Chekanov–Eliashberg dg-algebras and partially wrapped Floer cohomology

Johan Asplund

Uppsala University

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Outline

Introduction:

Paper I:

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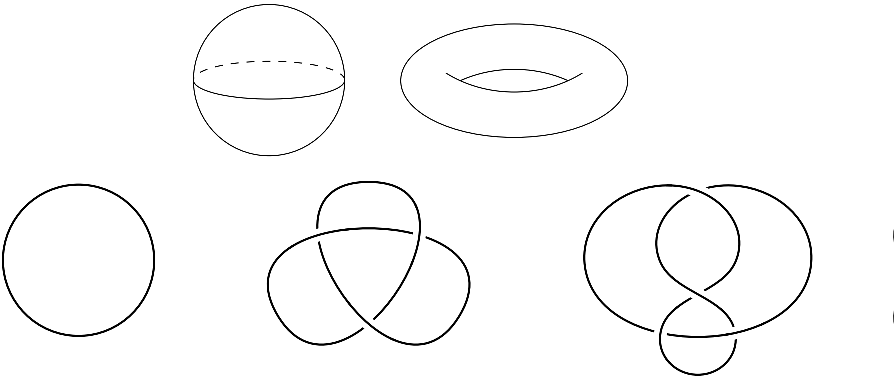
Introduction

Symplectic and contact geometry

Topology is about shapes



In symplectic geometry we can measure areas

Symplectic geometry

Symplectic manifold

A symplectic form on an even dimensional manifold M^{2n} is a two-form $\omega \in \Omega^2(M)$ satisfying

1. $d\omega = 0$
2. $\omega \underbrace{\wedge \cdots \wedge}_{n} \omega \neq 0$

We call (M, ω) a symplectic manifold.

Lagrangian submanifold

A half-dimensional submanifold $L^n \subset M^{2n}$ is called Lagrangian if $\omega|_{TL} = 0$.

Flexibility and rigidity

Flexibility

The h-principle.

Many interesting and important problems lie at the "boundary" of flexibility and rigidity.

Rigidity

Symplectic geometry poses many constraints and restrictions.

Gromov's J-holomorphic curves are often used to "measure" to what extent the geometry is constrained.

Rigidity: Gromov's non-squeezing theorem

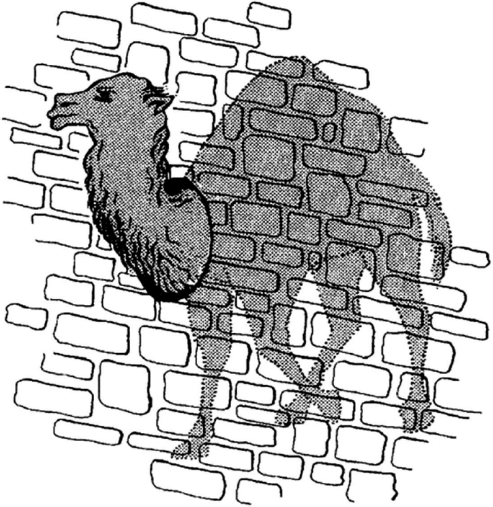


Fig. 1.3. The symplectic camel.

Contact geometry

Contact structure

A contact structure on an odd-dimensional manifold M^{2n+1} is a hyperplane distribution ξ such that locally $\xi = \ker \alpha$ and $\alpha \wedge (d\alpha)^{\wedge n} \neq 0$.

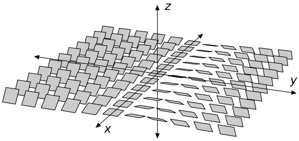


Figure: Standard contact structure on $\mathbb{R}^3$.

Legendrian submanifold

An n-dimensional submanifold $\Lambda^n \subset M^{2n+1}$ is called Legendrian if $T\Lambda \subset \xi$.

