. If $B = B(z, r)$ is a ball of radius r, we denote $\lambda B = B(z, \lambda r)$.

Lemma 8.5. *Let* $\lambda > 2$ *be given and let* W *be as above. Then for sufficiently large choices of* R, *there exists a ball* $B(z, r)$ *containing* W *so that* f *restricted to* λB *is conformal.*

Proof. Let $\{z_1, \ldots, z_{n_k}\}$ denote the critical points of f contained in A_k and let $\{f(z_1), \ldots, f(z_{n_k})\}$ denote the corresponding critical values. Let

$$\rho_k = \min\{|f(z_1)|, \ldots, |f(z_{n_k})|\}.$$

By our comments after the proof of Lemma 8.2, for R sufficiently large, we can make the ratio ρ_k/R_k as large as we would like, independent of k. Let A_k^* denote the annulus $\{z : \frac{1}{4}R_k \leq |z| \leq \frac{1}{2}\rho\}$.

Let W^* be the component of $f^{-1}(A_k^*)$ containing W. Then as we argued above, $f : W^* \to A_k^*$ is a conformal mapping. By Lemma 8.1, there exists a ball $B(z, r)$ containing W with $\mathrm{diam}(B(z, r)) = O(R_k^{-2}n_k^{-2})$. The modulus of $W^* \setminus B(z, r)$ can be made as large as we would like, independent of k, so for sufficiently large R it follows that $\lambda B(z, r) \subset W^*$ as well. The claim now follows from Lemma 3.2. □

9. C^1 Boundary Components

The purpose of this section is to show that the boundaries of Fatou components of f are C^1, and in many cases these boundary components are close to being circles.

We will say a set C is an *ϵ-approximate circle* if, for some translation C' of C, there exists a circle $C(0, r)$ and a mapping $h : C(0, r) \to C'$ which has the form $h(re^{i\theta}) = (r(\theta), \theta)$ in polar coordinates where $r : [0, 2\pi] \to \mathbb{R}$ is ϵ-Lipschitz. As an example, in Lemma 7.3, given any $\epsilon > 0$, we may choose R large enough so that the sets U_λ are ϵ-approximate circles.

Let Γ be a connected set in the plane. We say that a line L is *tangent* to $z \in \Gamma$ if for every $\alpha > 0$ there exists $r > 0$ such that the two-sided sector $S(z, L, \alpha) = \{x : d(x, L) \leq \alpha|x - z|\}$ contains all points of $\Gamma \cap B(z, r)$. A *unit tangent vector* τ_z based at z is a vector based at z with direction given by L. If Γ has a tangent at z there are two choices of unit tangent vector. We will say that Γ is a C^1-smooth curve if it has a unique tangent line at each point, and there is a choice of unit tangent vectors so that the direction of the unit tangent vectors varies continuously with $z \in \Gamma$.

We will need the following Lemma, which is Lemma 18.1 in [Bis17].

Lemma 9.1. *Suppose h is a holomorphic function on $A = \{z : 1 < |z| < 4\}$ and suppose that $|h|$ is bounded by ϵ on A. Let $H(z) = z^m(1 + h(z))$. For any fixed θ the radial segment $S(\theta) = \{re^{i\theta} : 3/2 \leq r \leq 5/2\}$ is mapped by H to a curve that makes angle at most $O(\epsilon/m)$ with any radial ray it meets.*

The lemma is proved by applying the Cauchy estimate to $z\frac{H'(z)}{H(z)}$; the argument of this expression measures the angle. Lemma 9.1 also implies the following stronger result. If Γ is an analytic Jordan arc in A, and τ_z is the unit tangent vector to

Γ at z, then the difference between the angle between τ_z and $S(\theta)$ and the angle between $\tau_{f(z)}$ and any ray it meets is $O(\epsilon/m)$. This follows immediately from the angle preserving property of holomorphic maps with nonzero derivative combined with Lemma 9.1.

Recall that by Theorem 6.12 the image of the annulus $V_k \subset A_k$ contains A_{k+1}. It follows from Lemma 9.1 and Lemma 8.2 that $W = f^{-1}(V_{k+1}) \subset V_k$ is a topological annulus, and the boundary components of W are $O(\epsilon_k)$-approximate circles (recall that the quantities ϵ_k were defined in Section 6, p.22). From here, with the additional help of Lemma 6.6, we can also deduce that the width of W is approximately $R_k/2n_k$. It turns out that this contracting, small angle distorting behavior is precisely what we need to prove the following theorem. See also Section 18 of [Bis17].

Theorem 9.2. *Let $\epsilon > 0$ be given. Then there exists R sufficiently large so that for all $k \geq 1$, there exists a C^1-smooth, ϵ-approximate circle $\Gamma_k \subset V_k$ contained in $\mathcal{J}(f)$ that surrounds the origin.*

Proof. Fix some $k \geq 1$ and define

$$\Gamma_{k,n} = \left\{ z \in V_k : \; f^j(z) \in A_{k+j} \, , \, j = 1, \ldots, n \right\}.$$

Since A_{k+n} is a round annulus, it has a natural foliation $\mathcal{U}_{k+n}$ of closed circles centered at the origin. $\Gamma_{k,n}$ has an induced foliation of closed analytic Jordan curves obtained by pulling back each element of $\mathcal{U}_{k+n}$ by f. The fact that these curves are indeed Jordan curves follows from Lemma 8.2 and the covering map behavior of $f^n : \Gamma_{k,n} \to A_{k+n}$.

Define $\Gamma_k = \cap_n \Gamma_{k,n}$. We claim that the compact connected set Γ_k satisfies the conditions of the theorem. We first produce candidate tangents for each $z \in \Gamma_k$. For each $z \in \Gamma_k$, let $\gamma_{k,n}(z)$ be the element of $\mathcal{U}_{k+n}$ that contains z. Let $\tau_n(z)$ denote the unit tangent vector at z to $\gamma_{k,n}(z)$. Then by Lemma 9.1 for $m \geq n$,

$$|\tau_n(z) - \tau_m(z)| = O\left(\sum_{l=n}^{m} \epsilon_l \right).$$

It follows that $\{\tau_n(z)\}$ is a Cauchy sequence, and $\tau_n(z)$ converges to some unit vector which we denote by $\tau(z)$. Then $\tau(z)$ is a tangent vector based at z to Γ_k. This follows from the fact that curves $\gamma_{n,k}$ are analytic, and therefore C^1, and the fact that $\tau(z)$ is defined to be the limit of the tangent vectors $\tau_n(z)$.

We need to check that τ_z varies continuously with z. For all $z, w \in \Gamma_k$,

$$|\tau(z) - \tau(w)| \leq |\tau(z) - \tau_n(z)| + |\tau_n(z) - \tau_n(w)| + |\tau_n(w) - \tau(w)|,$$

where $\tau_n(z)$ and $\tau_n(w)$ are tangent to $\gamma_{k,n}(z)$ and $\gamma_{k,n}(w)$, respectively. Let L denote the ray based at the origin that passes through z. Then L passes through

$\gamma_{k,n}(w)$ at exactly one point, which we denote by z'. We call the corresponding unit tangent vector $\tau_n(z')$. Therefore

$$|\tau_n(z) - \tau_n(w)| \leq |\tau_n(z) - \tau_n(z')| + |\tau_n(z') - \tau_n(w)|.$$

The distance between z and z' tends to 0 as $|z - w|$ tends to 0. So the fact that $\gamma_{k,n}(w)$ is analytic shows that $|\tau_n(z') - \tau_n(w)|$ can be made arbitrarily small if $|z-w|$ is sufficiently small. For the other term, the line L passes through each element of $\mathcal{U}_{n+k}$ once, and the corresponding unit tangent vectors vary continuously along L, which shows that $|\tau_n(z') - \tau_n(z)|$ tends to 0 for large n and $|z - w|$ small. Putting this all together, we see that $\tau(z)$ varies continuously.

Finally we show that $\Gamma_k \subset \mathcal{J}(f)$. For each n, $f^n : \Gamma_{k,n} \to A_{k+n}$. If $z \in \Gamma_k$, then z is contained in some set $S_{n,k}$ so that f^n is one-to-one on the interior of $S_{n,k}$ and so that $f^n : \overline{S_{n,k}} \to \overline{A_{k+n}}$. Note that $\operatorname{diam}(S_{n,k}) \to 0$ as $n \to \infty$. By definition of Γ_k, we have $f^n(z) \to \infty$ as $n \to \infty$. Let $w \in A_{k+n}$ be a zero of f. Then $f^{-n}(w)$ contains an element in $S_{n,k}$, and $f^{n+1}(w) \in B_f$, the basin of attraction containing the origin. It follows that in any neighborhood of z, $\{f^n\}$ cannot be an equicontinuous family. □

Now we turn to a systematic labeling of the Fatou components of f. For $k \geq 1$, define Ω_k to be the Fatou component containing B_{k-1}. Let D be the bounded complementary component of the Jordan annulus B_0. For $k \leq 0$, we may define Ω_k by taking appropriate preimages, namely,

$$\Omega_k = \{z \in D : f^j(z) \in D\,,\, j = 1, \ldots, k\,,\, f^{k+1}(z) \in \Omega_1\}.$$

By Theorem 9.2 each Ω_k is a distinct Fatou component for all integers k.

Lemma 9.3. *For all k, $f(\Omega_k) = \Omega_{k+1}$. In particular, each Ω_k is a multiply connected wandering domain.*

Proof. If $k \leq 0$, this is true by definition. For $k \geq 1$, we know that $f(B_{k-1}) \subset B_k \subset \Omega_{k+1}$ by Theorem 6.12. Since Ω_{k+1} is a connected component of the Fatou set it follows that $f(\Omega_k) \subset \Omega_{k+1}$. Since f has no asymptotic values and Ω_k is also a connected component of the Fatou set, we get $f(\Omega_k) = \Omega_{k+1}$. For the last comment, we just use the observation that each B_k surrounds the origin. □

Lemma 9.4. *Each Γ_k is a connected component of $\mathcal{J}(f)$. In fact, Γ_k is simultaneously the innermost boundary component of Ω_{k+1} and the outermost boundary component of Ω_k.*

Proof. Let $S_{n,k}$ be the sets from the proof of Theorem 9.2. For sufficiently large n, since Γ is C^1 and an approximate circle, Γ splits $S_{n,k}$ into exactly two connected components. Given an $S_{n,k}$ let S_o denote the component that is a subset of the unbounded complementary component of Γ_k, and we let S_i denote the component that is a subset of the bounded complementary component Γ_k.

