Chekanov–Eliashberg dg-algebras for singular Legendrians

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Based on joint work with Tobias Ekholm (arXiv:2102.04858)

Setup and main results

Setup

Let X be a $2n$-dimensional Weinstein manifold with ideal contact boundary ∂X.

$$\begin{array}{ccc} \Lambda \subset \partial X & & CE^*(\Lambda) \\ \text{smooth Legendrian} & \rightsquigarrow & \text{Chekanov–Eliashberg dg-algebra} \end{array}$$

Singular Legendrians

Let (V, λ) be a $(2n-2)$-dimensional Weinstein domain, together with a handle decomposition h.
Assume there is an embedding of V in ∂X such that it extends to a (strict) contact embedding

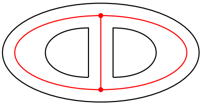
$$F \colon (V \times (-\varepsilon, \varepsilon)_z, dz + \lambda) \longrightarrow (\partial X, \alpha)$$

We call F a *Legendrian embedding of V in ∂X*.

Setup

Singular Legendrians

In particular, the union of the top dimensional strata of $\operatorname{Skel} V$ is Legendrian, and we will refer to $\operatorname{Skel} V$ as a “singular Legendrian” in ∂X.



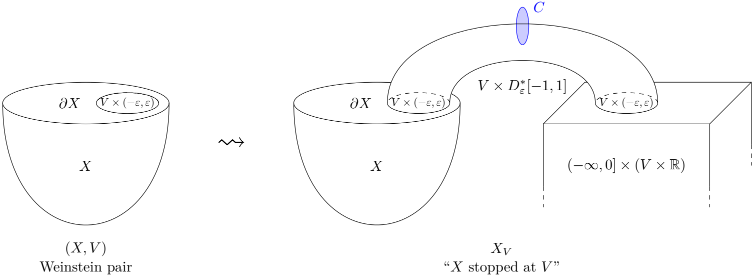
$\rightsquigarrow CE^*((V,h);X)$

$(V,h) \subset \partial X$
Legendrian embedding

Setup

Stopped Weinstein manifolds

We consider *stops* using a surgery description.



$C =$ union of co-core disks of top handles of $V \times D^*_\varepsilon[-1,1]$

Main results

Theorem A (A.–Ekholm)

There is a surgery isomorphism of A_∞-algebras

$$\Phi\colon\ CW^*(C; X_V) \longrightarrow CE^*((V,h); X)$$

Let $\Lambda \subset \partial X$ be a smooth Legendrian and let $(V(\Lambda), h(\Lambda))$ denote a small disk cotangent neighborhood of Λ with a handle decomposition with a single top handle.

Theorem B (A.–Ekholm)

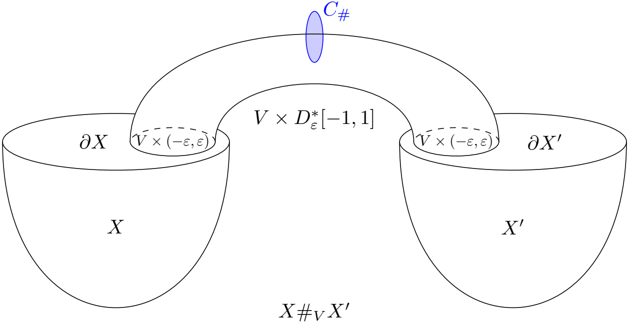
There is a quasi-isomorphism of dg-algebras

$$\Psi\colon\ CE^*((V(\Lambda), h(\Lambda)); X) \longrightarrow CE^*(\Lambda, C_{-*}(\Omega\Lambda); X)$$

Theorem A and B together prove a conjecture by Ekholm–Lekili and independently by Sylvan.

Main results

Now assume V is Legendrian embedded in the ideal contact boundary of X and X'. We can join X and X' together via V.

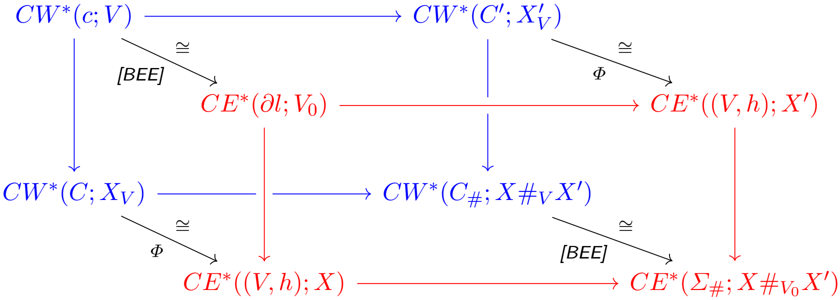


$C_\# =$ union of co-core disks of top handles of $V \times D^*_\varepsilon[-1,1]$.
$\Sigma_\# :=$ union of attaching spheres dual to $C_\#$.

