Fiber Floer cohomology and conormal stops

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Introduction

Introduction

Let S be a closed, orientable and spin n-manifold. Consider $(T^*S, \lambda = pdq)$ and $F = T^*_\xi S$ a cotangent fiber

Abbondandolo–Schwarz (2008) and Abouzaid (2012)

There is an isomorphism $HW^*(F) \cong H_{-*}(\Omega S)$ which intertwines the triangle product on $HW^*(F)$ with the Pontryagin product on $H_{-*}(\Omega S)$.

Main results

Let $K \subset S$ be a submanifold and consider its unit conormal bundle

$$\Lambda_K := \{(x,p) \mid x \in K, |p| = 1, \langle p, T_x K \rangle = 0\} \subset ST^*S\,.$$

- By adding at stop at Λ_K, we consider the partially wrapped Floer cohomology $HW^*_{\Lambda_K}(F)$.
- The based loop space ΩS is homotopy equivalent to the space BS of smooth piecewise geodesic loops.

Theorem A

Let $M_K := S \setminus K$. There is an isomorphism of A_∞-algebras $\Psi \colon CW^*_{\Lambda_K}(F) \longrightarrow C^{\text{cell}}_{-*}(B_\xi M_K)$.

In particular we have $HW^*_{\Lambda_K}(F) \cong H_{-*}(\Omega_\xi M_K)$.

Main results

Theorem B

Ψ induces an isomorphism of $\mathbb{Z}[\pi_1(M_K)]$-modules
$HW^*_{\Lambda_K}(F) \longrightarrow H_{-*}(\Omega_\xi M_K)$.

The smooth topology of K can be studied via Λ_K:

- *Legendrian contact homology* (LCH) of Λ_K is related to the Alexander polynomial of $K \subset \mathbb{R}^3$ and furthermore detects the unknot and torus knots (Ng 2008, Ekholm–Ng–Shende 2016, Cieliebak–Ekholm–Latschev–Ng 2016)
- Refined version of LCH of Λ_K is a complete knot invariant of $K \subset \mathbb{R}^3$ (ENS 2016)
- Legendrian isotopy class of Λ_K is a complete knot invariant of $K \subset \mathbb{R}^3$ (Shende 2016, ENS 2016)

Main results

Application

For certain codimension 2 knots $K \subset S^n$, $HW^*_{\Lambda_K}(F)$ is related to the Alexander invariant of K.

Theorem C

Let $n = 5$ or $n \geq 7$. Then there exists a codimension 2 knot $K \subset S^n$ with $\pi_1(M_K) \cong \mathbb{Z}$ such that $\Lambda_K \cup F \not\simeq \Lambda_{\mathit{unknot}} \cup F$.

In fact, our construction gives us for each n infinitely many codimension 2 knots K such that the links $\Lambda_K \cup F$ are pairwise not Legendrian isotopic.

Wrapped Floer cohomology

Wrapped Floer cohomology without Hamiltonian

Let $CW^*(F)$ denote the wrapped Floer cochain complex of the fiber $F \subset T^*S$.

Generators

- Reeb chords of $\partial F \subset ST^*S$
- One generator corresponding to a minimum of Morse function on F

Grading

(A shift of the) Conley–Zehnder index

Wrapped Floer cohomology without Hamiltonian

Parallel copies of F

Choose family of positive Morse functions

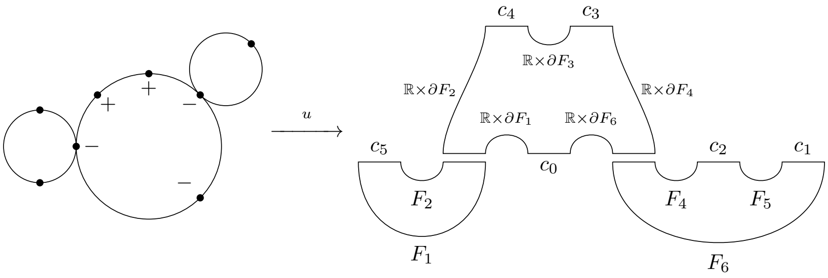
$$G_k \colon F \longrightarrow \mathbb{R}$$
$$g_k \colon \partial F \longrightarrow \mathbb{R}$$