Since $f^n(\overline{S_{n,k}}) = \overline{A_{n+k}}$, there exists a point $z \in S_o$ so that $|f^n(z)| = \frac{5}{2}R_{k+n}$. Then $f^{n+1}(z) \in B_{k+n+1}$ by Theorem 6.12. It follows that $f^{n+1}(z) \in \Omega_{k+n+2}$, and therefore $z \in \Omega_{k+1}$ by Lemma 9.3. So Γ_k is a boundary component for Ω_{k+1}.

There also exists a point $z \in S_i$ so that $|f^n(z)| = \frac{3}{2}R_{k+n}$. Then $f^{n+1}(z) \in B_{k+n}$, so that $f^{n+1} \in \Omega_{k+n+1}$ and $z \in \Omega_k$. So Γ_k is a boundary component for Ω_k as well.

Let Γ be the connected component of the Julia set containing Γ_k, and suppose that $\Gamma \setminus \Gamma_k$ was non-empty. Then $\Gamma \setminus \Gamma_k$ cannot have any components in the unbounded complementary component of Γ_k by the argument above. It cannot have a component in the bounded complementary component either. Suppose such a K existed. The argument above asserts that some iterate n of f takes some point $w \in K$ to the circle $C(0, 3/2R_{k+n})$, but the argument above would also assert that w would be in the Fatou set. □

By Lemma 8.1 and Lemma 9.3, we know that each Ω_k is multiply connected. The proof of Theorem 9.2 actually shows that each Ω_k is infinitely connected, since the sets $\Gamma_{n,k}$ contain preimages of petals (defined on p.18) contained in A_{n+k}, and these petals contain preimages of the basin of attraction containing the origin. We can break the complement of Ω_k into three types of regions Ω_k^a, Ω_k^0, and Ω_k^∞. Ω_k^0 is the region containing the origin and Ω_k^∞ is the unbounded region. The remaining regions Ω_k^a lie between the innermost and outermost boundary components of Ω_k. We define Ω_k^A to be the union of Ω_k and all the regions Ω_k^a, so that Ω_k^A is a Jordan annulus. With this notation, $\widehat{\Omega}_k$ is the union of Ω_k^0 and Ω_k^A.

Next, let $\Omega_k^a \subset R_k \cdot \Omega_{n_k}^p$, for some petal $\Omega_{n_k}^p$, be a complementary component of Ω_k. The boundary of Ω_k^a is in the Julia set, and Ω_k^a contains a zero of f. There exists an inverse image $f^{-1}(\Omega_k)$ inside of Ω_k^a, and we claim that the boundary of $f^{-1}(\Omega_k)$ inside of Ω_k^a is the boundary of Ω_k^a. f maps the boundary of $f^{-1}(\Omega_k)$ onto Γ_k, and this means that $f(\partial\Omega_k^a)$ belongs to the unbounded complementary component of Γ_k. Arguments similar to what we have done in this section shows that if this were to happen, then there exists n so that $f^n(\partial\Omega_k^a) \subset B_{n+k-1}$, and $\partial\Omega_k^a$ would have to be in the Fatou set.

So by Lemma 8.5, f maps Ω_k^a conformally onto some topological disk that contains the origin, and the boundary of that disk is the outermost boundary component of Ω_k. It follows that inside of Ω_k^a, there are conformal copies of Ω_j for $j \leq k$, and one conformal copy of B_f. This motivates the following definition.

Definition 9.5. We call a Fatou component ω of *k-type* if there exists m so that $f^m : \omega \to \Omega_k$ is a conformal mapping.

Such a value for k is unique, since conformal mappings as we defined them are injective. Note that by Lemma 3.2 and Lemma 8.5, we may arrange for the boundary of a Fatou component of k-type, $k \geq 1$, to be an ϵ-approximate circle for small ϵ. Since the orbits of all points in Ω_k tend to ∞ for all k, the same is true for the orbits of all points in a component of k-type. Later, we will prove that every Fatou component that escapes is of k-type for some k. Given this universality of Ω_k in the Fatou set of f, we create the following definition. Recall that B_f is the basin of attraction for f viewed as a polynomial-like mapping.

Definition 9.6. The *central series of Fatou components* is the union of B_f and all components Ω_k, where $k \in \mathbb{Z}$. The *central series of Fatou components truncated at* m is the union of B_f and all components Ω_k with $k \in \mathbb{Z}$ satisfying $k \leq m$.

As we will start to see in Section 10, the central series of Fatou components truncated at some m is the primary building block for the global Fatou set of f. That said, there are points in the Julia set that are not in the boundary of any Fatou component. Such points are called *buried points*. Let z in A , where A was defined in Section 8, be given, and suppose that every iterate of z remains in A. Then there exists integers k_n so that $f^n(z) \in A_{k_n}$ for all $n \geq 0$ (we interpret $f^0(z) = z$). Then we may define the *orbit sequence* of z to be $\alpha(z) = (k_0, k_1, k_2, \dots)$. The orbit sequence tells us exactly which A_k $f^n(z)$ belongs to for some iterate n. If z does not remain in A for all iterates, we may still write a finite orbit sequence similarly.

We will say that a point $z \in A$ *moves backwards* if there exists $j \geq 0$ so that $\alpha(z)$ satisfies $k_{j+1} \leq k_j$. A point $z \in A$ moves backwards finitely or infinitely often if there exists finite or infinitely many values j so that $\alpha(z)$ satisfies $k_{j+1} \leq k_j$. We will say a set $X \subset A$ moves backwards finitely often if the following two conditions hold:

(1) All $z \in X$ move backwards m times for some $m \geq 1$.
(2) Let $z \in X$ and let k_j be the mth entry of $\alpha(z)$ so that $k_{j+1} \leq k_j$. Then if $w \in X$, then $\alpha(z)$ and $\alpha(w)$ are equal up to the $j+1$-st index.

Informally, a set moves backwards finitely often if and only if all its elements move backwards finitely often at the exact same iterates.

Lemma 9.7. *Let $z \in A$ be given. If z is buried then it moves backwards infinitely often.*

Proof. Suppose that z is a buried point. Suppose for the sake of a contradiction z moves backwards only finitely often. By considering an iterate $f^n(z)$, we may assume z never moves backwards. z is in the Julia set, so under these circumstances if $z \in A_k$, then by Lemma 8.1, $z \in V_k$, and $f^n(z) \in V_{k+n}$ for all n. By the proof of Theorem 9.2, z must be on the boundary of Ω_k, so it is not buried. $\square$

We will see later that z is buried if and only if z moves backwards infinitely often. For the rest of the paper, we will refer to the points that move backwards infinitely often as Y.

10. A Detailed Description of the Dynamics of f

We can now offer a complete description of the Fatou and Julia set, along with several other dynamically interesting facts. We will need the following theorem, which we will prove in Sections 11-13. For convenience, we will often refer to a sum of diameters of the form below as an $(s+\epsilon_0)$-sum.

Theorem 10.1. *Let $\epsilon_0 > 0$ and $k \in \mathbb{Z}$ be given. Then R may be chosen large enough so that*

$$\sum_{\omega \subset \Omega_k^A} \operatorname{diam}(\omega)^{s+\epsilon_0} < \infty, \tag{10.1}$$

where the sum is taken over every Fatou component $\omega \subset \Omega_k^A$ which is of j-type for some $j \geq 1$.

We would like to emphasize that we are summing over all Fatou components contained inside of Ω_k^A of j-type for any $j \geq 1$, not just one fixed j. This theorem cannot be significantly strengthened to include all Fatou components of j-type for $j \in \mathbb{Z}$. Indeed, if B_f' is an inverse image of B_f contained inside of Ω_k^A, there exists infinitely many Fatou components surrounding B_f' with diameter larger than the diameter of B_f'. Compare to (20.1) in [Bis17], p. 455.

Let's first geometrically interpret what this sum means. Figure 7 shows a schematic diagram for the Fatou component Ω_k, $k \geq 1$. As discussed in Section 9, Ω_k is infinitely connected. The 'holes' in Figure 7 correspond to the components Ω_k^a. The innermost ring of holes is contained inside of the collection of petals $R_k \cdot \mathcal{P}_{n_k}$, defined in Section 8. There is exactly one hole for each individual petal $R_k \cdot \Omega_{n_k}^p$, and f maps the outermost boundary of each of these holes conformally onto the outermost boundary component of Ω_k with small distortion by Lemma 8.5. Therefore, each hole contains a preimage or copy of the central series of Fatou components truncated at k, and Lemma 8.5 says that this preimage is almost an affine rescaling of the central series truncated at k.

When $k \geq 1$, the other rings of holes of Ω_k are mapped to rings of holes in Ω_{k+1}. The second ring of holes maps into $R_{k+1} \cdot \mathcal{P}_{n_{k+1}}$, and in general, each ring of holes moves one ring inward under iteration by f. Therefore, all of the other rings of holes in Figure 7 are eventually mapped conformally into $R_j \cdot \mathcal{P}_{n_j}$ for some $j > k$, and therefore the same reasoning shows that those holes contain a preimage or copy of the central series of Fatou components truncated at j.

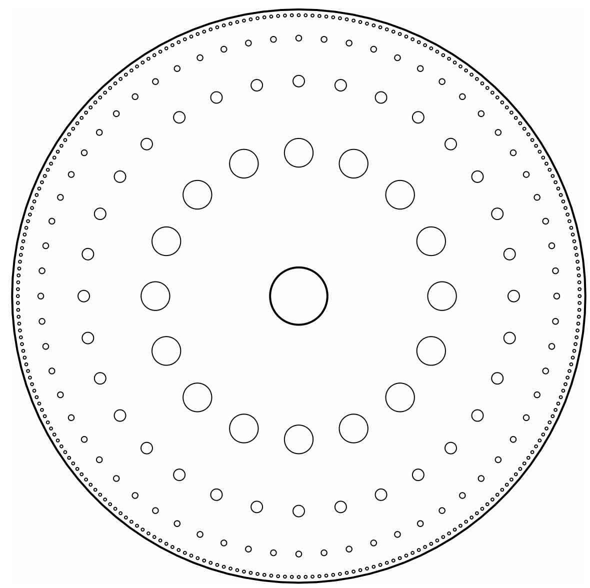

FIGURE 7. A schematic for a Fatou component Ω_k, $k \geq 1$. Every hole in Ω_k^A is bounded by the outermost boundary of a Fatou component of j-type, $j \geq k$. This forms the first layer of holes, which we see pictured above. Every hole contains a copy of the central series of Fatou components truncated at some $j \geq k$; the holes in all of these copies induce the second layer of holes in Ω_k^A. This introduces a third layer of holes, and the process continues inductively. The set Y of points that move backwards infinitely often coincides with the set of all points that are contained inside of a hole in every layer.