Weinstein domains

A Weinstein domain (M, λ, f) consists of

- A smooth manifold-with-boundary M.
- A one-form λ such that $d\lambda$ is a symplectic form on M.
- $\lambda|_{\partial M}$ is a contact form on ∂M.
- The vector field Z defined by $d\lambda(Z, -) = \lambda$ points outwards along ∂M.
- $f \colon M \longrightarrow \mathbb{R}$ Morse function such that Z is pseudo-gradient for f.

Floer theory and the Chekanov–Eliashberg dg-algebra

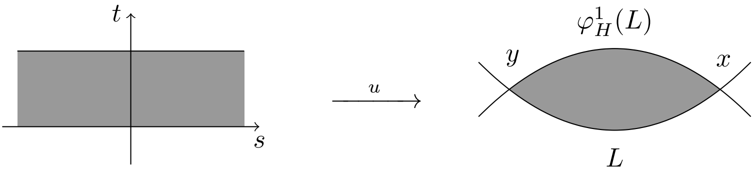
Floer theory

Goal: Study a Lagrangian submanifold L by studying an invariant of L known as wrapped Floer cohomology.

Let $\varphi_H^1(L)$ be a Hamiltonian push-off of L. Then consider a cochain complex of $\mathbb{Z}_2$-vector spaces

$$CW^*(L) := \mathbb{Z}_2 \left\langle L \cap \varphi_H^1(L) \right\rangle .$$

Let $\mathcal{M}(x, y)$ be the space of holomorphic maps



$$du + J \circ du \circ j = 0$$

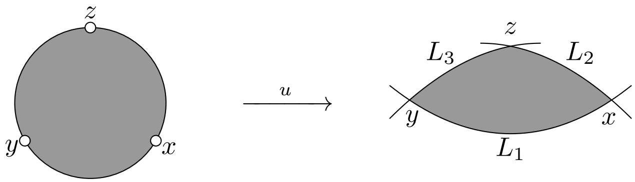
Wrapped Floer cohomology

Define

$$\partial x = \sum_{|y|=|x|+1} (\#_2 \mathcal{M}(x,y))y\,.$$

Then $(CW^*(L), \partial)$ is a cochain complex, and its cohomology is invariant under Hamiltonian isotopies of L.

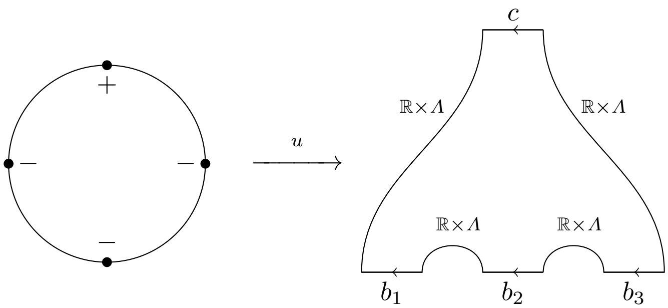
Can define multiplication maps and extend this to an A_∞-algebra



The Chekanov–Eliashberg dg-algebra

In a similar spirit, we can study Legendrian submanifolds $\Lambda \subset \partial M$ in the contact boundary of a Weinstein manifold.

Study J-holomorphic disks in the symplectization $\mathbb{R} \times \partial M$ with boundary on $\mathbb{R} \times \Lambda$. This produces a dg-algebra called the Chekanov–Eliashberg dg-algebra $CE^*(\Lambda)$.



Paper I

Setup and main results

Introduction

Let M be a closed, orientable and spin n-manifold. Consider $(T^*M, \lambda = pdq)$ and $F = T^*_\xi M$ a cotangent fiber.

Let $K \subset M$ be a submanifold and

$$\varLambda_K := \{(x,p) \mid x \in K, |p| = 1, \langle p, T_x K\rangle = 0\} \subset ST^*M\,.$$

- Construct a new Weinstein manifold $W_K = T^*M$ stopped at $\varLambda_K$. Denote the wrapped Floer cohomology of F in W_K by $HW^*_{\varLambda_K}(F)$.