- G_k has one minimum on F and no other critical points
- F_k = flow of F by the Hamiltonian vector field associated to G_k
- ∂F_k is a small pushoff of ∂F in the positive Reeb direction

Wrapped Floer cohomology without Hamiltonian

A_∞-structure

$\mu^k\colon CW^*(F)^{\otimes k} \longrightarrow CW^*(F)$ counts solutions to the equation $\overline{\partial}_J u = \frac{1}{2}\left(\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t}\right) = 0$ with k inputs, 1 output and switching boundary conditions

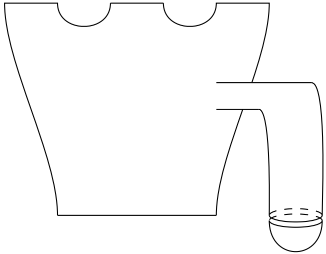


No cuvers can be multiply covered! Boundary bubbling is precluded for topological reasons.

Wrapped Floer cohomology without Hamiltonian

Anchoring

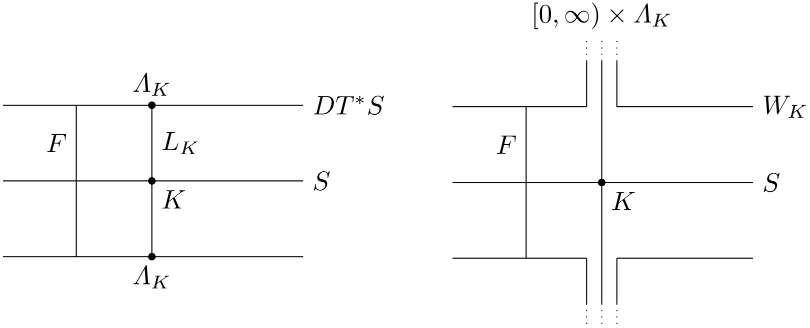
In reality, disks also have internal punctures asymptotic to Reeb orbits. Anchoring means that we cap off the Reeb orbits with holomorphic planes in the compact part of the Weinstein manifold.



Transversality of such moduli spaces requires abstract perturbations. Such a perturbation scheme based off of polyfolds was constructed in (Ekholm, 2019).

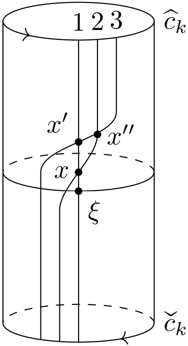
Partially wrapped Floer cohomology via surgery

Let $K \subset S$ be any submanifold and consider its unit conormal bundle $\Lambda_K \subset ST^*S$.



Attach a handle along Λ_K modeled on $D_\varepsilon T^*([0,\infty) \times \Lambda_K)$.
$CW^*_{\Lambda_K}(F)$ is defined as $CW^*(F)$ computed in W_K

Consider $(T^*S^1, \lambda = pdq)$. We compute $CW^*(F)$, and some curves contributing to μ^2.



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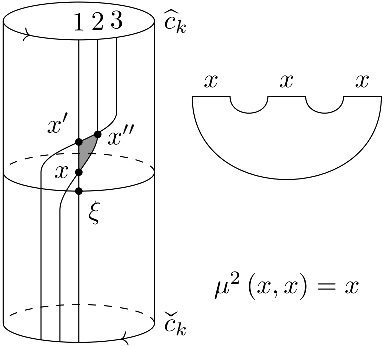
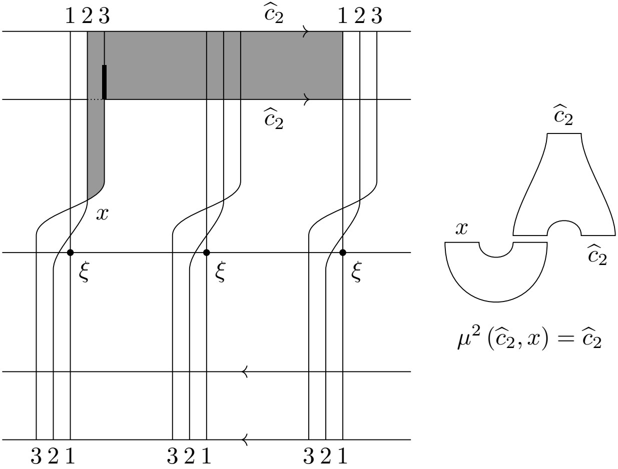