The discussion above also applies to Ω_k for $k \leq 0$, with some minor adjustments. In this case, Ω_k is defined to be the preimage $f^{(-1+k)}(\Omega_1)$ that surrounds the origin. Each complementary component Ω_k^a gets mapped conformally by f^{-k+1} onto some component Ω_1^a, so it also contains a preimage of the central series of Fatou components truncated at some $j \geq 1$. From this we deduce that the boundary of every complementary component of Ω_k contained in Ω_k^A is the boundary of some Fatou component of j-type, where $j \geq 1$.

To summarize the above discussion using the informal language of holes, we have replaced all of the holes in Ω_k by appropriate truncated copies of the central series of Fatou components. These truncated copies introduce a new second layer of holes contained in Ω_k, introduced by the every copy of the truncated central series of Fatou components. The reasoning above shows that every hole in this second layer contains a truncated copy of the central series of Fatou components, which introduces a third layer of holes in Ω_k, and so on.

From this discussion we see that the truncated copies of the central series of Fatou components serve as the main building block for the Fatou and Julia set of f. We fill all the holes in the central series by the appropriate truncated copies of the central series, and repeat this procedure for all the new, smaller holes that appear. Theorem 10.1 can now be interpreted as saying that the $(s+\epsilon_0)$-sum of diameters of *every* hole in *every* layer of holes contained in Ω_k^A is finite (in fact, it is slightly stronger, since many Fatou components may be contained inside of the same hole).

We can also see from this discussion that every hole in the mth layer described above moves backwards exactly m many times. Therefore, we see that the set Y of points that move backwards infinitely often coincides with the set of points that are contained inside a hole for every layer of holes in Ω_k^A. For each m, the mth layer of holes is a covering of $Y \cap \Omega_k^A$, so since the sum in Theorem 10.1 converges we will be able to deduce the following consequence.

Corollary 10.2. *For all k, $\dim_{\mathrm{H}}(Y \cap \Omega_k^A) \leq s + \epsilon_0$. In particular, we have $\dim_{\mathrm{H}}(Y) \leq s + \epsilon_0$.*

For the remainder of this section we will describe the dynamical consequences of Theorem 10.1 and Corollary 10.2. Recall that for an entire function f, we define the *escaping set* as

$$I(f) = \{z : |f^n(z)| \to \infty\}.$$

Choose some number S_0 so that there exists z with $|z| = S_0$ so that $z \in I(f)$ (for example, choose S_0 so that $|z| = S_0 \subset B_1$). Then define inductively

$$S_{n+1} = \max_{|z|=S_n} |f(z)|.$$

We define the *fast escaping set* as

$$A(f) = \{z : \text{there exists } k \geq 0 \text{ so that } |f^{n+k}(z)| \geq S_n \text{ for all } n \geq 0\}.$$

We define the *bungee set* as

$$BU(f) = \{z : \text{there exists } n_k \text{ and } n_j \text{ so that } |f^{n_k}(z)| \to \infty \text{ and } f^{n_j}(z) \text{ is bounded}\}.$$

Lastly, we define the *bounded orbit set* as

$$BO(f) = \{z : f^n(z) \text{ is bounded}\}.$$

Every point $z \in \mathbb{C}$ is either in $I(f)$, $BU(f)$, or $BO(f)$. All of the elements of $BU(f)$ belong to Y, and every point in $BO(f)$ that does not eventually map into K_f, the filled Julia set of the polynomial-like mapping f, also belongs to Y. Y also contains points in $I(f)$, but since these points move backwards infinitely often, they cannot belong to $A(f)$ for our function f.

We can also show that the set Y is precisely the set of buried points.

Corollary 10.3. *A point z moves backwards infinitely often if and only if it is buried.*

Proof. By Lemma 9.7, it only remains to show that moving backwards infinitely often implies the point is buried. First, note that such a point cannot belong to C, defined in the beginning of Section 8 as the inverse images of K_f.

Suppose z belonged to the boundary of some Fatou component ω. Since z moves backwards infinitely often, $z \in Y \cap \Omega_k^A$ for some integer k, and there exists a sequence $\omega_j(n)$ of components of $j(n)$-type contained in Ω_k^A so that $\operatorname{diam}(\omega_j(n)) \to 0$. These components $\omega_j(n)$ correspond exactly to the hole in the nth layer that z is contained in. Since we must have $\omega \subset \omega_j(n)$ for all n, this is a contradiction. $\square$

Corollary 10.4. *Every escaping Fatou component is of k-type for some unique integer k.*

Proof. Uniqueness has already been discussed when we defined components of k-type. Let ω be an escaping Fatou component, but suppose it is not of k-type. Then its boundary is in the Julia set, and by Corollary 10.3, every point on the boundary moves backwards finitely often. So by taking a large enough iterate of f it suffices to deal with the case that ω is a Fatou component which never moves backwards. In this case, since the boundary of ω is in the Julia set, Lemma 8.1 says that (by perhaps applying a larger iterate of f to ω) it is in a petal $R_k \cdot \Omega_{n_k}^p$, or in V_k for some $k \geq 1$. If it is in a petal, it moves backwards, so ω must be in V_k. But then $f^j(\omega) \subset V_{k+j}$ for all j, by the same reasoning. The proof of Theorem 9.2 shows that this is impossible. $\square$

Corollary 10.5. *The sets $I(f) \setminus A(f)$, $BU(f)$, and $BO(f) \setminus C$ all have Hausdorff dimension $\leq s + \epsilon_0 < 2$.*

Proof. $BU(f) \subset Y$ and $BO(f) \setminus C \subset Y$, and Y has Hausdorff dimension $\leq s + \epsilon_0$ by Corollary 10.2. If $z \in I(f) \setminus A(f)$, then $z \notin C$, nor can we have z inside a Fatou component or the boundary of a Fatou component of k-type for any integer k, since these sets belong to $A(f)$. Therefore z must be in Y, so z moves backwards infinitely often and it follows that z cannot belong to $A(f)$. Hence $I(f) \setminus A(f) \subset Y$, and the result follows. $\square$

We now know that Fatou components are either inverse images of the basin of attraction B_f of f viewed as a polynomial-like mapping, or are of k-type for some integer k. If ω is a preimage of B_f, we call it a *basin.* If ω is k-type for $k \geq 1$, we call ω *round.* If ω is k-type for $k \leq 0$ we call ω *wiggly.* As $k \to -\infty$, the boundaries of components of k-type trace more closely the fractal boundary of the basin of attraction. Round components are far enough away from the fractal boundary to still have approximately circular boundaries. Recall in Section 6 immediately after Lemma 6.9 we defined the summable parameters ϵ_k. In Section 9 we showed that the ϵ_k's determined the deviation from round circles of the innermost and outermost boundary components of Ω_k, $k \geq 1$, and for any $\delta > 0$, we could arrange for $\sum_{k=1}^{\infty} \epsilon_k < \delta$.

Corollary 10.6. *The boundary of any escaping Fatou component is the union of C^1 curves. If the Fatou component is round, the all of the boundary components are $O(\delta)$-approximate circles.*

Proof. All escaping components are of k-type for some k, so it suffices to show this for Ω_k by Lemma 8.5. Since all the boundary components of Ω_k are escaping by Corollary 10.3, they are the boundaries of components of j-type for $j \geq 1$. The components of j-type map conformally onto Ω_j, so their boundaries are also C^1, and by Lemma 8.5, they are approximate circles when $k \geq 1$. ☐

We can now offer a full description of the Julia set.

Theorem 10.7. *The Julia set can be decomposed into three pieces*

(1) *The buried points of f. Equivalently, the set Y of points which move backwards infinitely often.*
(2) *The C^1 components that escape to ∞. These components are always the boundary component of some Fatou component of k-type.*
(3) *Preimages of B_f, the Julia set of the quadratic-like map f.*

Corollary 10.8. *$J(f)$ has zero Lebesgue measure.*

Proof. The Julia set is the disjoint union of the the set of points that move backwards infinitely often which has Hausdorff dimension < 2 by Corollary 10.2, countably many C^1 curves, and countably many quasicircles with dimension strictly less than 2. ☐

We conclude the section by recording a lemma describing the nice geometry of the round Fatou components Ω_k, $k \geq 1$; we refer the reader again to Figure 7. The proof follows from some basic calculations using the fact that f looks like a power mapping on some portions of Ω_k, along with using Lemma 8.5. A similar discussion is found in Section 19 of [Bis17], and we omit the details.

Lemma 10.9 (The Shape of Round Fatou components)**.** *Choose some Fatou component Ω_k for some fixed $k \geq 1$. Define $d_j = 2(n_k + \cdots + n_{j-1})$ for $j > k$. Then Ω_k has the following geometric properties*

(1) *For all $j \geq k$, there are $n_j \cdot 2^{d_j}$ many boundary components of Ω_k which lie distance approximately $R_k \cdot 2^{-d_j}$ from the outermost boundary component of Ω_k. We call these components the jth ring of Ω_k.*
(2) *The boundary components in the jth ring of Ω_k are approximately distance $R_k n_j^{-1} 2^{-d_j}$ apart from each other and lie on a $O(\delta)$-approximately round circle.*
(3) *All boundary components of Ω_k arise in this manner for some $j \geq k$.*

11. The Proof of Theorem 10.1: A Labeling System for Holes

In this section, we formally construct a sequence of coverings $\mathcal{C}_m$ of $Y \cap A_1$ which correspond to the mth layer of holes in Ω_k^A described in the previous section. We will focus on the case in Theorem 10.1 with $k = 1$. It is straightforward to modify our techniques to other integers k.

Our initial covering $\mathcal{C}_0$ will have exactly one element, A_1, the annulus defined in Section 6. Notice that by the proof of Theorem 9.2, A_1 contains the outermost boundary component of Ω_1, so that $\widehat{A}_1$ contains $\widehat{\Omega}_1$. A_1 and Ω_1 both have diameter comparable to R_1. We first describe how to construct $\mathcal{C}_1$ from $\mathcal{C}_0$. For each $z \in A_1 \cap Y$, by definition, there is a smallest positive integer n so that $f^n(z) \in A_k$ for $k \leq n$. It is possible that $k \leq n$ is a negative integer. z belongs to one connected component of $f^{-n}(A_k)$, a Jordan annulus which is a proper subset of A_1. We denote such a component by W_k^n. The collection of all distinct components obtained by doing this procedure for all $z \in Y$ is denoted by $\mathcal{C}_1$.