Main results

Theorem A

Let $M_K := M \setminus K$*. There is an isomorphism of* A_∞*-algebras* $\Psi\colon CW^*_{\Lambda_K}(F) \longrightarrow C^{\mathrm{cell}}_{-*}(B_\xi M_K)$.

Remark

In particular we have an A_∞-quasi-isomorphism $CW^*_{\Lambda_K}(F) \cong C_{-*}(\Omega_\xi M_K)$, and the above result holds true for $K = \varnothing$. (cf. Abbondandolo–Schwarz 2008, Abouzaid 2012).

Theorem B

Ψ *induces an isomorphism of* $\mathbb{Z}[\pi_1(M_K)]$*-modules* $HW^*_{\Lambda_K}(F) \longrightarrow H_{-*}(\Omega_\xi M_K)$.

Application to knot theory

For certain codimension 2 knots $K \subset S^n$, we show that $HW^*_{\Lambda_K}(F)$ is related to the Alexander invariant of K.

Theorem C

Let $n = 5$ or $n \geq 7$. Then there exists a codimension 2 knot $K \subset S^n$ with $\pi_1(M_K) \cong \mathbb{Z}$ such that $\Lambda_K \cup \partial F \not\simeq \Lambda_{\textit{unknot}} \cup \partial F$.

Remark

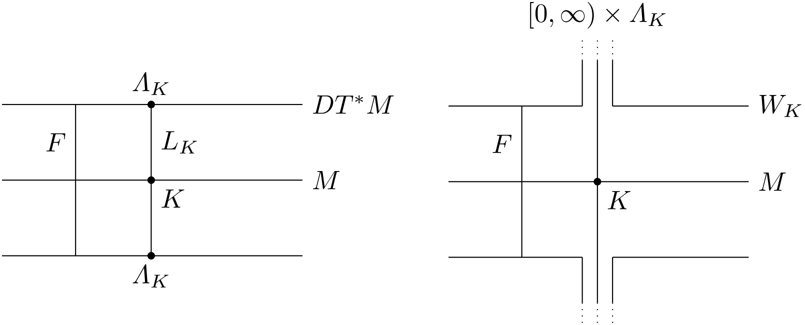
Chekanov–Eliashberg dg-algebra of $\Lambda_K \cup \partial F \subset ST^*\mathbb{R}^3$ for knots $K \subset \mathbb{R}^3$ is a complete knot invariant (Ekholm–Ng–Shende 2016).

Legendrian isotopy class of Λ_K is a complete knot invariant (Shende 2016).

Conormal stops and the A_∞-homomorphism

Surgery approach

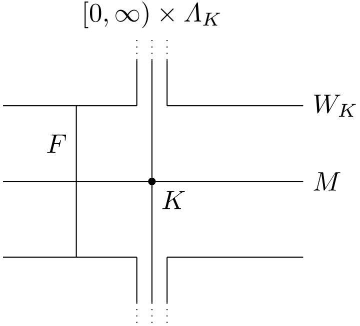
Let $K \subset M$ be any submanifold and consider its unit conormal bundle $\Lambda_K \subset ST^*M$.



Attach a handle along Λ_K modeled on $D_\varepsilon T^*([0,\infty) \times \Lambda_K)$.

The complement Lagrangian

$M \cap L_K = K$ is a clean intersection. Lagrangian surgery along K gives an exact Lagrangian $M_K \cong M \setminus K$

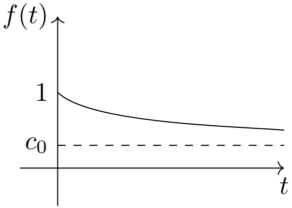


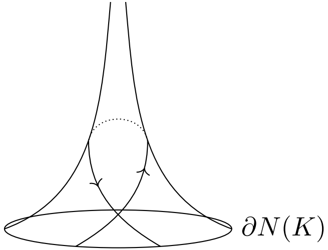
The complement Lagrangian

Metric on M_K

Pick a generic metric g on M away from $N(K)$. Then define

$$h = \begin{cases} dt^2 + f(t)\, g|_{\partial N(K)} & \text{in } [0,\infty) \times \partial N(K) \\ g & \text{in } M \setminus N(K) \end{cases}$$