$$\mu^2(x, x) = x$$

Consider $(T^*S^1, \lambda = pdq)$. We compute $CW^*(F)$, and some curves contributing to μ^2.



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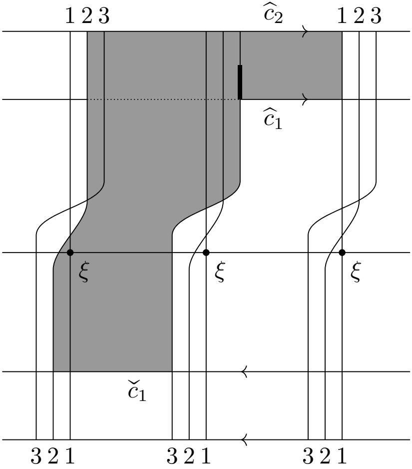
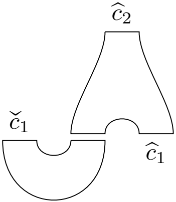
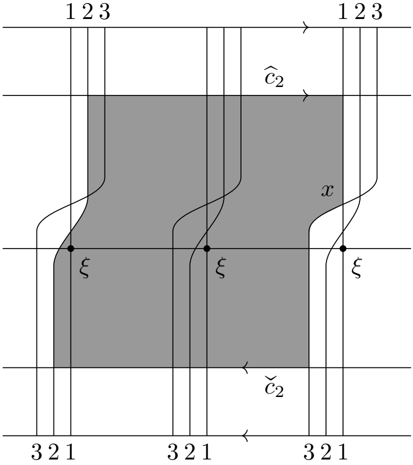
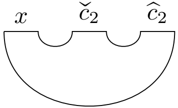




$\mu^2(\widehat{c}_2, \check{c}_1) = \widehat{c}_1$

Consider $(T^*S^1, \lambda = pdq)$. We compute $CW^*(F)$, and some curves contributing to μ^2.

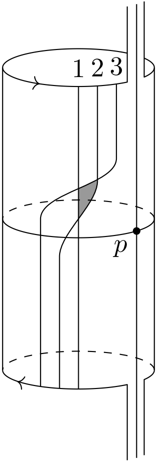




$$\mu^2(\widehat{c}_2, \check{c}_2) = x$$

Example: T^*S^1 stopped at Λ_p

Consider $(T^*S^1, \lambda = pdq)$ stopped at Λ_p, where $p \in S^1$ is a point.

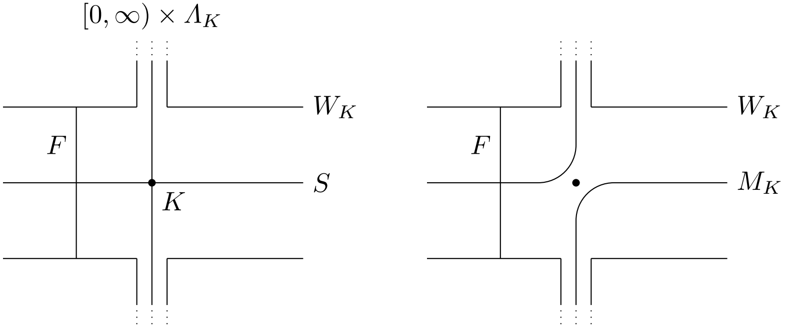


- $CW^*_{\Lambda_p}(F) = \langle x \rangle$
- Only non-trivial product: $\mu^2(x, x) = x$.