Before proceeding further, we would like to remark on some potentially confusing notation. The convention of referring to elements in $\mathcal{C}_1$ as W_k^n is ambiguous. Indeed, there are 2^N many elements of $\mathcal{C}_1$ that could be called W_1^1, one for each petal $R_1 \cdot \Omega_{n_1}^p$. With this ambiguity in mind, we will adopt the convention that the notation $W_k^n \in \mathcal{C}_1$ always refers to a single element. We will say that an element $X \in \mathcal{C}_1$ is *of the form* W_k^n if it could be denoted by W_k^n using the procedure above for some $z \in X$. Therefore every element of $\mathcal{C}_1$ is of the form W_k^n for some positive integer n and some integer $k \leq n$. This slight abuse of notation will not be an issue in what follows; we will exclusively refer to either single elements W_k^n or the collection of all elements of the form W_k^n exclusively. We will make it clear in each context if we are referring to a single element versus a collection of elements.

With this in mind we now describe how to obtain $\mathcal{C}_2$ from $\mathcal{C}_1$. Let $W_k^n \in \mathcal{C}_1$ be given, and choose $z \in Y \cap W_k^n$. Then $f^n(z) \in A_k$. But since $z \in Y$, there exists a smallest q so that $f^{n+q}(z) \in A_j$ for $j \leq k + q - 1$. z is contained in a component of $f^{-(n+q)}(A_j) \subset A_1$, which we denote by W_j^{n+q}. Therefore, for all $z \in Y \cap W_k^n$ there is a Jordan annulus $W_j^{n+q} \subset W_k^n$ containing z which moves backwards twice. We define $\mathcal{C}_2$ to be the collection of all distinct components obtained by applying this procedure to each element of $\mathcal{C}_1$.

We proceed inductively to construct $\mathcal{C}_{m+1}$ from $\mathcal{C}_m$. Let $z \in W_{k_m}^{n_m} \cap Y$ for some element $W_{k_m}^{n_m} \in \mathcal{C}_m$. Then since $z \in Y$, there exists a smallest q so that $f^{n_m+q}(z) \in A_j$ for $j \leq k_m + q - 1$. z is a member of some component of $f^{-(n_m+q)}(A_j)$, and we denote this component by $W_j^{n_m+q}$. Therefore, for all $z \in Y \cap W_{k_m}^{n_m}$, there is a Jordan annulus $W_j^{n_m+q} \subset W_{k_m}^{n_m}$ which moves backwards $m+1$ many times. $\mathcal{C}_{m+1}$ is the collection of all distinct components obtained by applying this procedure to each element of $\mathcal{C}_m$.

We summarize several properties of the coverings $\mathcal{C}_m$ in the Lemma below.

Lemma 11.1. *The collection of sets $\mathcal{C}_m$, $m \geq 0$, has the following properties.*

1. *$\mathcal{C}_m$ is a countable union of Jordan annuli which cover $Y \cap \Omega_1^A$.*
2. *$\mathcal{C}_{m+1}$ is a refinement of $\mathcal{C}_m$, i.e, every element in $\mathcal{C}_{m+1}$ is a subset of an element in $\mathcal{C}_m$.*
3. *Let $W_k^n \in \mathcal{C}_m$. Then W_k^n moves backwards m times.*
4. *If $W_k^n \in \mathcal{C}_m$, then W_k^n contains the outermost boundary component of exactly one Fatou component of k-type that moves backwards m times, but not $m+1$ times.*
5. *Suppose ω is a Fatou component of k-type contained in Ω_1^A. Then there is a positive integer m so that ω moves backwards exactly m times. Moreover, there exists a unique element $W_k^n \in \mathcal{C}_m$ so that W_k^n contains the outermost boundary of ω.*

Proof. We only need to discuss (4) and (5). If $W_k^n \in \mathcal{C}_m$, then by definition $f^n(W_k^n) = A_k$ and W_k^n moves backwards m many times. A_k contains the outermost boundary of Ω_k, so W_k^n must contain a Fatou component of k-type that moves backwards m times. Since $f(\Omega_k) = \Omega_{k+1}$, this component does not move backwards again, which gives uniqueness, since all other Fatou components whose outermost boundary is contained in W_k^n will move backwards more than m many times.

For (5), the existence of such an m is guaranteed by the definition of k-type and the fact that $f(\Omega_k) = \Omega_{k+1}$. The existence of W_k^n follows from the fact that the outermost boundary component of ω moves backwards m many times. So the outermost boundary of $f^n(\omega)$ is contained in A_k, so it will be contained in some W_k^n in $\mathcal{C}_m$. □

We move on to describing how to change the covering $\mathcal{C}_m$ by topological annuli into a simpler covering $\widehat{\mathcal{C}}_m$ by topological balls. The elements of $\widehat{\mathcal{C}}_m$ are going to be topological disks that cover the holes mentioned in Section 9. Let $W_k^n \in \mathcal{C}_1$ be given. Then $\widehat{W}_k^n$ is a topological disk with the same diameter as W_k^n. $\widehat{W}_k^n$ is contained inside of exactly one element $\widehat{W}_n^n$, with $W_n^n \in \mathcal{C}_1$. This means that the Fatou component of k-type that $\widehat{W}_k^n$ contains which moves backwards exactly once is a subset of the polynomial hull of the Fatou component of n-type that moves backwards exactly once that $\widehat{W}_n^n$ contains. $\widehat{\mathcal{C}}_1$ is the collection of all topological disks of the form $\widehat{W}_n^n$, $n \geq 1$. $\widehat{\mathcal{C}}_1$ has the following maximality property: If $\widehat{W}_n^n \in \widehat{\mathcal{C}}_1$, there is no $W_k^n \in \mathcal{C}_1$ so that $\widehat{W}_n^n$ is a proper subset of $\widehat{W}_k^n$. This implies that $\widehat{\mathcal{C}}_1$ covers all the points that $\mathcal{C}_1$ does. We will continue to use the convention that $\widehat{W}_k^n$ refers to a single element, and refer to the collection of all elements that could be labeled the same way as the elements of the form $\widehat{W}_k^n$.

To inductively obtain $\widehat{\mathcal{C}}_{m+1}$, assume that $\widehat{\mathcal{C}}_m$ has been constructed and satisfies the maximality property that no element $\widehat{W}_k^n \in \widehat{\mathcal{C}}_m$ is a proper subset of $\widehat{W}_j^n$ where $W_j^n \in \mathcal{C}_m$. Start with take $\widehat{W}_k^n \in \widehat{\mathcal{C}}_m$. Then $k \geq 1$; otherwise $\widehat{W}_k^n$ is a proper subset of $\widehat{W}_{k+1}^n$ for some $W_{k+1}^n \in \mathcal{C}_m$. $\widehat{W}_k^n$ contains a sequence of components $W_j^n \in \mathcal{C}_m$ for $j \leq k$. Fix W_j^n, and consider the elements of $\mathcal{C}_{m+1}$ contained inside of W_j^n of the form W_{j+q-1}^{n+q}. If $j \geq 1$, then all $q \geq 1$ occur. If $j \leq 0$, then q must satisfy $q \geq 2-j$. Either way, for each valid choice of q, the polynomial hulls of the components of the form W_{k+q-1}^{n+q} determine the elements of $\widehat{\mathcal{C}}_{m+1}$ inside of W_j^n. Doing this for all $j \leq k$, we obtain all of the elements of $\widehat{\mathcal{C}}_{m+1}$ contained in $\widehat{W}_k^n$. $\widehat{\mathcal{C}}_{m+1}$ is the collection of all elements obtained in this way for each $\widehat{W}_k^n \in \mathcal{C}_m$.

We summarize the properties of the coverings $\widehat{\mathcal{C}}_m$ below.

Lemma 11.2. *The collection of sets $\widehat{\mathcal{C}}_m$, $m \geq 0$, has the following properties.*

1. *$\widehat{\mathcal{C}}_m$ is a countable collection of topological disks which cover $Y \cap \Omega_1^A$.*
2. *$\widehat{\mathcal{C}}_{m+1}$ is a refinement of $\widehat{\mathcal{C}}_m$.*
3. *Let $\widehat{W}_k^n \in \widehat{\mathcal{C}}_m$. Then $\widehat{W}_k^n$ moves backwards m many times.*
4. *If $\widehat{W}_k^n \in \widehat{\mathcal{C}}_m$. Then $k \geq 1$, and $\widehat{W}_k^n$ contains the outermost boundary of exactly one Fatou component of k-type that moves backwards m many times, but not $m+1$ many times.*
5. *Maximality: if $\widehat{W}_k^n \in \widehat{\mathcal{C}}_m$, then there does not exist a different element $W_j^n \in \mathcal{C}_m$ so that $\widehat{W}_k^n$ is a proper subset of $\widehat{W}_j^n$.*

Proof. The reasoning for (4) follows similarly as it does in Lemma 11.1. The only thing left to be checked is maximality. But this follows from the fact that inside of each $W_j^n \subset \widehat{W}_k^n$ in the construction above, we refined by using components of the form W_{j+q-1}^{n+q}. □

Example 11.3. We would like illustrate what happens when we refine $\widehat{W}_n^n \in \widehat{\mathcal{C}}_1$ into components that belong in $\widehat{\mathcal{C}}_2$. This example is actually quite universal and will motivate the technical lemmas in the next two sections. The idea behind this construction, pulling back a construction on the central series of Fatou components to all other Fatou components of k-type, will be a important theme in the following sections. Let A_k be the annuli defined in Sections 6 and 8 for $k \in \mathbb{Z}$.

When $k \geq 1$, we may refine A_k similarly to how we constructed $\mathcal{C}_1$ and $\widehat{\mathcal{C}}_1$. We call this new collection of topological disks $\mathcal{V}_k$, and we denote individual elements of $\mathcal{V}_k$ by V_k^n, where n is the largest integer so that V_k^n contains a copy of the central series of Fatou components truncated at n which moves backwards exactly once. Similar to our conventions above, we denote a single element of $\mathcal{V}_k$ as V_k^n despite the ambiguity of the notation, and refer to a collection of elements of $\mathcal{V}_k$ as being of the form V_k^n. Figure 9 gives a schematic illustration of $\mathcal{V}_k$'s for some $k \geq 1$.

When $k \leq 0$ and $z \in A_k \cap Y$, then $f^{1-k}(z) \in A_1 \cap Y$, and therefore $z \in \widehat{W}_n^n \subset A_1$ for some $\widehat{W}_n^n \in \widehat{\mathcal{C}}_1$. In this case, define V_k^n to be the component of $f^{-(1-k)}(\widehat{W}_n^n)$ that contains z. Doing this for all z, we obtain a countable collection of topological disks which we denote by $\mathcal{V}_k$. We make the same notational conventions as we do when $k \geq 1$; individual elements are referred to as V_k^n and a collection of components is of the form V_k^n if each individual component could be labeled as V_k^n. Figure 10 gives a schematic illustration of $\mathcal{V}_k$'s for some $k << 0$.

