A Transcendental Julia Set of Dimension 1

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2 October 2017

Some History

1. (Baker, 1975): Julia set of a transcendental entire function contains a continuum. So we always have $\dim(J(f)) \geq 1$.
2. (McMullen, 1987): Studied two families of transcendental entire functions:

$$\{f(z) = \lambda e^z : \lambda \neq 0\}, \quad \dim J(f) = 2$$

$$\{g(z) = \sin(az + b) : a \neq 0\}, \quad J(f) \text{ has positive area.}$$

3. (Stallard, 1997-2000): Constructed examples in $\mathcal{B}$ with Hausdorff dimension d for all $d \in (1, 2]$.

The Main Theorem

Theorem (Bishop, 2011)

There exists a transcendental entire function f so that $J(f)$ has Hausdorff dimension 1.

What is f?

The function f is a *family* of infinite products

$$f(z) = F_0(z) \cdot \prod_{k=1}^{\infty} F_k(z).$$

Each f is determined by fixed parameters $\{N \in \mathbb{N}, \lambda > 1, R > 1, S \subset \mathbb{N}\}$.

$$F_0(z) = N\text{th iterate of } p_\lambda(z) = \lambda(2z^2 - 1),$$

$$F_k(z) = \left(1 - \frac{1}{2}\left(\frac{z}{R_k}\right)^{n_k}\right).$$

Here, $\{R_k\}$ and $\{n_k\}$ are defined in terms of $\{N, \lambda, R, S\}$ and increase rapidly to ∞.

What is f?

To illustrate, choose parameters $N = 5$, $R = \lambda = 10$. Then

$$F_0(z) = (2\lambda)^{2^N - 1} z^{2^N} + \text{lower order terms}$$

We define $\{n_k\}$ in terms of N by

$$n_k = 2^{N+k-1}$$

We define $\{R_k\}$ so that we have growth at least

$$R_{k+1} \geq 2R_k^2.$$

Then, for example

$$F_4(z) = \left(1 - \left(\frac{z}{1,600,000,000}\right)^{512}\right).$$

Why $F_0(z)$?

The Julia set of $p_\lambda(z)$, and therefore of $F_0(z)$, is a Cantor set in $[-1,1]$. It's dimension tends to 0 as $\lambda \to \infty$.

$\{R_k\}$ and $\{n_k\}$ are chosen to increase sufficiently quickly, so that on $D = B(0, 1/2R)$,

$$\prod_{k=1}^{\infty} F_k(z) \approx 1.$$

Therefore on D

$$f(z) \approx F_0(z) = (p_\lambda(z))^{\circ N}.$$

Why $F_0(z)$?

It follows that f has some invariant Cantor set E in D of small dimension.

This Cantor set above will be in the Julia set, but its small dimension will not impact its Hausdorff dimension.

Finally, outside of D, the infinite product part of f will be the dominant term.

Why $F_k(z)$?

First, we decompose $\mathbb{C}$ into annuli. Define

$$A_k = \{z : \frac{1}{4}R_k \leq |z| \leq 4R_k\}, \qquad B_k = \{z : 4R_k \leq |z| \leq \frac{1}{4}R_{k+1}\}.$$

Further, we will need to define for k for negative indices. If $k \geq 0$:

$$A_0 = \{z \in D : f(z) \in A_1\}$$

$$A_{-k} = \{z \in D : f(z), \dots f^k(z) \in D, f^{k+1}(z) \in A_1\}$$

In this way we can define $A = \cup_{k \in \mathbb{Z}} A_k$. Finally we will need to define

$$V_k = \{z : 3/2R_k \leq |z| \leq 5/2R_k\} \subset A_k.$$

Why $F_k(z)$?

One of the key features of $F_k(z)$ is that it may be written in terms of $T_2(z^m)$, where

$$T_2(z) = 2z^2 - 1.$$

By rescaling T_2 appropriately, we obtain the function

$$H_m(z) = -T_2(r_2 z^m + z_2) = z^m(2 - z^m).$$

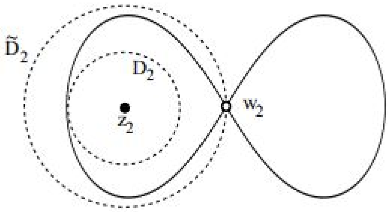

Figure: Level set of $|T_2(z)| = 1$.

Why $F_k(z)$? (cont.)

All of this gives us a good local model for f.

Lemma

For all z:

$$F_k(z) = \frac{1}{2}\left(\frac{R_k}{z}\right)^{n_k} \cdot H_{n_k}\left(\frac{z}{R_k}\right).$$

And for all $z \in A_k$

$$f(z) = C_k \cdot H_{n_k}\left(\frac{z}{R_k}\right) \cdot (1 + O(R_k^{-1}))$$

where C_k depends on all of the initial starting parameters.

Roughly, f looks like H_{n_k} on A_k.

Why $F_k(z)$?(cont.)

H_m has conformal mapping properties we can describe explicitly.

