HOMOTOPY RIGIDITY OF NEARBY LAGRANGIAN COCORES

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Abstract. Notes from a talk given at the conference Homotopy Theory and Floer Homology at Institut Mittag-Leffler in June 2025.

Everything reported in these notes are joint work with Yash Deshmukh and Alex Pieloch.

1. Results

Let X be a Weinstein manifold. Assume that X is stably polarized, i.e., we have fixed a lift:

$$\begin{array}{ccc} & & BO \\ & \nearrow & \downarrow{\scriptstyle -\otimes\mathbb{C}} \\ X & \xrightarrow{TX} & BU \end{array}$$

Any Weinstein manifold X^{2n} is built via attachment of index k Weinstein handles

$$\left(D^k \times D^{2n-k}, \frac{1}{2}\sum_{j=1}^{n-k}(x_j dy_j - y_j dx_j) + \sum_{j=n-k+1}^{n}(2x_j dy_j + y_j dx_j)\right), \quad 0 \le k \le n$$

If $k \le n-1$ it is called *subcritical*. If $k = n$, we call it a *critical handle*, see Figure 1.

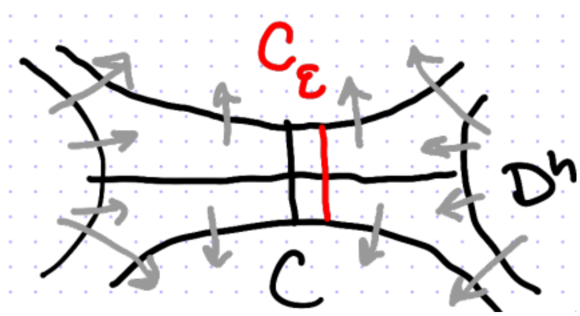

Figure 1. The local model of a critical handle.

We call $C = \{0\} \times D^n$ the *Lagrangian cocore disk*. Here $C_\varepsilon = \{\varepsilon\} \times D^n$, $\varepsilon \in D^n \smallsetminus \{0\}$ for some ε near 0. It is called the *shifted cocore*.

From now on, fix a Weinstein handlebody presentation of X.

There exists a retract $\pi\colon X \to \operatorname{Core} X$ to an n-dimensional CW-complex $\operatorname{Core} X$, that inherits a cell structure from the handlebody presentation of X. We denote the k-skeleton of $\operatorname{Core} X$ by $(\operatorname{Core} X)_k$.

Definition 1.1. $L \subset X$ is a *nearby cocore* if

- L is exact and conical at ∞
- $\partial_\infty L = \partial_\infty C_\varepsilon$ for some shifted cocore
- $L \cap \boldsymbol{C} = \varnothing$, where $\boldsymbol{C} := \bigcup$ cocore disks

Example 1.2. An exact conical Lagrangian $L \subset T^*Q$ such that $\partial_\infty L = \partial_\infty T^*_{q_1} Q$ for some $q_1 \in Q$ and $L \cap T^*_{q_2} Q = \varnothing$, for some $q_2 \ne q_1$, is a nearby cocore

Motivated by the nearby Lagrangian conjecture, we have the following:

Question 1.3. *Is a nearby cocore Hamiltonian isotopic to a Lagrangian cocore?*

A weaker homotopical version:

Question 1.4. *Is the retract* $\pi|_L \colon L \to \operatorname{Core} X \setminus \pi(\boldsymbol{C})$ *null-homotopic?*

Theorem 1.5 (A.–Deshmukh–Pieloch)**.** *Let* $k \geq 1$ *and let* $n \geq 2k+2$*. Suppose that* X^{2n} *is a stably polarized Weinstein sector with a chosen Weinstein handlebody decomposition such that there is no index* i *Weinstein handle where* $n-k+1 \leq i \leq n-1$*. If* $L \subset X$ *is a nearby cocore, the following composition is null-homotopic*

$$L \xrightarrow{\pi|_L} \operatorname{Core} X \setminus \pi(\boldsymbol{C}) \longrightarrow (\operatorname{Core} X)_{n-k}/(\operatorname{Core} X)_{n-k-1}.$$

Corollary 1.6 (A.–Deshmukh–Pieloch)**.** *Let* X *be either:*

- $T^*\mathbb{C}\mathrm{P}^2$*, or*
- *a tree plumbing with each vertex being* $T^*\mathbb{R}^n$*,* T^*S^n *or* $T^*(S^{n-k}\times\mathbb{R}^k)$ *for* $k \in \{1,4,5,12,61\}$ *and* $n \geq 2k+2$ *(where* k *for all vertices).*

Let $L \subset X$ *be a nearby cocore. Then*

$$\pi|_L \colon L \longrightarrow \operatorname{Core} X \setminus \pi(\boldsymbol{C})$$

is null-homotopic.

