Open problems in Floer homotopy theory*

These are some participant-suggested problems from the problem session at the IML workshop on "homotopy theory and Floer homology" in June 2025, moderated by Sheel Ganatra.

For simplicity each problem is cited to the person who suggested the first version of it, but most had contributions from many others, who have mostly not been cited. Various things have been paraphrased or re-ordered.

1 Applications

Question 1 (Hiro Lee Tanaka)**.** *Is there a purely symplectic way to construct the p-adic numbers via a Floer-theoretic inverse limit, such as through Rabinowitz Floer theory?*

Context. Rabinowitz Floer theory can be used to describe formal neighbourhoods in algebraic geometry, cf. [GGV22].

Question 2 (Ciprian Bonciocat)**.** *Are there applications of Floer homotopy theory to:*

1. *understanding minimal numbers of intersection/fixed points, including in degenerate situation? (cf. [Bla24, HP24]).*
2. *constraining the topology of clean intersections, using a spectral version of the Pozniak spectral sequence [Poź94]? (cf. Blakey's talk).*

Question 3 (Thomas Kragh)**.** *Is there an example of a pair of Lagrangians L_1, L_2 that are isomorphic in a chain-level Fukaya category, but not in a spectral Fukaya category? (cf. P.'s talk).*

Remark. There are infinitely many Legendrians distinguished by generating function spectra but not generating function homology (cf. [TT]).

Question 4 (Alex Oancea)**.** *Can we better understand the cone length of a manifold M (a lower bound for the stable Morse number, cf. [CLOT03]) from Floer homotopy theory?*

Remark. The minimal number of fixed points of a Hamiltonian diffeomorphism of a symplectically aspherical closed symplectic manifold is known to equal the cup length in ordinary cohomology [Rud99, OR99, CLOT03].

Question 5 (Mohammed Abouzaid)**.** *Using Floer homotopy theory, can we reprove the result of [ACGK24] (which used generating functions) that for a closed exact Lagrangian in T^*S^n, its stable Gauss map is null-homotopic?*

Question 6 (Egor Shelukhin)**.** *Can we prove, using generating functions, Ekholm-Smith's results [ES18] about Whitney sphere links? (cf. Asplund's talk).*

Question 7 (Semon Rezchikov)**.** *Show Morave K-theory is $\mathbb{E}_1$ but not $\mathbb{E}_2$, using geometric arguments related to Baas-Sullivan theory (cf. [Baa73])?*

Remark (Stefan Schwede)*.* For p and odd prime, $\mathbb{S}/(p^n\mathbb{S})$ is $\mathbb{E}_{n-1}$ but not $\mathbb{E}_\infty$ [Bur22]- can one prove this geometrically?

Remark (Mohammed Abouzaid)*.* Mironov [Mir75, Mir80, Mir78] has studied multiplicative structures on cohomology theories through this lens.

*These notes were taken by Noah Porcelli, who bears responsibility for all errors therein.

2 Computational aspects

Question 8 (Kenny Blakey). *1. What computational tools are available for computing Floer homotopy types?*

2. What spectra can be realised as Floer homotopy types?

3. Which spectrally enriched categories can arise as spectral Fukaya categories?

Question 9 (Abigail Ward). *Which of the standard computational tools from ordinary Floer theory sill apply? For example:*

1. *Are there fibre sequences from Lagrangian surgery?*
2. *Is there a Dehn twist exact sequence?*
3. *Do the results of [GPS20, GPS24b, GPS24a] or [BEE12] go through spectrally?*

Question 10 (Dan Pomerleano). *Is there mirror symmetry over spectra, e.g. KU, MU, $\mathbb{S}$?*

Question 11 (Egor Shelukhin). *Prove that for a flexible or stably displaceable Liouville domain, spectral symplectic cohomology vanishes.*

Question 12 (Sheel Ganatra). *Produce examples with interesting symplectic cohomology over a spectrum E, e.g. where $SH(X;\mathbb{Z}) = 0$ but $SH(X;E) \neq 0$ for some other E (cf. examples sketched by Large).*

3 Equivariant aspects

Question 13 (Semon Rezchikov). *Understand better the structure/appearance of homotopical/equivariant characteristic p operators in (spectral?) enumerative geometry.*

For example, there is a quantum K-theory version of Jae Hee Lee's [Lee25] "quantum Steenrod = p-curvature" conjecture (citation unknown).