Main results

Theorem C (A.–Ekholm)

Below, the front face is a pushout. After passing to cohomology, the diagram commutes and the back face is a pushout.



The Chekanov–Eliashberg dg-algebra

CE^* for smooth Legendrians

Setup

Let X be a $2n$-dimensional Weinstein manifold with ideal contact boundary ∂X. ($c_1(X) = 0$)
Let $\Lambda \subset \partial X$ be a smooth Legendrian with vanishing Maslov class.

- α contact form on ∂X
- R_α Reeb vector field, defined by $\begin{cases} d\alpha(R_\alpha, -) = 0 \\ \alpha(R_\alpha) = 1 \end{cases}$

Consider $\mathcal{R} = \{\text{Reeb chords of } \Lambda\}$ and let $\Lambda = \bigsqcup_{i=1}^n \Lambda_i$. Then $\mathcal{R}_{ij} \subset \mathcal{R}$ is the set of Reeb chords from Λ_i to Λ_j.
Let $\mathbb{F}$ be a field. Let $\{e_i\}_{i=1}^n$ be such that

- $e_i^2 = e_i$
- $e_i e_j = 0$ if $i \neq j$

CE^* for smooth Legendrians

Graded algebra

Define $\boldsymbol{k} := \bigoplus_{i=1}^{n} \mathbb{F} e_i$. Then $\mathcal{R}$ is a $\boldsymbol{k}$-$\boldsymbol{k}$-bimodule via

$$e_i \cdot c = \begin{cases} c, & \text{if } c \in \mathcal{R}_{ji} \\ 0, & \text{otherwise} \end{cases} \qquad c \cdot e_i = \begin{cases} c, & \text{if } c \in \mathcal{R}_{ij} \\ 0, & \text{otherwise} \end{cases}$$

Then define

$$CE^*(\Lambda) := \boldsymbol{k} \langle \mathcal{R} \rangle \ .$$

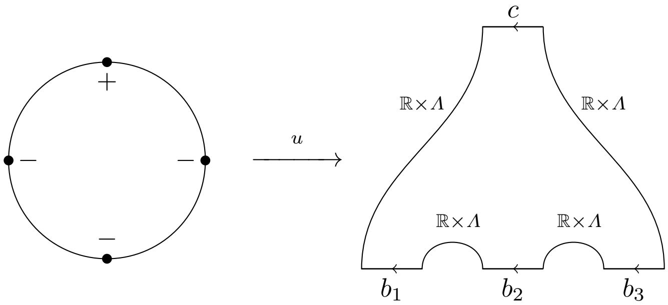
Grading is given by

$$|c| = -\operatorname{CZ}(c) + 1 \, .$$

CE^* for smooth Legendrians

Differential

$\partial\colon CE^*(\Lambda) \longrightarrow CE^*(\Lambda)$ counts (anchored) rigid J-holomorphic disks in $\mathbb{R} \times \partial X$ with boundary on $\mathbb{R} \times \Lambda$ with 1 positive puncture, and several negative punctures.



A curve giving the term $\partial c = b_1 b_2 b_3 + \cdots$.

CE^* for singular Legendrians

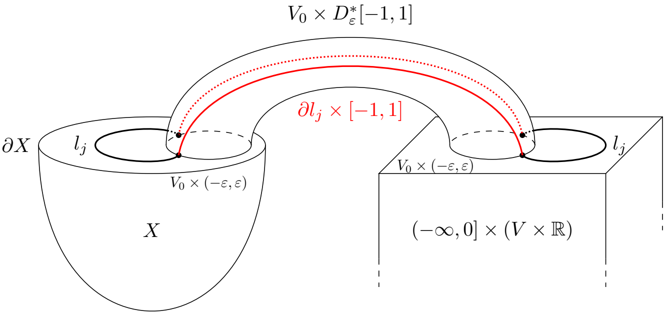
Assume V^{2n-2} is a Weinstein domain which is Legendrian embedded in ∂X with handle decomposition h and $c_1(V) = 0$. Let V_0 denote its subcritical part.
Let

$$l := \bigcup_{j=1}^{m} l_j = \text{union of core disks of top handles}$$

$$\partial l := \bigcup_{j=1}^{m} \partial l_j = \text{union of the attaching spheres of top handles}$$

CE^* for singular Legendrians

Now attach $V_0 \times D^*_\varepsilon[-1,1]$ to $V_0 \times (-\varepsilon, \varepsilon) \subset \partial X$ to construct X_{V_0}.