Let's turn our attention back to $\widehat{W}_n^n \in \widehat{\mathcal{C}}_1$, $\widehat{W}_n^n \subset \Omega_1^A$. $\widehat{W}_n^n$ corresponds to some complementary component in Ω_1^A that contains a central series of Fatou components truncated at n. Let ω_j, $j \leq n$ denote the Fatou components of j-type that make up this copy of the central series. The nth iterate of f conformally maps this copy of the central series inside $\widehat{W}_n^n$ to the actual central series of Fatou components truncated at n. Therefore elements of $\widehat{\mathcal{C}}_2$ contained inside of $\widehat{W}_n^n$ are precisely the preimages of the elements of $\mathcal{V}_j$ for $j \leq n$.

12. The s-Sum of the Fatou Components: Refining away from B_f

In this section, we show that our refining and polynomial hull taking refinements to the coverings in the previous section result in a decreased $(s + \epsilon_0)$-sum of the diameters. First we need an estimate comparing the diameter of W_k^n to the diameter of the component W_{k-1}^n that it surrounds whenever $k \geq 1$. See Figure 8. The following is inequality (17.1) in [Bis17]. We include the proof to emphasize an important detail, and because it is the simplest situation that illustrates how Lemma 3.2 will be used in the next two sections. Define $R_0 := \mathrm{diam}(\Omega_0)$.

Lemma 12.1. *Let $k \geq 1$, $\alpha \geq s$. Suppose that W_{k-1}^n, W_k^n are both elements of $\mathcal{C}_m$ for some m, and suppose that $W_{k-1}^n \subset \widehat{W}_k^n$. Then R may be chosen large enough so that*

$$\mathrm{diam}(W_{k-1}^n)^\alpha \leq \frac{1}{8}\,\mathrm{diam}(W_k^n)^\alpha.$$

Proof. We have $f^n(W_k^n) = A_k$ and $f^n(W_{k-1}^n) = A_{k-1}$, and f^n is conformal on a topological ball B containing W_k^n. Indeed, if $n > 1$, then $f^{n-1}(W_k^n)$ is contained in some petal $R_j \cdot \Omega^p_{n_j}$, and the proof of Lemma 8.5 says that we may take f to be conformal on some ball $B' \subset R_j \cdot \Omega^p_{n_j}$ of unit size which contains $f^{n-1}(W^n_k)$. Then we can take B to be the appropriate connected component of $f^{-(n-1)}(B')$. If $n = 1$, we may take $B' = B$ directly from $R_1 \cdot \Omega^p_{n_1}$. In either case, by Lemma 3.2 there exists a constant C so that

$$\frac{\operatorname{diam}(W^n_{k-1})^\alpha}{\operatorname{diam}(W^n_k)^\alpha} \leq C \frac{\operatorname{diam}(A_{k-1})^\alpha}{\operatorname{diam}(A_k)^\alpha} \leq C \frac{R^\alpha_{k-1}}{R^\alpha_k} \leq C \frac{1}{\sqrt{R^\alpha_1}}.$$

When R is sufficiently large, this proves the lemma. □

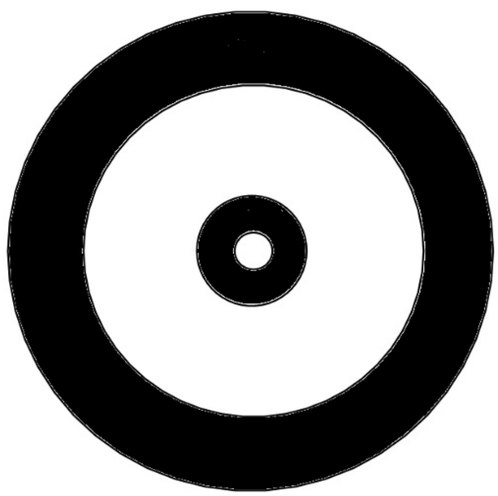

FIGURE 8. A schematic for Lemma 12.1. W^n_k and W^n_{k-1} belong to the same covering $\mathcal{C}_m$, and W^n_{k-1} is contained inside of the bounded complementary component of W^n_k. When R is large and $k \geq 1$, both components are approximately round and the diameter of W^n_{k-1} is controlled in terms of the diameter of W^n_k.

The next lemma is more complicated. It says that at any stage, when we refine a component W^n_k for $k \geq 1$, we can control the sum of the diameters of the refined components inside W^n_k in terms of the diameter of W^n_k. See Figure 8. This Lemma corresponds inequality (17.2) in [Bis17]. We will clarify the part of the proof with an estimate related to the "petal map"; the rest of the proof is the same.

Lemma 12.2. *Let $k \geq 1$, $\alpha \geq s$. Suppose that $W^n_k \in \mathcal{C}_m$ with $k \geq 1$ for some m. Let $\widehat{W}^{n+q}_{k+q-1}$ be components of $\widehat{\mathcal{C}}_{m+1}$ contained inside of W^n_k. Let $W^n_k(q)$ denote the* *components of the form* $\widehat{W}^{n+q}_{k+q-1}$ *for a fixed* $q \geq 1$. *Then*

$$\sum_{q\geq 1} \sum_{W_k^n(q)} \operatorname{diam}(\widehat{W}^{n+q}_{k+q-1})^\alpha \leq \frac{1}{8} \operatorname{diam}(W_k^n)^\alpha$$

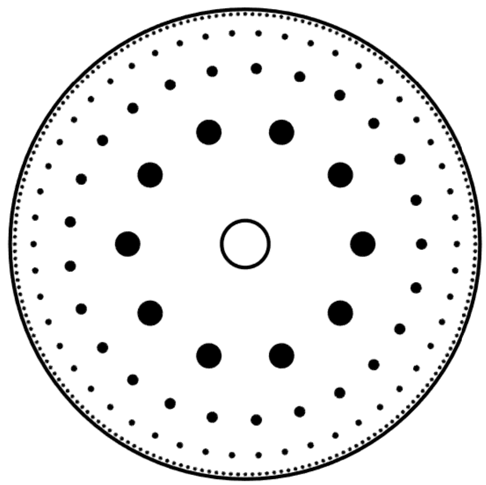

FIGURE 9. A schematic for Lemma 12.2. W_k^n belongs to the covering $\mathcal{C}_m$ and is the annulus bounded by the innermost and outermost boundary curves. The filled components correspond to elements $\widehat{W}^{n+q}_{k+q-1} \in \widehat{\mathcal{C}}_{m+1}$ for $q = 1, 2, 3, 4$. Lemma 12.2 says when R is large enough and $k \geq 1$, the α-sum of these components in $\widehat{\mathcal{C}}_{m+1}$ can be controlled in terms of $\operatorname{diam}(W_k^n)^\alpha$.

Proof. We have $\widehat{W}^{n+q}_{k+q-1} \subset W_k^n \subset A_1$. Then $f^n(\widehat{W}^{n+q}_{k+q-1}) \subset B \subset A_k$, where B is a ball where f^q is conformal, constructed similarly by a pullback of some ball B' of unit size as in the proof of Lemma 12.1. By one application of Lemma 3.2 for all $q \geq 1$ we have

$$\frac{\operatorname{diam}(\widehat{W}^{n+q}_{k+q-1})^\alpha}{\operatorname{diam}(W_k^n)^\alpha} \leq C \frac{\operatorname{diam}(f^n(\widehat{W}^{n+q}_{k+q-1}))^\alpha}{R_k^\alpha}. \tag{12.1}$$

If $q = 1$, then $f^n(\widehat{W}^{n+1}_k)$ belongs to a petal $R_k \cdot \Omega^p_{n_k}$ and $\operatorname{diam}(f^n(\widehat{W}^{n+1}_k)) = O(R_k^{-2} n_k^{-2})$ by Lemma 8.1. Therefore for sufficiently large R, (12.1) yields

$$\frac{\operatorname{diam}(\widehat{W}^{n+1}_k)^\alpha}{\operatorname{diam}(W_k^n)^\alpha} \leq \frac{1}{R_k^\alpha}.$$

If $q \geq 2$, we need to be a little more careful. Another application of Lemma 3.2 with B as above gives

$$\frac{\operatorname{diam}(f^n(\widehat{W}^{n+q}_{k+q-1}))^\alpha}{\operatorname{diam}(B)^\alpha} \leq C\frac{\operatorname{diam}(f^{n+q-1}(\widehat{W}^{n+q}_{k+q-1}))^\alpha}{\operatorname{diam}(B')^\alpha}.$$

Then since $\operatorname{diam}(f^{n+q-1}(\widehat{W}^{n+q}_{k+q-1})) = O(R^{-2}_{k+q-1}n^{-2}_{k+q-1})$ by Lemma 8.1. Therefore if R is large enough, we combine with (12.1) to obtain

$$\frac{\operatorname{diam}(\widehat{W}^{n+q}_{k+q-1})^\alpha}{\operatorname{diam}(W^n_k)^\alpha} \leq \frac{1}{R^\alpha_k}.$$

The rest of the lemma follows similarly to the proof of Lemma 16.3 on page 449 in [Bis17]. □

13. The s-Sum of the Fatou Components: Refining near B_f and Conclusions

Let $\widehat{W}^n_k \in \widehat{\mathcal{C}}_m$ be given. Lemma 12.2 only works for components in $\widehat{\mathcal{C}}_{m+1}$ contained in the annular regions $W^n_j \in \mathcal{C}_m$ with $W^n_j \subset \widehat{W}^n_k$ and $j \geq 1$. However, there are also components of $\mathcal{C}_{m+1}$ contained in $W^n_j \in \mathcal{C}_m$ for $j < 1$ that are contained in $\widehat{W}^n_k$. The methods of the above two lemmas do not work as $j \to -\infty$. We handle this difficulty by using a Whitney type decomposition.

First, we need to recall some notation from Section 11. Let A_1 be as above and $k \leq 0$. Then A_k is the preimage $f^{-(1-k)}(A_1)$ that surrounds the origin. Choose $z \in Y \cap A_k$. Then $f^{-k+1}(z) \in A_1 \cap Y$, and therefore $z \in \widehat{W}^n_n \subset A_1$ for some $\widehat{W}^n_n \in \widehat{\mathcal{C}}_1$. Then define V^n_k to be the component of $f^{k-1}(\widehat{W}^n_n)$ that contains z. Let $\mathcal{V}_k$ denote the set of all V^n_k's contained in A_{-k}. Let $\mathcal{V} = \cup_{k\leq 0}\mathcal{V}_k$. See Figure 9.