The A_∞-homomorphism

The complement Lagrangian

$S \cap L_K = K$ is a clean intersection. Lagrangian surgery along K gives an exact Lagrangian $M_K \cong S \setminus K$

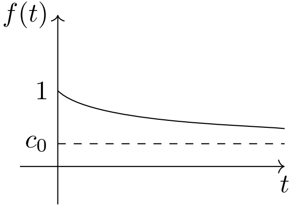


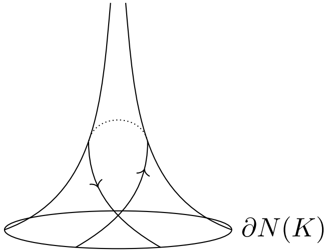
The complement Lagrangian

Metric on M_K

Pick a generic metric g in S away from $N(K)$. Then define

$$h = \begin{cases} dt^2 + f(t)\, g|_{\partial N(K)} & \text{in } [0,\infty) \times \partial N(K) \\ g & \text{in } S \setminus N(K) \end{cases}$$



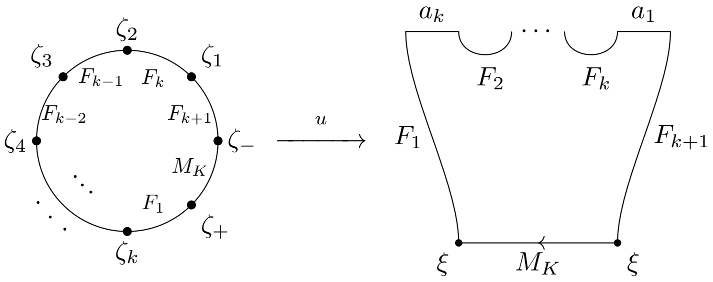


Moduli space of half strips

Let

- $k \geq 1$
- $\boldsymbol{a} = a_1, \ldots, a_k$ generators of $CW^*_{\Lambda_K}(F)$

Consider $\mathcal{M}(\boldsymbol{a})$ moduli space of holomorphic maps



Moduli space of half strips

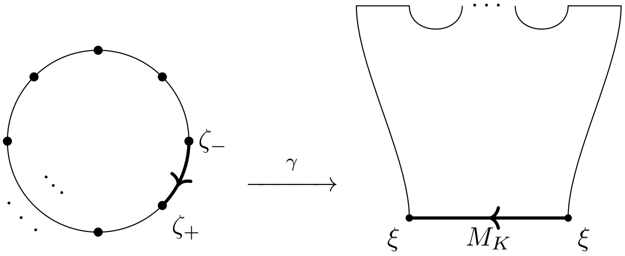
The moduli space $\mathcal{M}(\boldsymbol{a})$ has the following properties:

- Transversely cut out.
- Compact after adding broken disks. Denote the compactification by $\overline{\mathcal{M}}(\boldsymbol{a})$.
- $\dim \overline{\mathcal{M}}(\boldsymbol{a}) = -1 + k - \sum_{i=1}^{k} |a_i|$
- There exists a family of fundamental chains $[\overline{\mathcal{M}}(\boldsymbol{a})]$, which is compatible with orientations and the boundary stratification.

The evaluation map

$$\begin{aligned} \mathrm{ev}\colon \overline{\mathcal{M}}(\boldsymbol{a}) &\longrightarrow \Omega_\xi M_K \\ u &\longmapsto \gamma\,. \end{aligned}$$

Restriction of u to the boundary arc between the punctures $\zeta_\pm$



The A_∞-homomorphism

We define

$$\begin{aligned} \Psi_k \colon CW^*_{\Lambda_K}(F)^{\otimes k} &\longrightarrow C^{\mathsf{cell}}_{-*}(\Omega_\xi M_K) \\ a_k \otimes \cdots \otimes a_1 &\longmapsto \mathrm{ev}_*([\overline{\mathcal{M}}(\boldsymbol{a})]) \end{aligned}$$

Proposition

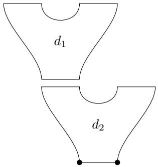
$\Psi := \{\Psi_k\}_{k=1}^\infty$ is an A_∞-homomorphism.