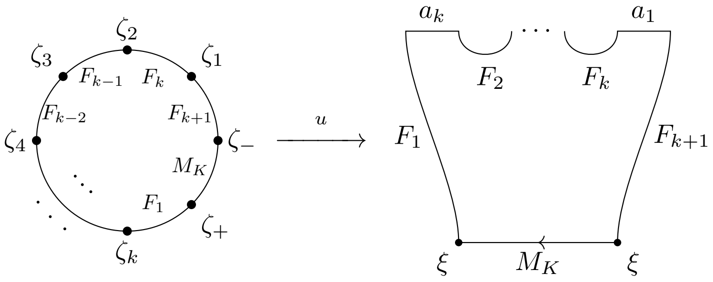




Moduli space of half strips

Let $k \geq 1$ and $\boldsymbol{a} = \{a_1, \ldots, a_k\}$ a set of generators of $CW^*_{\Lambda_K}(F)$.

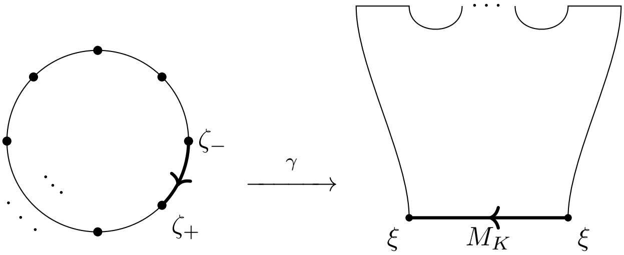
Consider $\mathcal{M}(\boldsymbol{a})$ moduli space of holomorphic maps



The evaluation map

$$\begin{aligned} \mathrm{ev}\colon \overline{\mathcal{M}}(\boldsymbol{a}) &\longrightarrow \Omega_\xi M_K \\ u &\longmapsto \gamma\,. \end{aligned}$$

Restriction of u to the boundary arc between the punctures $\zeta_\pm$



The A_∞-homomorphism

We define

$$\begin{aligned} \Psi_k \colon CW^*_{\Lambda_K}(F)^{\otimes k} &\longrightarrow C^{\text{cell}}_{-*}(\Omega_\xi M_K) \\ a_k \otimes \cdots \otimes a_1 &\longmapsto \mathrm{ev}_*([\overline{\mathcal{M}}(\boldsymbol{a})]) \end{aligned}$$

Proposition

$\Psi := \{\Psi_k\}_{k=1}^{\infty}$ *is an* A_∞*-homomorphism.*

Proof of main theorem

Correspondence between generators

Lemma

There is a one-to-one correspondence

$$\left\{ \begin{array}{c} \textit{Reeb chords of } \partial F \\ \textit{of index } -\lambda \textit{ and action } A \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Geodesic loops in } M_K \\ \textit{of index } \lambda \textit{ and length } A \end{array} \right\}$$

Proof.

Let u_0 be the trivial holomorphic strip over the Reeb chord a and consider $\beta := \text{cutoff}(|p|) \cdot \frac{pdq}{|p|}$. Then

$$0 = \int_D u_0^* d\beta = \mathfrak{a}(a) - L(\gamma)\,.$$

□

Towards the proof of the main theorem

Lemma

u_0 is transversely cut out

Proof of lemma.

Let $v \in \ker D_{u_0}$ and $\varepsilon > 0$

$$\begin{cases} u_\varepsilon := u_0 + \varepsilon v + O(\varepsilon^2) \\ \gamma_\varepsilon := \operatorname{ev}(u_\varepsilon) \end{cases} .$$

Towards the proof of the main theorem

Proof of lemma.