Lemma 13.1. *Fix $\epsilon_0 > 0$, and let $\alpha \geq s + \epsilon_0$. With the notation as above, there exists a constant C (depending on ϵ_0) so that for sufficiently large R,*

$$\sum_{V^n_k \in \mathcal{V}} \operatorname{diam}(V^n_k)^\alpha \leq C2^N R^\alpha_0.$$

Proof. For each Ω_k, $k \leq 0$, $f : \Omega^A_k \to \Omega^A_{k+1}$ is a 2^N-to-1 covering map, so each Ω_k can be decomposed into $2^{N(-k+1)}$ pieces each of which maps conformally onto Ω^A_1 minus a slit, denoted as Ω^S_1. We may choose the slit to be contained in the right half plane of $\mathbb{C}$. This process breaks Ω_k into $2^{N(-k+1)}$ pieces and we denote this collection by $\mathcal{Q}_k$. Define $\mathcal{Q} = \cup_{k\leq 0}\mathcal{Q}_k$. We also choose to define $\mathcal{Q}$ by the dynamics of f, so that each $Q \in \mathcal{Q}_k$ maps onto some $Q' \in \mathcal{Q}_{k+1}$. To accomplish this, it suffices to choose an appropriate decomposition of Ω^A_0, and then define the decomposition of Ω_k for $k < 0$ by inverse images, similar to Example 4.1. Also see Figure 9.

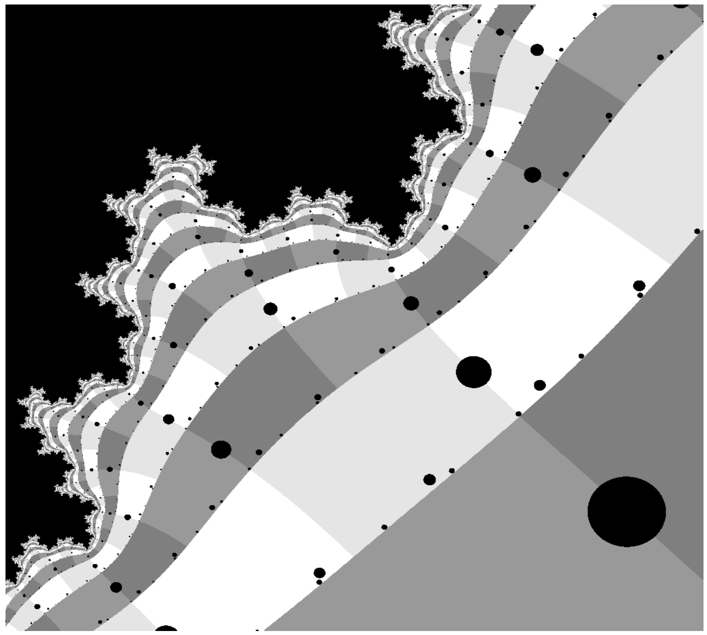

FIGURE 10. A schematic for Lemma 13.1. We see a portion of B_f, and we see annular regions representing the innermost and outermost boundaries of Fatou components Ω_k for $k \leq 0$. The complimentary components of Ω_k contained in Ω_k^A are filled in and correspond to the elements in $\mathcal{V}_k$. Inside each annular region, we have depicted the pieces of Ω_k that make up the collection $\mathcal{Q}$, which is a Whitney type decomposition of the unbounded complementary component of J_f.

With this procedure, it is not necessarily true that each V_k^n is compactly contained in some $Q \in \mathcal{Q}$. Doing the same procedure above, but with a slit in the left half plane of $\mathbb{C}$, we obtain a similar but "rotated" collection $i\mathcal{Q} = \cup_{k \leq 0} i\mathcal{Q}_k$. Then each V_k^n is compactly contained inside at least one element of $\mathcal{Q}_k$ or $i\mathcal{Q}_k$.

We claim $\mathcal{Q}$ and $i\mathcal{Q}$ form Whitney type decompositions of the unbounded complementary component of the Julia set J_f of the polynomial-like map f. Indeed, $(f, \widehat{\Omega}_0, \widehat{\Omega}_1)$ is a degree 2^N polynomial-like mapping, and is therefore quasiconformally conjugate in $\widehat{\Omega}_0 \setminus \overline{\mathbb{D}}$ to z^{2^N}. Moreover, elements of $\mathcal{Q}$ correspond under this conjugacy to a Whitney-type decomposition of the unbounded complement of the unit circle, as in Example 4.1. The same discussion applies to $i\mathcal{Q}$. It follows from

Lemma 4.5 that $\mathcal{Q}$ and $i\mathcal{Q}$ are Whitney type decompositions of a neighborhood of the unbounded complementary component of $\mathcal{J}_f$.

Let Q in $\mathcal{Q}_k$ be given. Let $V_k^n \in \mathcal{V}_k$ be compactly contained in Q. Then some iterate of f conformally maps V_k^n onto some $\widehat{W}_n^n \in \widehat{\mathcal{C}}_1$, and by Lemma 3.2

$$\frac{\operatorname{diam}(V_k^n)^\alpha}{\operatorname{diam}(Q)^\alpha} \leq C \frac{\operatorname{diam}(\widehat{W}_n^n)^\alpha}{\operatorname{diam}(\Omega_1)^\alpha}.$$

Next we sum over all V_k^n that are compactly contained inside of Q. We denote this by $V_k^n \subset\subset Q$. We obtain

$$\sum_{V_k^n \subset\subset Q} \frac{\operatorname{diam}(V_k^n)^\alpha}{\operatorname{diam}(Q)^\alpha} \leq C \sum_{\widehat{W}_n^n \in \widehat{\mathcal{C}}_1} \frac{\operatorname{diam}(\widehat{W}_n^n)^\alpha}{\operatorname{diam}(\Omega_1)^\alpha} \leq \frac{C}{R_1^\alpha} \sum_{\widehat{W}_n^n \in \widehat{\mathcal{C}}_1} \operatorname{diam}(\widehat{W}_n^n)^\alpha.$$

We now sum over all V_k^n that are compactly contained in some $Q \in \mathcal{Q}_k$. Then

$$\sum_{V_k^n} \operatorname{diam}(V_k^n)^\alpha \leq \frac{C}{R_1^\alpha} \sum_{\widehat{W}_n^n \in \widehat{\mathcal{C}}_1} \operatorname{diam}(\widehat{W}_n^n)^\alpha \cdot \sum_{Q \in \mathcal{Q}_k} \operatorname{diam}(Q)^\alpha.$$

Similarly, if we sum over all V_k^n that are compactly contained in some $Q' \in i\mathcal{Q}_k$ we obtain

$$\sum_{V_k^n} \operatorname{diam}(V_n^n)^\alpha \leq \frac{C}{R_1^\alpha} \sum_{\widehat{W}_n^n \in \widehat{\mathcal{C}}_1} \operatorname{diam}(\widehat{W}_n^n)^\alpha \cdot \sum_{Q' \in i\mathcal{Q}_k} \operatorname{diam}(Q)^\alpha.$$

Note that every V_k^n is compactly contained in at least one element of $\mathcal{Q}$ or $i\mathcal{Q}$. So

$$\sum_{V_k^n \in \mathcal{V}_k} \operatorname{diam}(V_n^n)^\alpha \leq \frac{C}{R_1^\alpha} \left(\sum_{Q \in \mathcal{Q}_k} \operatorname{diam}(Q)^\alpha + \sum_{Q' \in i\mathcal{Q}_k} \operatorname{diam}(Q')^\alpha \right) \sum_{\widehat{W}_n^n \in \widehat{\mathcal{C}}_1} \operatorname{diam}(\widehat{W}_n^n)^\alpha.$$

Finally, this estimate is true for all k, so summing over all k gives us,

$$\sum_{V_k^n \in \mathcal{V}} \operatorname{diam}(V_k^n)^\alpha \leq \frac{C}{R_1^\alpha} \sum_{\widehat{W}_n^n \in \widehat{\mathcal{C}}_1} \operatorname{diam}(\widehat{W}_n^n)^\alpha \cdot \left(\sum_{Q \in \mathcal{Q}_k} \operatorname{diam}(Q)^\alpha + \sum_{Q' \in i\mathcal{Q}_k} \operatorname{diam}(Q')^\alpha \right).$$

Since $\mathcal{Q}$ and $i\mathcal{Q}$ are Whitney type decompositions of the unbounded complementary component of J_f, and $\alpha \geq s + \epsilon_0$, the α-sum of elements of $\mathcal{Q}$ and $i\mathcal{Q}$ converges and are comparable to the α-sum of elements in $\mathcal{Q}_0$. Recall that $R_0 = \operatorname{diam}(\Omega_0)$. Then by this observation and Lemma 12.2

$$\sum_{V_k^n \in \mathcal{V}} \operatorname{diam}(V_n^n)^\alpha \leq \frac{C \cdot 2^N R_0^\alpha}{R_1^\alpha} \cdot \sum_{\widehat{W}_n^n \in \mathcal{C}_1} \operatorname{diam}(\widehat{W}_n^n)^\alpha \leq C 2^N R_0^\alpha.$$

whenever R is sufficiently large. □

If $\widehat{W}_k^n \in \widehat{\mathcal{C}}_m$, then some iterate of f maps $\widehat{W}_k^n$ conformally onto $\widehat{A}_k$. The components of $\widehat{\mathcal{C}}_{m+1}$ that are contained inside of $W_j^n \in \mathcal{C}_m$ with $W_j^n \subset \widehat{W}_k^n$ for $j < 1$ get conformally mapped onto the elements of $\mathcal{V}$. This allows us to prove the following more general lemma.

Lemma 13.2. *Fix $\epsilon_0 > 0$ and $\alpha \geq s + \epsilon_0$. Consider an element of the form $\widehat{W}_k^n \in \widehat{\mathcal{C}}_m$ for some m. Let W_j^n for $j \leq 1$ be the elements of $\mathcal{C}_m$ which are contained in $\widehat{W}_k^n$. Let $W_j^n(q)$ denote the components of the form $\widehat{W}_{j+q-1}^{n+q}$ in $\widehat{\mathcal{C}}_{m+1}$ which are contained in W_j^n (we define $W_j^n(q)$ to be empty if $j+q-1 \leq 0$). Then there exists a sufficiently large R so that the α-sum of all the components $\widehat{W}_{j+q-1}^{n+q} \in \widehat{\mathcal{C}}_{m+1}$ contained in W_j^n for $j < 1$ satisfies*

$$\sum_{j=0}^{-\infty}\sum_{q=1}^{\infty}\sum_{W_j^n(q)} \operatorname{diam}(\widehat{W}_{j+q-1}^{n+q})^\alpha \leq \frac{1}{8}\operatorname{diam}(W_1^n). \tag{13.1}$$

Proof. Choose some $\widehat{W}_k^n \in \widehat{\mathcal{C}}_m$. Then $f^n(\widehat{W}_k^n) = \widehat{A}_k$, and this mapping is conformal. The elements being summed in (13.1) are mapped conformally onto $\mathcal{V}$. Therefore, by Lemma 3.2,

$$\begin{aligned}
\sum_{j=0}^{-\infty}\sum_{q=1}^{\infty}\sum_{W_j^n(q)} \operatorname{diam}(\widehat{W}_{j+q-1}^{n+q})^\alpha &\leq C\frac{\operatorname{diam}(W_1^n)^\alpha}{\operatorname{diam}(A_1)^\alpha}\sum_{V_k^n\in\mathcal{V}}\operatorname{diam}(V_k^n)^\alpha \\
&\leq \frac{C\cdot 2^N\cdot R_0^\alpha}{R_1^\alpha}\cdot\operatorname{diam}(W_1^n)^\alpha \\
&\leq \frac{1}{8}\operatorname{diam}(W_1^n)^\alpha,
\end{aligned}$$

whenever R is large enough. □

We now have everything we need to prove Theorem 10.1.