Define

$$\Sigma(h) := l \sqcup_{\partial l \times \{-1\}} (\partial l \times [-1,1]) \sqcup_{\partial l \times \{1\}} l$$

CE^* for singular Legendrians

Definition

We define the Chekanov–Eliashberg dg-algebra of a Legendrian embedding of (V, h) *in* ∂X *as*

$$CE^*((V, h); X) := CE^*(\Sigma(h); X_{V_0}) .$$

Theorem A

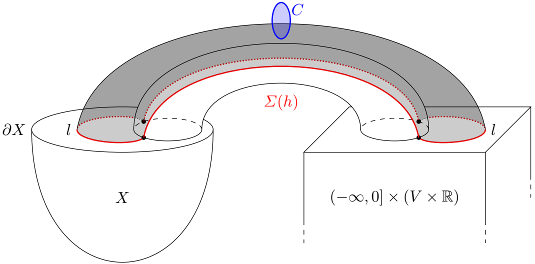
There is a surgery isomorphism of A_∞*-algebras*

$$\Phi\colon\ CW^*(C; X_V) \longrightarrow CE^*((V, h); X)$$

Proof of the surgery formula

Proof of Theorem A.

Follows immediately from the definition together with the Bourgeois–Ekholm–Eliashberg surgery formula.



$$CW^*(C; X_V) \cong CE^*(\Sigma(h); X_{V_0}) = CE^*((V,h); X)$$

□

Description of generators

Lemma

For any $\mathfrak{a} > 0$, there is some $\varepsilon > 0$ small enough (size of the stop) so that we have the following one-to-one correspondence

$$\left\{ \begin{matrix} \text{Reeb chords of } \Sigma(h) \subset \partial X_{V_0} \\ \text{of action } < \mathfrak{a} \end{matrix} \right\}$$

$$\Big\updownarrow \text{1:1}$$

$$\left\{ \begin{matrix} \text{Reeb chords of } l \subset \partial X \\ \text{of action } < \mathfrak{a} \end{matrix} \right\} \cup \left\{ \begin{matrix} \text{Reeb chords of } \partial l \subset \partial V_0 \\ \text{of action } < \mathfrak{a} \end{matrix} \right\}$$

Lemma

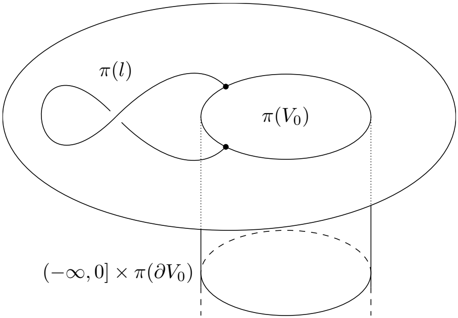
There is a dg-subalgebra of $CE^*((V,h);X)$ which is freely generated by Reeb chords of $\partial l \subset \partial V_0$ and canonically isomorphic to $CE^*(\partial l; V_0)$.

Computations and examples

Special case: $\partial X = P \times \mathbb{R}$

Assume $V \subset P \times \mathbb{R}$ is a Legendrian embedding so that $\pi(V_0) \subset P$ is embedded. Consider

$$P^\circ := (P \setminus \pi(V_0)) \sqcup_{\pi(\partial V_0)} \left((-\infty, 0] \times \pi(\partial V_0)\right)$$



Special case: $\partial X = P \times \mathbb{R}$

Then we can consider $CE^*(l; P^\circ \times \mathbb{R})$, where l is the Legendrian lift of $\pi(l) \subset P^\circ$.

Proposition

There is an isomorphism of dg-algebras

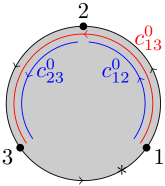
$$CE^*(l; P^\circ \times \mathbb{R}) \cong CE^*((V,h); \mathbb{R} \times (P \times \mathbb{R}))\,.$$

Upshot

Can compute $CE^*(l; P^\circ \times \mathbb{R})$ and hence $CE^*((V,h); \mathbb{R} \times (P \times \mathbb{R}))$ by projecting l and holomorphic curves to P°.
(cf. An–Bae)

Computations

Example (n points in the circle, I_n)



Let $X = \mathbb{R}^2$ and $\Lambda = n$ pts $\subset \partial X = S^1$.