Then

1. From u_0: $\mathfrak{a}(a) = L(\gamma_0)$
2. Can show: $\mathfrak{a}(a) - L(\gamma_\varepsilon) > 0$

This implies

$$L(\gamma_\varepsilon) - L(\gamma_0) < 0 \implies E_{**}(\pi_* v, \pi_* v) < 0\,.$$

where

$$\pi_* \colon \ \ker D_{u_0} \longrightarrow \{w \in T_{\gamma_0} BM_K \mid E_{**}(w,w) < 0\}\ .$$

By unique continuation, π_* is injective and finally $\dim\ker D_{u_0} \leq \operatorname{ind}\gamma_0 = \operatorname{ind} D_{u_0}$. □

Paper II

Setup and main results

Setup

Let X be a $2n$-dimensional Weinstein manifold.

Singular Legendrians

Let (V, λ) be a $(2n-2)$-dimensional Weinstein domain.

Assume there is an embedding of V in ∂X such that it extends to a (strict) contact embedding

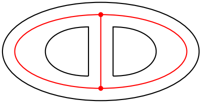
$$F\colon (V \times (-\varepsilon, \varepsilon)_z, dz + \lambda) \longrightarrow (\partial X, \alpha)$$

We call F a *Legendrian embedding of V in ∂X*.

Setup

Singular Legendrians

The union of the top dimensional strata of $F(\operatorname{Skel} V) \subset \partial X$ is Legendrian.



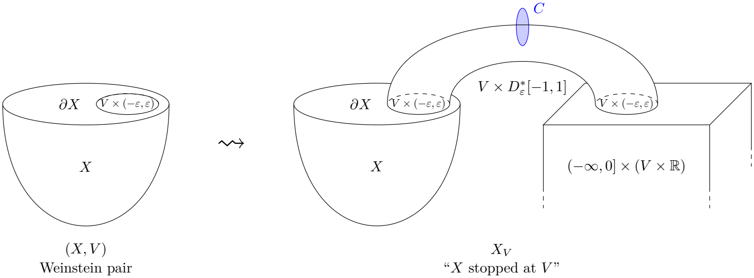
$(V, h) \subset \partial X$
Legendrian embedding

$\rightsquigarrow CE^*((V, h); X)$

Setup

Stopped Weinstein manifolds

We attach a *stop* to X along $V \times (-\varepsilon, \varepsilon)$.



$C =$ union of co-core disks of top handles of $V \times D^*_\varepsilon[-1, 1]$

Main results

Theorem A (A.–Ekholm)

There is a surgery isomorphism of A_∞-algebras

$$\Phi\colon\ CW^*(C;X_V) \longrightarrow CE^*((V,h);X)$$

Let $\Lambda \subset \partial X$ be a smooth Legendrian and let $(V(\Lambda),h(\Lambda))$ denote a small disk cotangent neighborhood of Λ with a handle decomposition with a single top handle.

Theorem B (A.–Ekholm)

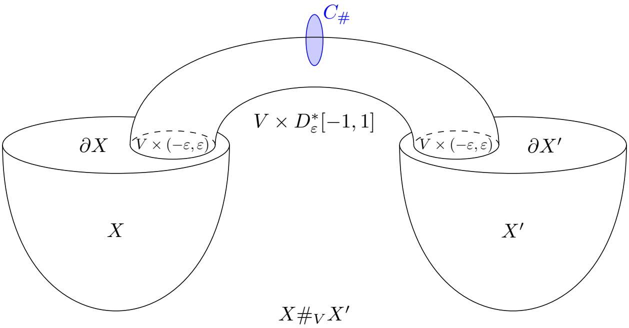
There is a quasi-isomorphism of dg-algebras

$$\Psi\colon\ CE^*((V(\Lambda),h(\Lambda));X) \longrightarrow CE^*(\Lambda, C_{-*}(\Omega\Lambda);X)$$

Theorem A and B together prove a conjecture by Ekholm–Lekili and independently by Sylvan.

Main results

Now assume V is Legendrian embedded in ∂X and $\partial X'$. We can join X and X' together along V.



$C_\# =$ union of co-core disks of top handles of $V \times D^*_\varepsilon[-1,1]$.
$\Sigma_\# :=$ union of attaching spheres dual to $C_\#$.