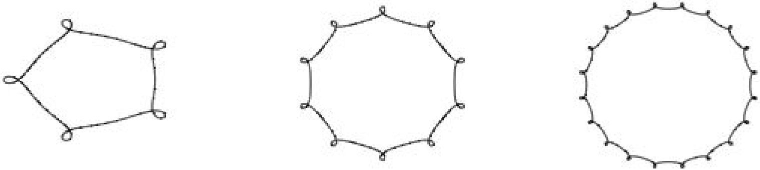

FIGURE 4. Level sets of the form $\{z : |T_2(z^m)| = 1\}$, for $m = 5, 10, 20$.

Figure: Level set of $|T_z(z^m)| = 1$.

f is $m-1$ on the inner disk to $\mathbb{D}$ with a critical point at 0.

Why $F_k(z)$?(cont.)

H_m has conformal mapping properties we can describe explicitly.

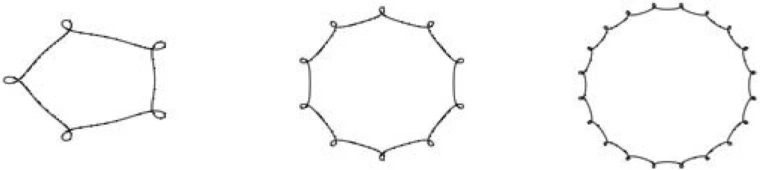

FIGURE 4. Level sets of the form $\{z : |T_2(z^m)| = 1\}$, for $m = 5, 10, 20$.

Figure: Level set of $|T_z(z^m)| = 1$.

f is conformal from the little loops to $\mathbb{D}$ - we call these the petals.

The general itinerary of $f(z)$

Now that we know what each component looks and acts like, we can move on to describing the dynamics more explicitly.

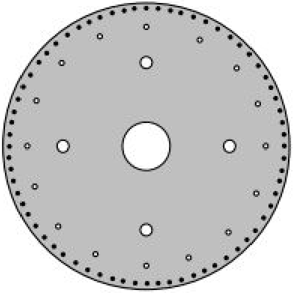

Figure: A Model Fatou Component

Itinerary: Points in B_k

On B_k, $f(z) \approx z^{2n_k}$. So on this component, f behaves rather orderly. In fact, we have

Lemma

For all k, we have $f(B_k) \subset B_{k+1}$.

Preceding inductively, points that start or end up in any of the B_k's travel locally uniformly to ∞.

Corollary

For all k, $B_k \subset \mathcal{F}(f)$. Furthermore, $J(f) \subset A \cup E$.

Itinerary: Points in A_k

We have a similar lemma to the previous one for the A_k's:

Lemma

For all $k \in \mathbb{Z}$, $A_{k+1} \subset f(A_k)$.

Since the zeros of f are in the A_k's, there is the possibility of $f(z) \in A_j$ for $z \in A_k$ and $j < k$. So we consider two cases:

1. The set Y of points z that go backwards infinitely often.
2. The set Z of points z that eventually only move forwards.

Illustration of A_k's possible itineraries

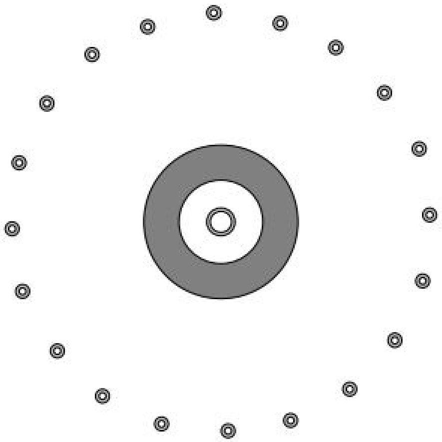

Figure: Possible preimages of A_k

Itinerary: Points in Z

Theorem

Z is the union of C^1 closed Jordan curves.

This part of the Julia set, therefore, has Hausdorff Dimension 1.

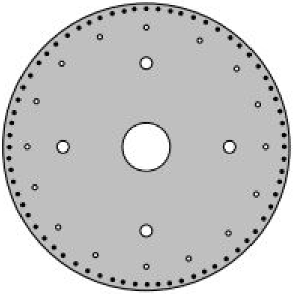

Figure: A Model Fatou Component

Itinerary: Points in Y

Theorem

For any $\alpha > 0$, by choosing the initial parameters R, λ and N sufficiently large, we have $\dim Y \leq \alpha$.

The set Y is a Cantor set of points determined by the nested loops in Z.

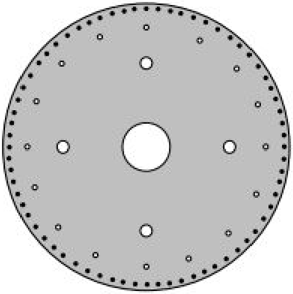

Figure: Notice the Geometry of the Holes

Summary

The Julia set consisted roughly of three different components:

1. The Cantor Repellor E, chosen with small dimension
2. The set Y of points that go backwards infinitely often, can be chosen with small dimension.
3. The set Z of fast escaping points, which has dimension 1.

Therefore, the Julia set of f must have dimension 1.

What did we Omit?

Much more can be said about the dynamics of Bishop's example.

1. By using the parameter $S \subset \mathbb{N}$, we can give f arbitrarily slow growth.
2. The packing dimension (upper Minkowski dimension 2.0) is also 1.
3. $J(f)$ has locally finite Hausdorff measure, and other measure theoretic properties.

References

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Thanks to Chris Bishop for the figures.