The A_∞-homomorphism

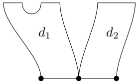
Proof.

We look at the boundary of $\overline{\mathcal{M}}(\boldsymbol{a})$ of dimension d. The codimension 1 boundary is covered by strata of the form:

(a)



(b)



Strata of the form (a) contributes to $\partial \Psi_k$ with terms of the form

$$\sum_{r+s+t=k} \Psi_{r+1+t}(\mathrm{id}^{\otimes r} \otimes \mu^s \otimes \mathrm{id}^{\otimes t}) .$$

Note: Since we use cubical chains, there is only contribution when $d_1 = 0$.

The A_∞-homomorphism

Proof.

Strata of the form (b) contributes to $\partial\Psi_k$ with terms of the form

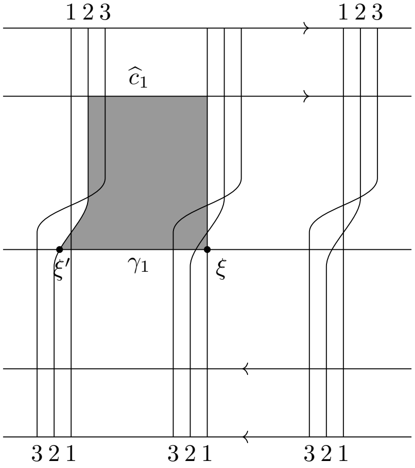
$$\sum_{k_1+k_1=k} P(\Psi_{k_2} \otimes \Psi_{k_1}) \, .$$

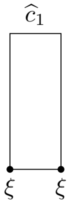
Collecting all the terms therefore gives us

$$\partial\Psi_k = \sum_{r+s+t=k} \Psi_{r+1+t}(\mathrm{id}^{\otimes r} \otimes \mu^s \otimes \mathrm{id}^{\otimes t}) + \sum_{k_1+k_1=k} P(\Psi_{k_2} \otimes \Psi_{k_1})$$

□

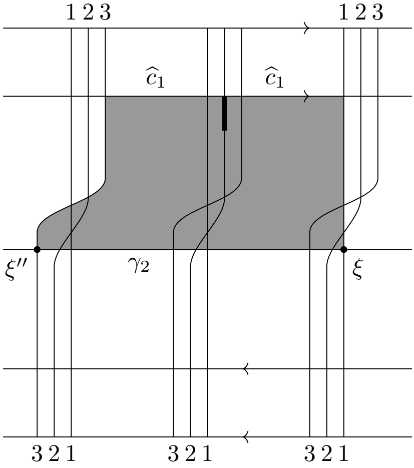
There is one smooth geodesic loop in S^1 per homotopy class. $C_{-*}^{\text{cell}}(BS^1) = \langle \gamma_k \rangle_{k=-\infty}^{\infty}$. We consider some curves contributing to Ψ_1 and Ψ_2.

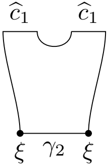




$\Psi_1(\widehat{c}_1) = \gamma_1$

There is one smooth geodesic loop in S^1 per homotopy class. $C_{-*}^{\mathsf{cell}}(BS^1) = \langle \gamma_k \rangle_{k=-\infty}^{\infty}$. We consider some curves contributing to $\varPsi_1$ and $\varPsi_2$.





$\varPsi_2(\widehat{c}_1, \widehat{c}_1) = \gamma_2$

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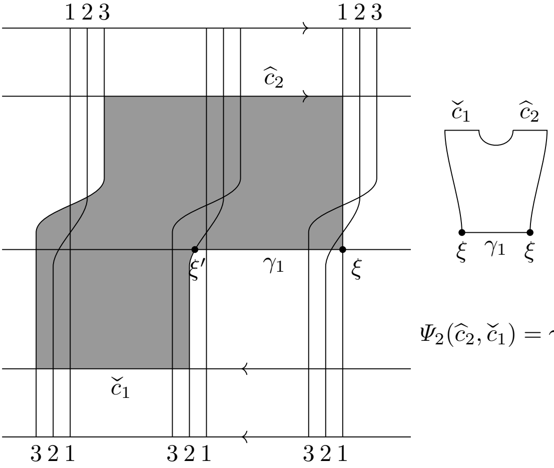
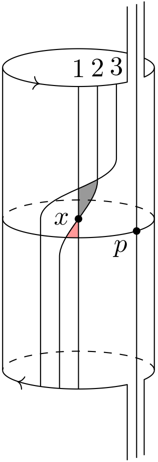