Main results

Theorem C (A.–Ekholm)

The diagram below is a pushout diagram.

$$\begin{array}{ccc} CE^*(\partial l; V_0) & \xrightarrow{\textit{incl.}} & CE^*((V,h);X') \\ \downarrow{\scriptstyle \textit{incl.}} & \ulcorner & \downarrow{\scriptstyle \textit{incl.}} \\ CE^*((V,h);X) & \xrightarrow{\textit{incl.}} & CE^*(\Sigma_{\#}(h); X \#_{V_0} X') \end{array}$$

The Chekanov–Eliashberg dg-algebra

CE^* for singular Legendrians

Assume V^{2n-2} is a Weinstein domain which is Legendrian embedded in ∂X with handle decomposition h. Let V_0 denote its subcritical part.

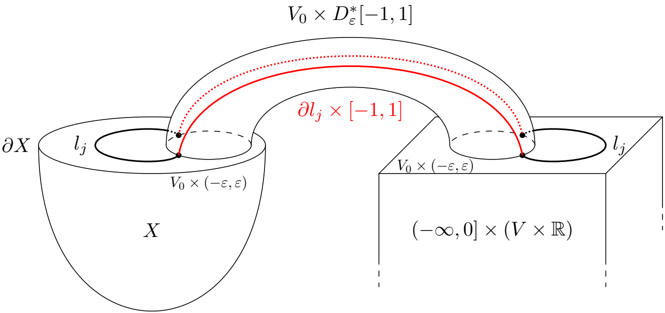
Let

$$l := \bigcup_{j=1}^{m} l_j = \text{union of core disks of top handles in } h$$

$$\partial l := \bigcup_{j=1}^{m} \partial l_j = \text{union of the attaching spheres of top handles in } h$$

CE^* for singular Legendrians

Now attach $V_0 \times D^*_\varepsilon[-1,1]$ to $V_0 \times (-\varepsilon, \varepsilon) \subset \partial X$ to construct X_{V_0}.



Define

$$\Sigma(h) := l \sqcup_{\partial l \times \{-1\}} (\partial l \times [-1,1]) \sqcup_{\partial l \times \{1\}} l$$

CE^* for singular Legendrians

Definition

We define the Chekanov–Eliashberg dg-algebra of a Legendrian embedding of (V, h) in ∂X as

$$CE^*((V, h); X) := CE^*(\Sigma(h); X_{V_0}) \, .$$

Theorem A

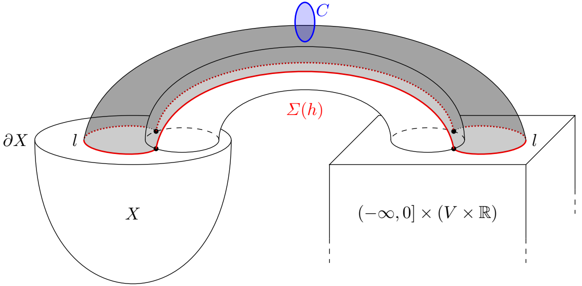
There is a surgery isomorphism of A_∞-algebras

$$\Phi \colon CW^*(C; X_V) \longrightarrow CE^*((V, h); X)$$

Proof of the surgery formula

Proof of Theorem A.

Follows immediately from the definition together with the Bourgeois–Ekholm–Eliashberg surgery formula.



$$CW^*(C; X_V) \cong CE^*(\Sigma(h); X_{V_0}) = CE^*((V, h); X)$$

□

Description of generators

Lemma

For any $\mathfrak{a} > 0$, there is some $\varepsilon > 0$ small enough (size of the stop) so that we have the following one-to-one correspondence

$$\left\{ \begin{matrix} \text{Reeb chords of } \Sigma(h) \subset \partial X_{V_0} \\ \text{of action } < \mathfrak{a} \end{matrix} \right\}$$

$$\updownarrow \text{1:1}$$

$$\left\{ \begin{matrix} \text{Reeb chords of } l \subset \partial X \\ \text{of action } < \mathfrak{a} \end{matrix} \right\} \cup \left\{ \begin{matrix} \text{Reeb chords of } \partial l \subset \partial V_0 \\ \text{of action } < \mathfrak{a} \end{matrix} \right\}$$