In fact, a strengthening is obtained by employing an h-principle:

Corollary 1.7. *With* X *as in Corollary 1.6, any nearby cocore* $L \subset X$ *is smoothly isotopic, relative to its boundary, to the shifted cocore in the complement of all Lagrangian cocores.*

Remark 1.8.

- Ekholm–Smith [ES18] proved the conclusion of Corollary 1.6 for $X = T^*\mathbb{R}^n$, $n \geq 4$.
- Côté–Dimitroglou Rizell [CDR22] proved that nearby cocores in T^*(compact Riemann surface) are Hamiltonian isotopic to the cotangent fiber.

2. Strategy

Let X_0 denote the subcritical part of X (after removing critical handles). Let $L \subset X$ be a nearby cocore. This yields an exact Lagrangian filling L of a Legendrian unknot $\Lambda_0 \subset \partial_\infty X_0$ in a Darboux ball in $\partial_\infty X_0$, see Figure 2.

Remark 2.1. Theorem 1.5 could be rephrased in terms of exact Lagrangian fillings of Legendrian unknots $\Lambda_0 \subset \partial_\infty X_0$ as above.

One can in fact show that L has to be diffeomorphic to D^n if $n \neq 4$ (and homeomorphic to D^4 when $n = 4$). Using wrapped Floer cohomology with $\mathbb{Z}_2$-grading and $\mathbb{Z}_2$-coefficients one can first show that L is relative pin and must have trivial Maslov class. Using wrapped Floer cohomology with $\mathbb{Z}$-coefficients then shows that L is an integer homology disk, and in fact is simply connected by [EL23, Theorem 70].

Let $\widehat{X}_0$ be X_0 with a critical Weinstein handle attached along Λ_0, see Figure 3. Let $\widehat{L} = L \cup_\partial \text{core disk}$ and $\widehat{C} = C_\varepsilon \cup_\partial \text{core disk}$.

The new cocore disk F generates the wrapped Fukaya category $\mathcal{W}(\widehat{X}_0;\mathbb{Z})$, so we know using classical tools that $\widehat{L} \cong \widehat{C}$ in $\mathcal{W}(\widehat{X}_0;\mathbb{Z})$. There is in fact a spectral lift of this result.

Proposition 2.2 (A.–Deshmukh–Pieloch)**.** *There are objects supported on* $\widehat{L}, \widehat{C} \subset \widehat{X}_0$ *such that* $\widehat{L} \cong \widehat{C}$ *in* $\mathcal{W}(\widehat{X}_0; MO\langle k+2\rangle)$*.*