Proof of Theorem 10.1. It is sufficient to show that the $(s+\epsilon_0)$-sum of the elements in $\widehat{\mathcal{C}}_{m+1}$ is at most half the $(s+\epsilon_0)$-sum of the elements in $\widehat{\mathcal{C}}_m$, because then the $(s+\epsilon_0)$-sum is geometric. To accomplish this, it suffices to show that for any $\widehat{W}_k^n \in \widehat{\mathcal{C}}_m$, the $(s+\epsilon_0)$-sum of all the elements of $\mathcal{C}_{m+1}$ contained in $\widehat{W}_k^n$ is at most half of $\operatorname{diam}(W_k^n)^{s+\epsilon_0}$.

To that end, let $\widehat{W}_k^n \in \widehat{\mathcal{C}}_m$ be given. Using the notation of the previous lemmas, the $(s+\epsilon_0)$-sum of all the elements of $\mathcal{C}_{m+1}$ contained in $\widehat{W}_k^n$ is represented by

$$\begin{aligned}
I = \sum_{j=k}^{-\infty}\sum_{q\geq 1}\sum_{W_j^n(q)} \operatorname{diam}(\widehat{W}_{j+q-1}^{n+q})^{s+\epsilon_0} &= \sum_{j=1}^{k}\sum_{q\geq 1}\sum_{W_j^n(q)} \operatorname{diam}(\widehat{W}_{j+q-1}^{n+q})^{s+\epsilon_0} \\
&\quad +\sum_{j=0}^{-\infty}\sum_{q\geq 1}\sum_{W_j^n(q)} \operatorname{diam}(\widehat{W}_{j+q-1}^{n+q})^{s+\epsilon_0}.
\end{aligned}$$

By Lemma 13.2 we have

$$\sum_{j=0}^{-\infty}\sum_{q\geq 1}\sum_{W_j^n(q)} \operatorname{diam}(\widehat{W}_{j+q-1}^{n+q})^{s+\epsilon_0} \leq \frac{1}{8}\operatorname{diam}(W_1^n)^{s+\epsilon_0}.$$

Combining this with Lemma 12.2 to estimate the other sum, we have

$$I \leq \frac{1}{4}\sum_{j=1}^{k}\operatorname{diam}(W_j^n)^{s+\epsilon_0}.$$

Then repeatedly using Lemma 12.1 we can conclude that $I \leq \frac{1}{2}\operatorname{diam}(W_k^n)^{s+\epsilon_0}$. $\square$

Corollary 13.3. *Let* $\epsilon_0 > 0$. *Then for sufficiently large* R, *we have* $\dim_{\mathrm{H}}(Y) \leq s+\epsilon_0$.

Proof. Each $\widehat{\mathcal{C}}_m$ is a covering of $Y \cap A_1$. The proof of Theorem 10.1 shows that the $(s+\epsilon_0)$-sum of all components of all $\widehat{\mathcal{C}}_m$'s converges, and therefore the $(s+\epsilon_0)$-sum of the elements in $\widehat{\mathcal{C}}_m$ tends to 0 as $m \to \infty$. Therefore $\dim_{\mathrm{H}}(Y \cap A_1) \leq s+\epsilon_0$. The same arguments in these sections can be modified to show that $\dim_{\mathrm{H}}(Y \cap A_k) \leq s+\epsilon_0$ for all $k \in \mathbb{Z}$. Therefore $\dim_{\mathrm{H}}(Y) \leq s+\epsilon_0$. $\square$

14. The Packing Dimension of $\mathcal{J}(f)$ is < 2

In this section, we prove that the packing dimension of $\mathcal{J}(f)$ can be taken to be arbitrarily close to its Hausdorff dimension. We do this by estimating the local upper Minkowski dimension using Theorem 4.4. To do this, we will need to show the $(s+\epsilon_0)$-sum of a Whitney type decomposition of the complement of the Julia set contained in the neighborhood Ω_1^A is finite. This will require separating the components into wiggly, round, and basin components, and performing some useful decompositions of these components.

The following result follows from the results of Sullivan in [Sul83] (see Theorems 3 and 4). Recall that if f is polynomial-like, K_f denotes its filled Julia set.

Theorem 14.1. *Let* $f : U \to V$ *be a hyperbolic polynomial-like map. Then we have* $\dim_{\mathrm{P}}(\partial K_f) = \dim_{\mathrm{H}}(\partial K_f) = \overline{\dim}_{\mathrm{M}}(\partial K_f)$.

In particular, this result applies to f when viewed as a polynomial like map. So this result applies to J_f, the quasicircle Julia set of f.

In order to apply Theorem 4.4, we need to decompose the Fatou components of f into simpler pieces. First we collect the following lemmas proved in Section 20 of [Bis17]. The first lemma will allow us to break the infinitely connected Fatou components into simpler, annular regions and still conclude the convergence of a t-sum of a Whitney type decomposition.

Lemma 14.2. *Let Ω be a bounded open set containing disjoint open subsets $\{\Omega_j\}$ so that $\Omega \backslash \cup_j \Omega_j$ has zero Lebesgue measure. Let $W(\Omega)$ be a Whitney type decomposition of Ω and $W(\Omega_j)$ be a Whitney type decomposition for Ω_j. Then for $t \in (1,2]$ we have*

$$\sum_{Q \in W(\Omega)} \operatorname{diam}(Q)^t \leq \sum_j \sum_{Q \in W(\Omega_j)} \operatorname{diam}(Q)^t.$$

Figure 11 illustrates how Lemma 14.2 will be implemented. By Theorem 10.9, the complementary components of the Fatou component Ω_1 contained in Ω_1^A are arranged in approximately circular rings, which can be connected by an approximate circle. Doing this for every ring of complementary components, we obtain a countable union of Jordan annuli. This can be done for every Fatou component $\omega \subset \Omega_1^A$. We call this procedure *necklacing* the Fatou component.

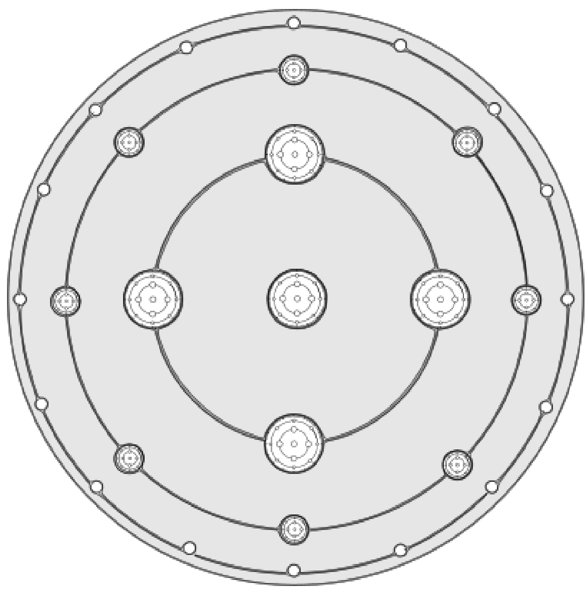

FIGURE 11. A schematic for the necklacing construction. Holes of Ω_k in the same circular ring are connected via approximate circles, and this construction can be repeated for all $\omega \subset \Omega_k^A$ by pulling back this construction. The result is the multiply connected Fatou components are now decomposed into topological annuli, which can be straightened into round annuli by a biLipshitz map. Lemmas 14.2 and 14.3 say that it suffices to estimate the critical exponent for a Whitney type decomposition of the complement of the "necklaced" Julia set of f.

Lemma 14.3. *If $f : \Omega_1 \to \Omega_2$ is L-biLipschitz, and let $W(\Omega_1)$ and $W(\Omega_2)$ be Whitney type decompositions for Ω_1 and Ω_2. Then for any $t \in (0,2]$, there is a constant C depending only on L and the constants defining the Whitney type* *decompositions so that we have*

$$\frac{1}{C}\sum_{Q'\in W(\Omega_2)}\operatorname{diam}(Q')^t=\sum_{Q\in W(\Omega_1)}\operatorname{diam}(Q)^t\leq C\sum_{Q'\in W(\Omega_2)}\operatorname{diam}(Q')^t.$$

Proof. The image of $W(\Omega_1)$ under f is a Whitney type decomposition; indeed, L-BiLipschitz maps are always L^2-quasiconformal. So the t-sums of $W(f(\Omega_1))$ and $W(\Omega_2)$ are comparable depending on L and the constants defining the Whitney type decompositions. If $Q\in W(\Omega_1)$ then $1/L\operatorname{diam}(Q)\leq\operatorname{diam}(f(Q))\leq L\operatorname{diam}(Q)$. Therefore the t-sums of $W(f(\Omega_1))$ and $W(\Omega_1)$ are comparable depending on L, and the result follows. □

We will use Lemma 14.3 to map the decomposed round components onto round annuli, where we can estimate the t-sum directly.

Lemma 14.4. *Let $A=A(\rho,\rho(1+\delta))=\{z:\rho\leq|z|\leq\rho(1+\delta)\}$ be a round annulus with $\delta,\rho>0$. Let $W(A)$ denote a Whitney type decomposition of A. Then for $t>1$,*

$$\sum_{Q\in W(A)}\operatorname{diam}(Q)^t\leq O\left(\frac{1}{1-2^{t-1}}\delta^{t-1}\rho^t\right).$$

Proof. We first construct a suitable Whitney type decomposition of $A(1,1+\delta)$; see Figure 12. The result will follow from the observations in Section 4 and applying the map $z\mapsto\rho z$. For the given $\delta>0$ there exists N so that $\delta\in[2^{-N},2^{-N+1})$. Let

$$S=\{z=re^{2\pi i\theta}:r\in(1+\delta/4,1+3\delta/4),\theta\in(0,2^{-N})\}.$$

Both $\operatorname{diam}(S)$ and $\operatorname{dist}(S,\partial A)$ are comparable to 2^{-N}. Then rotating S exactly 2^N many times gives the first layer of the Whitney type decomposition which we denote by $\mathcal{S}_1$. To get subsequent layers, we define

$$S_1=\{z=re^{i\theta}:R\in(R+\delta/8,R+\delta/4),\theta\in(0,2^{-N-1})\}.$$

$$S_1'=\{z=re^{2\pi i\theta}:R\in(R+3\delta/4,R+7\delta/8),\theta\in(0,2^{-N-1})\}.$$

Then $\operatorname{diam}(S_1)$ and $\operatorname{dist}(S_1,\partial A)$ are both comparable to 2^{-N-1}, and the same is true for S_1'. Then rotating S_1 and S_1' 2^{N+1} many times generates $\mathcal{S}_2$, the second layer of the Whitney type decomposition. Proceeding inductively, we obtain a Whitney type decomposition $W(A)$.