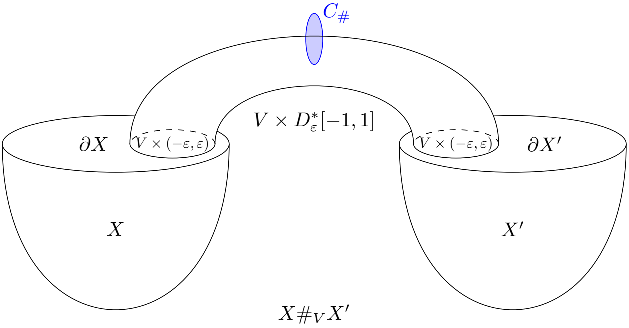
Lemma

There is a dg-subalgebra of $CE^*((V,h);X)$ which is freely generated by Reeb chords of $\partial l \subset \partial V_0$ and canonically isomorphic to $CE^*(\partial l; V_0)$.

Proof of the pushout diagrams

Joining Weinstein manifolds along V

Recall the construction of $X \#_V X'$. Assume V is Legendrian embedded in the ideal contact boundary of X and X'. We can join X and X' together via V.



Joining Weinstein manifolds along V

Theorem C (A.–Ekholm)

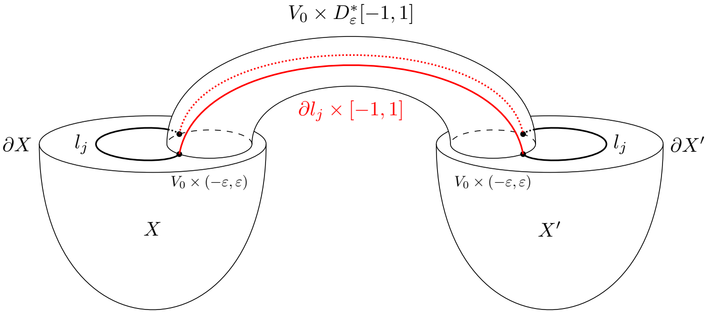
The diagram below is a pushout diagram.

$$\begin{array}{ccc} CE^*(\partial l; V_0) & \xrightarrow{\text{incl.}} & CE^*((V,h); X') \\ \downarrow{\scriptstyle \text{incl.}} & \ulcorner & \downarrow{\scriptstyle \text{incl.}} \\ CE^*((V,h); X) & \xrightarrow{\text{incl.}} & CE^*(\Sigma_{\#}(h); X \#_{V_0} X') \end{array}$$

Proof of the pushout diagram for CE^*

Proof of Theorem C.

Consider $X \#_{V_0} X'$, and $\Sigma_\#(h) \subset \partial(X \#_{V_0} X')$ the attaching spheres obtained by joining l on either side by $\partial l \times [-1,1]$ through the handle.



Proof of the pushout diagram for CE^*

Proof of Theorem C.

By the description of the generators and holomorphic curves we obtain

$$CE^*(\Sigma_\#(h); X\#_{V_0}X') \cong CE^*((V,h);X) *_{CE^*(\partial l;V_0)} CE^*((V,h);X')$$

which means that the diagram

$$\begin{array}{ccc} CE^*(\partial l; V_0) & \xrightarrow{\text{incl.}} & CE^*((V,h);X') \\ \downarrow{\scriptstyle \text{incl.}} & \ulcorner & \downarrow{\scriptstyle \text{incl.}} \\ CE^*((V,h);X) & \xrightarrow{\text{incl.}} & CE^*(\Sigma_\#(h); X\#_{V_0}X') \end{array}$$

is a pushout diagram.
Key observation: $CE^*((V,h);X) \subset CE^*(\Sigma_\#(h); X\#_{V_0}X')$ is a dg-subalgebra since curves can not “cross” the handle. □

Thank you!