Let $V = T^*\Lambda \subset S^1$. The only generators of $CE^*((V,h);\mathbb{R}^2)$ are Reeb chords in S^1 of the top handles $l = \Lambda$

- c^0_{ij} for $1 \leq i < j \leq n$
- c^p_{ij} for $1 \leq i, j \leq n$

The differential ∂ is given by

$$\partial(c^0_{ij}) = (-1)^* \sum_{k=1}^{n} c^0_{kj} c^0_{ik}$$

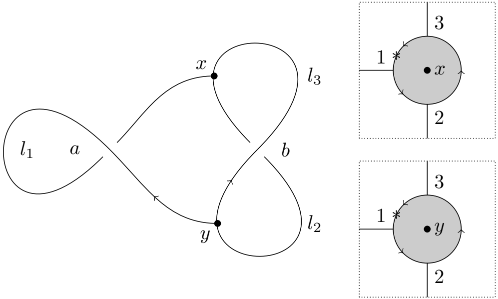
$$\partial(c^1_{ij}) = \delta_{ij} + (-1)^* \sum^{n} c^1_{kj} c^0_{ik} + (-1)^* \sum^{n} c^0_{kj} c^1_{ik}$$

Computations

Example (Link of Lagrangian arboreal A_2-singularity)

Let $X = \mathbb{R}^4$ and $\Lambda \subset S^3$. Then $V = T^*\Lambda$ has 0-handles x and y and 1-handles l_1, l_2 and l_3.

Generators are Reeb chords of l: a and b, and generators of $\partial l \subset \partial V_0$: $\{x^p_{ij}\}$ and $\{y^p_{ij}\}$.

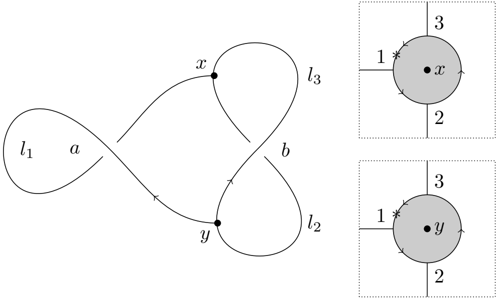


Computations

Example (Link of Lagrangian arboreal A_2-singularity)

The dg-subalgebra $CE^*(\partial l; V_0)$ consists of two copies of I_3. The differential of a and b is as follows

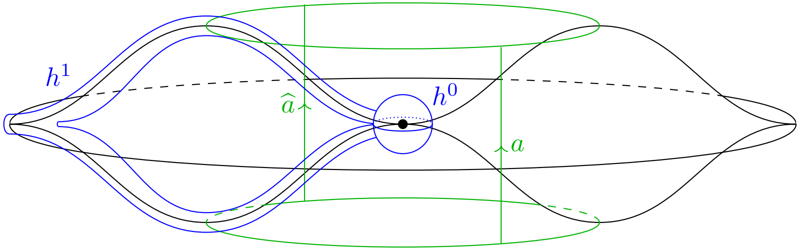
$$\partial a = e_1 + y_{31}^1 b x_{12}^0 + y_{31}^1 x_{12}^0 - y_{21}^1 x_{12}^0, \qquad \partial b = x_{23}^0 - y_{23}^0$$



Computations

Example (Singular torus)

Let $X = \mathbb{R}^6$ and $\Lambda \subset S^5$ is given by the following front.

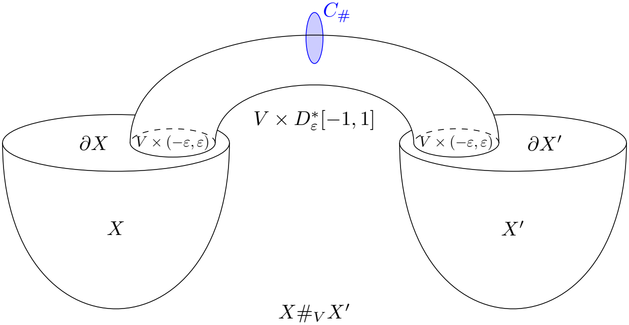


The intersection $l \cap \partial h^0$ is a standard Hopf link in S^3.
The dg-subalgebra $CE^*(\partial l; V_0)$ is generated by the generators of the Hopf link together with with a copy of I_2.
Suitable augmentation of $CE^*(\partial l; V_0)$ gives Chekanov–Eliashberg dg-algebra of nearby smooth tori obtained by smoothing.