The t-sum for the elements in $\mathcal{S}_1$ is comparable to $2^{N(1-t)}$, and the t-sum for the elements in $\mathcal{S}_{n+1}$ is comparable to $2^{(N+n)(1-t)}$. Therefore, the t-sum is geometric and comparable to the quantity

$$\frac{2^{N(1-t)}}{1-2^{1-t}}.$$

2^N and $1/\delta$ are both comparable so we obtain

$$\sum_{Q \in W(A)} \operatorname{diam}(Q)^t = O\left(\frac{1}{1-2^{1-t}} \delta_j^{t-1}\right).$$

□

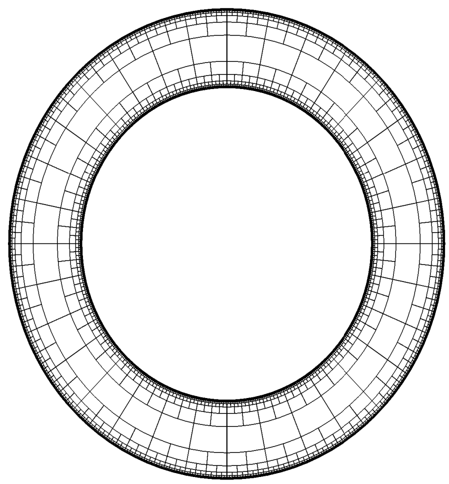

FIGURE 12. An illustration of a Whitney type decomposition of the annulus described in Lemma 14.4

The following is the fundamental estimate for round Fatou components. See Theorem 20.3 in [Bis17].

Theorem 14.5. *Let $W(\omega_k)$ be a Whitney type decomposition for a component ω_k of k-type, $k \geq 1$, and let $t \geq s + \epsilon_0$. Then*

$$\sum_{Q \in W(\omega_k)} \operatorname{diam}(Q)^t = O\left(\frac{1}{1-2^{1-t}} \operatorname{diam}(\omega_k)^t\right).$$

Proof. Perform the necklacing decomposition on ω_k. By Lemma 14.2, it suffices to estimate the t-sum of a Whitney type decomposition of all of these Jordan annuli. By Lemma 10.9, these annuli are biLipschitz equivalent to round annuli $A(r, r(1+\delta_j))$, where r is the diameter of Ω_k. Lemma 10.9 part (1) and Lemma 8.5 shows that δ_j tends very rapidly to 0 and are summable. Therefore, by Lemmas 14.3 and 14.4, we have,

$$\sum_{Q \in W(\omega_k)} \operatorname{diam}(Q)^t = O\left(\frac{1}{1-2^{1-t}} \sum_{j=1}^{\infty} \delta_j^{t-1} \operatorname{diam}(\omega_k)^t\right) = O\left(\frac{1}{1-2^{1-t}} \operatorname{diam}(\omega_k)^t\right).$$

□

Next we show how to control the critical exponent for the boundaries of the central series of Fatou components.

Lemma 14.6. *Let $W(\mathcal{J}(f))$ be a Whitney type decomposition of the complement of $\mathcal{J}(f)$, and let W denote the elements of $W(\mathcal{J}(f))$ contained in $B_f \cup (\cup_{k=-\infty}^{n} \Omega_k)$. Then*

$$\sum_{Q \in W} \operatorname{diam}(Q)^{s+\epsilon_0} < \infty.$$

Proof. We split the $(s+\epsilon_0)$-sum into three pieces

$$\sum_{Q \in W} \operatorname{diam}(Q)^{s+\epsilon_0} = I + II + III.$$

I is the sum of the $Q \in W$ contained in B_f, II is the sum of the $Q \in W$ contained in round Ω_k, and III is the sum of the $Q \in W$ contained in wiggly Ω_k.

I converges by definition of the critical exponent. The $Q \in W$ which satisfy $Q \in B_f$ form a Whitney type decomposition of the bounded complementary component of B_f. Since $\overline{\dim_{\mathrm{M}}}(J_f) = s$, I converges.

II converges by Lemma 14.5.

III requires some work. The necklacing construction pulls back to Ω_k for $k \leq 0$. Let $\mathcal{Q} = \cup_{k \leq 0} \mathcal{Q}_k$ be the collection of pieces from Lemma 13.2. The boundaries of the elements $Q \in \mathcal{Q}$ further decompose the necklaced versions of Ω_k into necklaced quadrilateral-type pieces. By Lemma 14.2, it is sufficient to estimate the $(s+\epsilon_0)$-sum II by estimating a $(s+\epsilon_0)$-sum for a Whitney type decomposition W of the complement of all of the necklaced quadrilateral-type pieces of Ω_k.

Choose some $Q \in \mathcal{Q}_k$ (note that $k \leq 0$). Lemma 3.2 implies that for all $S \in W$, $S \subset Q$,

$$\frac{\operatorname{diam}(S)^{s+\epsilon_0}}{\operatorname{diam}(Q)^{s+\epsilon_0}} \leq C \frac{\operatorname{diam}(f^{-k+1}(S))^{s+\epsilon_0}}{\operatorname{diam}(\Omega_1)^{s+\epsilon_0}}.$$

Since f is conformal on Q, the Whitney type decomposition for the necklaced pieces of Ω_k get mapped to a Whitney type decomposition for Ω_1^S, the slit version of Ω_1. Similar to the proofs of Theorem 14.5 and Theorem 14.4, the $(s+\epsilon_0)$-sum of a Whitney type decomposition for the slit Ω_1^S is finite and comparable $\operatorname{diam}(\Omega_1)^{s+\epsilon_0}$. Summing over all $S \subset Q$ and applying Corollary 4.7 we have

$$\sum_{S \in W, S \subset Q} \operatorname{diam}(S)^{s+\epsilon_0} \leq \frac{C}{R_1^{s+\epsilon_0}} \operatorname{diam}(Q)^{s+\epsilon_0}.$$

Therefore, by summing over all $S \in W$, we sum over all $Q \in \mathcal{Q}$ and get

$$\sum_{S \in W} \operatorname{diam}(S)^{s+\epsilon_0} \leq \frac{C}{R_1^{s+\epsilon_0}} \sum_{Q \in \mathcal{Q}} \operatorname{diam}(Q)^{s+\epsilon_0} < \infty,$$

because $\mathcal{Q}$ is a Whitney type decomposition for the unbounded complementary component of J_f. $\square$

Let $W(\Omega_1^A)$ denote a Whitney type decomposition for all Fatou components contained inside of Ω_1^A.

Theorem 14.7. *The $(s+\epsilon_0)$-sum of the Whitney type decomposition $W(\Omega_1^A)$ above converges.*

Proof. Similar to Lemma 14.6, the Fatou components have three types, and we decompose the $(s+\epsilon_0)$-sum into three pieces. We write

$$\sum_{Q\subset W(\Omega_1^A)} \operatorname{diam}(Q)^{s+\epsilon_0} = I + II + III.$$

Here, I represents the $(s+\epsilon_0)$-sum in basins, II represents the $(s+\epsilon_0)$-sum in round components, and III represents the $(s+\epsilon_0)$-sum in wiggly components.

First we estimate I. Note that if a basin $B_f' = f^{-m}(B_f)$ for some m, then B_f' moves backwards l many times, so it is is contained in a unique $\widehat{\omega}$ where ω is a Fatou component of 1-type that moves backwards l many times. $f^m : \widehat{\omega} \to \widehat{\Omega}_1$ is conformal, so for all $Q \subset B_f'$,

$$\sum_{Q\in B_f'} \operatorname{diam}(Q)^{s+\epsilon_0} \le C \operatorname{diam}(\omega)^{s+\epsilon_0} \sum_{f^m(Q)\in B_f} \operatorname{diam}(f^m(Q))^{s+\epsilon_0}.$$

By Corollary 4.7, the sum above is comparable to an $(s+\epsilon_0)$-sum of a fixed Whitney type decomposition for B_f, which we discussed converges in the proof of Lemma 14.6. Therefore, by summing over all B_f' and all components ω of 1-type containing B_f' and applying Theorem 10.1, we get

$$I \le C \sum_{B_f'\subset\omega} \operatorname{diam}(\omega)^{s+\epsilon_0} < \infty.$$

II converges by Theorem 14.5 and Theorem 10.1.

Finally we estimate III. For every component ω of 1-type, $\widehat{\omega}$ contains a unique sequence of components ω_k of k-type for $k \le 0$ so that there exists m so that $f^m(\omega) = \Omega_1$ and $f^m(\omega_k) = \Omega_k$, and f^m is conformal on $\widehat{\omega}$. Let ω be of 1-type and ω_k be its associated sequence of wiggly components. Corollary 4.7 and the proof of Lemma 14.6 show that

$$\sum_{k=0}^{-\infty} \sum_{Q\subset\omega_k} \operatorname{diam}(Q)^{s+\epsilon_0} \le C \operatorname{diam}(\omega)^{s+\epsilon_0} \cdot \sum_{Q\subset\mathcal{Q}} \operatorname{diam}(Q)^{s+\epsilon_0} \le C \operatorname{diam}(\omega)^{s+\epsilon_0}.$$

Again, Theorem 10.1 allows us to conclude that $III < \infty$. $\square$

Corollary 14.8. *The upper Minkowski dimension, and hence the packing dimension of $\mathcal{J}(f) \cap A_1$ is at most $s+\epsilon_0$.*

Proof. The above argument shows that the critical exponent for the Whitney decomposition of $\mathcal{J}(f) \cap A_1$ is less than or equal to $s+\epsilon_0$. Since $\mathcal{J}(f)$ has zero Lebesgue measure, Theorem 4.4 says that the upper Minkowski dimension of $\mathcal{J}(f) \cap A_1$ is also less than or equal to $s+\epsilon_0$. By results in [RS05], since f has no exceptional values, the local upper Minkowski dimension is constant and coincides with the packing dimension, so that $\dim_{\mathrm{P}}(J(f)) \leq s+\epsilon_0$. □

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