Question 14 (Ivan Smith). *Can one use Floer homotopy theory (perhaps in the presence of normal polarisations) to prove $\mathbb{Z}/p$-localisation analogues of the $\mathbb{Z}/2$ version proved in [SS10]?*

More generally, what are further symplectic applications of equivariant homotopy theory?

Question 15 (Mohammed Abouzaid). *In equivariant stable homotopy theory, is there a framework for localisation arguments when working over MU?*

Question 16 (Hiro Lee Tanaka). *Can one construct G-spectra, and functors between them, using flow categories? For example:*

1. *Can one construct $RO(G)$-graded Bredon homology or Mackey functors in this set-up?*
2. *Can one re-prove the Segal conjecture $\mathbb{S}^{tC_p} \simeq \hat{\mathbb{S}}_p$ using flow categories?*

4 Foundational aspects

Question 17 (Semon Rezchikov). *1. Define a formal tensor product "$\otimes$" of flow categories, via a clean combinatorial construction.*

2. Prove Künneth for Morse flow categories. Letting $\mathscr{M}^f$ denote the Morse flow category of a Morse-Smale pair, this is then the statement that for $f_i : M_i \to \mathbb{R}$ Morse-Smale pairs,

$$\mathscr{M}^{f_0 \oplus f_1} \cong \mathscr{M}^{f_0} \text{``}\otimes\text{''} \mathscr{M}^{f_1}$$

3. Show the Cohen-Jones-Segal construction is monoidal with respect to this structure.

Remark (Hiro Lee Tanaka). One approach may be to take fibre products over the stack Broken, cf. [LT18].

Question 18 (Ciprian Bonciocat)**.** *Find a good set-up (e.g. model category, ∞-category, ...) for pro-spectra, that works well for Floer homotopy.*

Desiderata. 1. Weak equivalences to be maps that induce isomorphisms on homology.

2. Has self-duality.

3. Is compatible with a Cohen-Jones-Segal realisation functor, with duality corresponding to reversing the action filtration on the flow category side.

Remark. One can only "reverse the action filtration" in the framework of [AB24] under a finiteness hypothesis.

Remark. This is related to asking for a spectral version of a Tate vector space (cf. Alexandru Oancea's talk).

Remark. There is also a parametric version of this question.

Question 19 (Hiro Lee Tanaka)**.** *What is a curved stable ∞-category?*

Context. There are concrete models for curved A_∞ category categories (in chain complexes), which apply in the context of Fukaya categories of compact symplectic manifolds.

Question 20 (P.)**.** *What is the notion of generating function that produces an MU-module instead of an $\mathbb{S}$-module?*

Remark (Hiro Lee Tanaka). One may be able to reverse-engineer this from [JT24, NS22].

Question 21 (Sheel Ganatra)**.** *Construct the spectral Fukaya category.*

Question 22 (Ciprian Bonciocat)**.** *Write down a model for THH of an algebra object in* Flow *(and construct a monoidal structure on* Flow*, so that one can make sense of this statement).*

References

[AB24] Mohammed Abouzaid and Andrew J. Blumberg. Foundation of floer homotopy theory i: Flow categories, 2024. Preprint, available at arXiv:2404.03193. 3

[ACGK24] Mohammed Abouzaid, Sylvain Courte, Stéphane Guillermou, and Thomas Kragh. Twisted generating functions and the nearby lagrangian conjecture, 2024. Preprint, available at arXiv:2011.13178. 1

[Baa73] Nils Andreas Baas. On bordism theory of manifolds with singularities. *Math. Scand.*, 33:279–302, 1973. 1