$\Psi_2(\widehat{c}_2, \check{c}_1) = \gamma_1$

Example: T^*S^1 stopped at Λ_p

Consider $(T^*S^1, \lambda = pdq)$ stopped at Λ_p, where $p \in S^1$ is a point.
Recall: $CW^*_{\Lambda_p}(F) = \langle x \rangle$ with only non-trivial product $\mu^2(x,x) = x$.



- $S^1 \setminus \{p\} \cong \mathbb{R}$ and $C^{\text{cell}}_{-*}(B\mathbb{R}) \cong \langle \text{const loop} \rangle$.
- The A_∞-isomorphism is given by $\Psi_1(x) = \text{const loop}$.

Proof of main theorem

Correspondence between generators

Each geodesic in M_K of index λ corresponds to a generator of $C_{-*}^{\text{cell}}(BM_K)$ in degree λ.

Length filtrations

- For $c \in CW^*_{\Lambda_K}(F)$ a Reeb chord, define $\mathfrak{a}(c) := \int_c \lambda$
- For $\sigma \in C_{-*}^{\text{cell}}(BM_K)$, define $\mathfrak{a}(\sigma) := \max_{x\in[0,1]^*} L(\sigma(x))$, where

$$L(\gamma) = \int_0^1 |\gamma(t)|\, dt\,.$$

Lemma

There is a one-to-one correspondence

$$\left\{ \begin{matrix} \textit{Reeb chords of } \partial F \\ \textit{of index } -\lambda \textit{ and action } A \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \textit{Geodesic loops in } M_K \\ \textit{of index } \lambda \textit{ and length } A \end{matrix} \right\}$$

Correspondence between generators

Proof.

We consider the "trivial holomorphic strip" over a Reeb chord a:

$$T_{(q,p)} \operatorname{im} u_0 = \operatorname{span}\{\text{Reeb}, \text{Liouville}\}$$

Consider the 2-form $d\beta$ where

$$\beta := \text{cutoff}(|p|) \cdot \frac{pdq}{|p|} .$$

Integrating $d\beta$ over any holomorphic half strip u asymptotic to a yields $\mathfrak{a}(a) - L(\gamma) \geq 0$. In the case of u_0 we have equality. □

Towards the proof of the main theorem

Lemma

u_0 is transversely cut out

Corollary

For any $a \in CW^*_{\Lambda_K}(F)$ we have $\mathfrak{a}(a) = \mathfrak{a}(\Psi_1(a))$.

Proof of lemma.

Let $v \in \ker D_{u_0}$ and $\varepsilon > 0$

$$\begin{cases} u_\varepsilon := \exp_{u_0}(\varepsilon v) \\ \gamma_\varepsilon := \mathrm{ev}(u_\varepsilon) \end{cases} .$$

Towards the proof of the main theorem

Proof of lemma.

Then

1. From u_0: $\mathfrak{a}(a) = L(\gamma_0)$
2. Can show: $\mathfrak{a}(a) - L(\gamma_\varepsilon) > 0$

This implies

$$L(\gamma_\varepsilon) - L(\gamma_0) < 0 \implies E_{**}(\pi_* v, \pi_* v) < 0\,.$$

By unique continuation, the restriction

$$\pi_* \colon \ \ker D_{u_0} \longrightarrow \{w \in T_\gamma BM_K \mid E_{**}(w,w) < 0\}\ ,$$

is injective, and finally $\dim \ker D_{u_0} \leq \operatorname{ind} \gamma_0 = \operatorname{ind} D_{u_0}$. □

Outline of proof of the main theorem

Theorem

For any $A > 0$, there is an isomorphism of A_∞-algebras $\mathcal{F}_A CW^*_{\Lambda_K}(F) \cong \mathcal{F}_A C^{\mathit{cell}}_{-*}(BM_K)$.