Proof of the pushout diagrams

Joining Weinstein manifolds along V

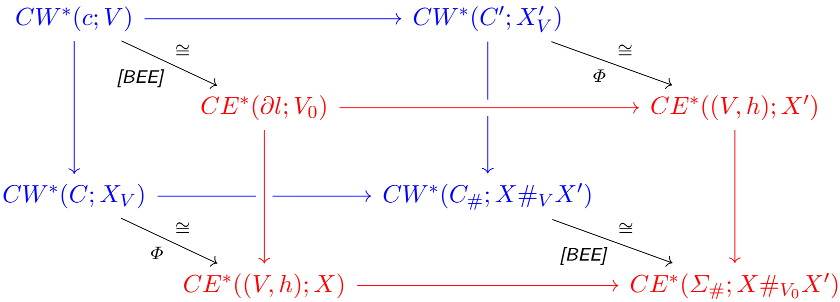
Recall the construction of $X \#_V X'$. Assume V is Legendrian embedded in the ideal contact boundary of X and X'. We can join X and X' together via V.



Joining Weinstein manifolds along V

Theorem C (A.–Ekholm)

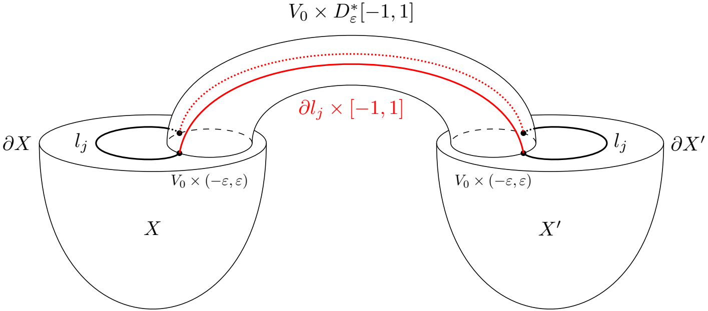
Below, the front face is a pushout. After passing to cohomology, the diagram commutes and the back face is a pushout.



Proof of the pushout diagram for CE^*

Proof of Theorem C.

Consider $X \#_{V_0} X'$, and $\Sigma_{\#}(h) \subset \partial(X \#_{V_0} X')$ the attaching spheres obtained by joining l on either side by $\partial l \times [-1, 1]$ through the handle.



Proof of the pushout diagram for CE^*

Proof of Theorem C.

By the description of the generators we obtain

$$CE^*(\Sigma_\#(h); X\#_{V_0}X') \cong CE^*((V,h);X) *_{CE^*(\partial l; V_0)} CE^*((V,h);X')$$

which means that the diagram

$$\begin{array}{ccc} CE^*(\partial l; V_0) & \xrightarrow{\text{incl.}} & CE^*((V,h);X') \\ \downarrow{\scriptstyle \text{incl.}} & \ulcorner & \downarrow{\scriptstyle \text{incl.}} \\ CE^*((V,h);X) & \xrightarrow{\text{incl.}} & CE^*(\Sigma_\#(h); X\#_{V_0}X') \end{array}$$

is a pushout.

Key observation: $CE^*((V,h);X) \subset CE^*(\Sigma_\#(h); X\#_{V_0}X')$ since curves can not "cross" the handle. □

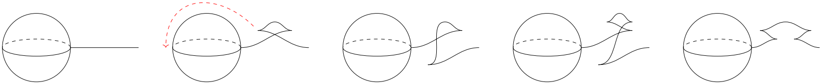
Stop removal

Corollary (Stop removal)

Let $X' := V \times D_1^*[-1,1]$ *equipped with the Liouville vector field* $Z_V + x\partial_x + y\partial_y$. *Then* $CW^*(C_\#; X\#_V X')$ *has trivial cohomology.*

Proof.

The key is to observe that after rounding corners $V \times \{(-1,0)\} \subset \partial\left(V \times D_1^*[-1,1]\right)$ is loose (meaning that each core disk l_j of every top handle of V admits a loose chart).



Stop removal

Proof.

Since we can create loose charts it means that there is at least one generator $b \in CE^*((V,h);X')$ such that $\partial b = 1$.

Use

$$\begin{array}{ccc} CE^*(\partial l; V_0) & \xrightarrow{\text{incl.}} & CE^*((V,h);X') \\ \downarrow{\text{incl.}} & \ulcorner & \downarrow{\text{incl.}} \\ CE^*((V,h);X) & \xrightarrow{\text{incl.}} & CE^*(\Sigma_\#(h); X\#_{V_0}X') \end{array}$$

to conclude that the same is true for $CE^*(\Sigma_\#(h); X\#_{V_0}X')$.

By surgery we therefore have

$$CW^*(C_\#; X\#_V X') \cong CE^*(\Sigma_\#(h); X\#_{V_0}X') \cong 0$$

. □

Thank you!