[BEE12] Frédéric Bourgeois, Tobias Ekholm, and Yasha Eliashberg. Effect of Legendrian surgery. *Geom. Topol.*, 16(1):301–389, 2012. With an appendix by Sheel Ganatra and Maksim Maydanskiy. 2

[Bla24] Kenneth Blakey. Floer homotopy theory and degenerate lagrangian intersections, 2024. Preprint, available at arXiv:2410.11478. 1

[Bur22] Robert Burklund. Multiplicative structures on moore spectra, 2022. 1

[CLOT03] Octav Cornea, Gregory Lupton, John Oprea, and Daniel Tanré. *Lusternik-Schnirelmann category*, volume 103 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003. 1

[ES18] Tobias Ekholm and Ivan Smith. Nearby Lagrangian fibers and Whitney sphere links. *Compos. Math.*, 154(4):685–718, 2018. 1

[GGV22] Sheel Ganatra, Yuan Gao, and Sara Venkatesh. Rabinowitz fukaya categories and the categorical formal punctured neighborhood of infinity, 2022. Preprint, available at arXiv:2212.14863. 1

[GPS20] Sheel Ganatra, John Pardon, and Vivek Shende. Covariantly functorial wrapped Floer theory on Liouville sectors. *Publ. Math. Inst. Hautes Études Sci.*, 131:73–200, 2020. 2

[GPS24a] Sheel Ganatra, John Pardon, and Vivek Shende. Microlocal Morse theory of wrapped Fukaya categories. *Ann. of Math. (2)*, 199(3):943–1042, 2024. 2

[GPS24b] Sheel Ganatra, John Pardon, and Vivek Shende. Sectorial descent for wrapped Fukaya categories. *J. Amer. Math. Soc.*, 37(2):499–635, 2024. 2

[HP24] Amanda Hirschi and Noah Porcelli. Lagrangian intersections and cuplength in generalised cohomology theories, 2024. Preprint, available at arXiv:2211.07559. 1

[JT24] Xin Jin and David Treumann. Brane structures in microlocal sheaf theory. *J. Topol.*, 17(1):Paper No. e12325, 68, 2024. 3

[Lee25] Jae Hee Lee. Quantum steenrod operations of symplectic resolutions, 2025. Preprint, available at arXiv:2312.02100. 2

[LT18] Jacob Lurie and Hiro Lee Tanaka. Associative algebras and broken lines, 2018. Preprint, available at arXiv:1805.09587. 2

[Mir75] O. K. Mironov. Existence of multiplicative structures in the theory of cobordism with singularities. *Izv. Akad. Nauk SSSR Ser. Mat.*, 39(5):1065–1092, 1219, 1975. 1

[Mir78] O. K. Mironov. Multiplications in cobordism theories with singularities and the Steenrod-tom Dieck operations. *Izv. Akad. Nauk SSSR Ser. Mat.*, 42(4):789–806, 1978. 1

[Mir80] O. K. Mironov. Steenrod-Dieck operations and obstructions to admissible multiplications in cobordism theories with singularities. *Uspekhi Mat. Nauk*, 35(3(213)):203–206, 1980. 1

[NS22] David Nadler and Vivek Shende. Sheaf quantization in weinstein symplectic manifolds, 2022. Preprint, available at arXiv:2007.10154. 3

[OR99] J. Oprea and Y. B. Rudyak. On the Lusternik-Schnirelmann category of symplectic manifolds and the Arnold conjecture. *Math. Z.*, 230(4):673–678, 1999. 1

[Poź94] M. Poźniak. *Floer Homology, Novikov Rings and Clean Intersections*. University of Warwick, 1994. 1

[Rud99] Y. B. Rudyak. On analytical applications of stable homotopy (the Arnold conjecture, critical points). *Math. Z.*, 230(4):659–672, 1999. 1

[SS10] Paul Seidel and Ivan Smith. Localization for involutions in Floer cohomology. *Geom. Funct. Anal.*, 20(6):1464–1501, 2010. 2

[TT] Lisa Traynor and Hiro Lee Tanaka. In preparation. 1