Finally take colimits as $A \to \infty$, and we obtain the main theorem.

Applications

$\mathbb{Z}[\pi_1(M_K)]$-module structure in cohomology

The fundamental group of M_K acts on $HW^*_{\Lambda_K}(F)$ and $H_*(\Omega M_K)$ as follows

$$\begin{aligned}\pi_1(M_K) \times H_*(\Omega M_K) &\longrightarrow H_*(\Omega M_K)\\ ([\gamma], \sigma) &\longmapsto P(\sigma \otimes \sigma_\gamma)\end{aligned}$$

$$\begin{aligned}\pi_1(M_K) \times HW^*_{\Lambda_K}(F) &\longrightarrow HW^*_{\Lambda_K}(F)\\ ([\gamma], a) &\longmapsto \mu^2(a \otimes a_\gamma)\end{aligned}$$

$\mathbb{Z}[\pi_1(M_K)]$-module structure in cohomology

Theorem B

Ψ induces an isomorphism of $\mathbb{Z}[\pi_1(M_K)]$-modules $HW^*_{\Lambda_K}(F) \longrightarrow H_{-*}(\Omega M_K)$.

Remark

Note that there is a quasi-isomorphism $C_0^{\mathrm{cell}}(BM_K) \cong \mathbb{Z}[\pi_1(M_K)]$. We can also prove that $CW^*_{\Lambda_K}(F)$ is an A_3-module over $C_0^{\mathrm{cell}}(BM_K)$ ("group action up to homotopy, but no higher coherent homotopies"), and that Ψ induces an isomorphism of A_3-modules on the chain level.

The Alexander invariant

We now let $n = 5$ or $n \geq 7$ and $K \subset S^n$ a codimension 2 knot.

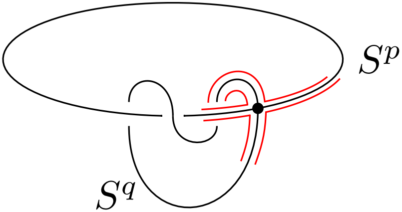
- Alexander invariant of K is $H_*(\widetilde{M}_K)$ viewed as a $\mathbb{Z}[t^{\pm 1}]$-module
- It is used to show non-triviality of certain knots K with $\pi_1(M_K) \cong \mathbb{Z}$.

Plumbing of spheres

Let $n = p + q + 1$ where $p \geq 2$, $q \geq 4$ or $p = q = 2$. Consider $S^p, S^q \subset S^n$.

The Alexander invariant

Plumbing of spheres



Define K to be the boundary of the plumbing above.

Relation to $HW^*_{\Lambda_K}(F)$

- Path-loop fibration
- Leray–Serre spectral sequence

The Alexander invariant and partially wrapped Floer cohomology

Theorem C

Let $n = p + q + 1$ with p and q as above. Then there exists a codimension 2 knot $K \subset S^n$ with $\pi_1(M_K) \cong \mathbb{Z}$ such that $\Lambda_K \cup F \not\simeq \Lambda_{\text{unknot}} \cup F$.

Proof.

By using the Leray–Serre spectral sequence, we can compute that in either case

$$H_p(\widetilde{M}_K) \cong \mathbb{Z}[t^{\pm 1}]/(t-1) \otimes_{\mathbb{Z}[t^{\pm 1}]} HW^{1-p}_{\Lambda_K}(F)\,.$$

Implies $HW^{1-p}_{\Lambda_K}(F) \not\cong HW^{1-p}_{\Lambda_{\text{unknot}}}(F)$ and hence $\Lambda_K \cup F \not\simeq \Lambda_{\text{unknot}} \cup F$. □

Thank you!