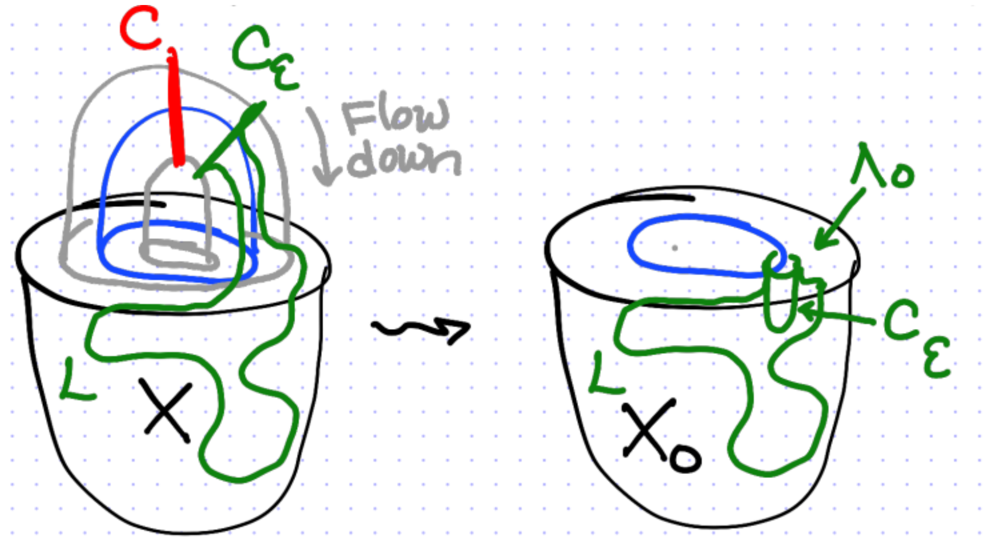

FIGURE 2. Construction of an unknot filling from a nearby cocore.

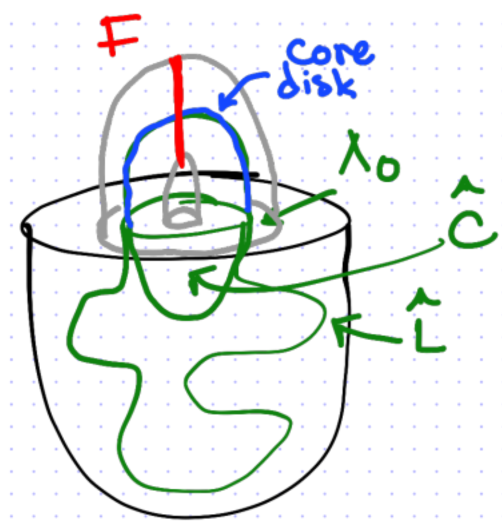

FIGURE 3. Attaching a critical handle to X_0.

Remark 2.3. Here $MO\langle k\rangle$ denotes the Thom spectrum of the k-connected cover of the orthogonal group. In particular we have e.g. $O\langle 3\rangle = Spin$ and $O\langle 7\rangle = String$.

In this proposition, $\mathcal{W}(\widehat{X}_0; R)$ denotes the wrapped Donaldson–Fukaya category with R-coefficients, for any commutative (E_∞) ring spectrum R [Lar21, PS24a, PS24b, ADP24].

Objects: $L \subset \widehat{X}_0$ exact conical Lagrangians equipped with grading (null-homotopy of the Maslov class) and R-brane structure, i.e., a choice of null-homotopy of the composition

$$L \xrightarrow{\mathcal{G}_L} U/O \xrightarrow{\simeq} B^2O \times B\mathbb{Z} \xrightarrow{B^2J} B^2GL_1(\mathbb{S}) \longrightarrow B^2GL_1(R)$$

Morphisms: $HW(L, K; R)$ is an R-module constructed via the Cohen–Jones–Segal construction of an R-oriented flow category associated to the R-branes L and K.

Using the open-closed map, we have:

Proposition 2.4 ([ADP24, PS24a]). *If L and K are two compact R-orientable R-branes such that $L \cong K$ in $\mathcal{W}(\widehat{X}_0; R)$, there exist R-fundamental classes on L and K such that $[L]_R = [K]_R$ in $H_n(\widehat{X}_0; R)$.*

Since $\widehat{L}$ and $\widehat{C}$ topologically are spheres for $n \geq 2k+2$, there is a unique tangential $O\langle k+2\rangle$-structure on its tangent bundle, i.e., a homotopically unique lift

$$\begin{array}{ccc} & & BO\langle k+2\rangle \\ & \nearrow & \downarrow \\ L & \xrightarrow{TL} & BO \end{array}$$

This yields an $MO\langle k+2\rangle$-fundamental class $[\widehat{L}] \in H_n(\widehat{X}_0; MO\langle k+2\rangle)$. Combining these two results yields that there exists an $MO\langle k+2\rangle$-fundamental class $[\widehat{C}] \in H_n(\widehat{X}_0; MO\langle k+2\rangle)$ such that

$$[\widehat{L}] = [\widehat{C}] \in H_n(\widehat{X}_0; MO\langle k+2\rangle).$$

Next, $\pi\colon X_0 \to \operatorname{Core} X_0$ admits an extension $\widehat{\pi}\colon \widehat{X}_0 \to \operatorname{Core} X_0$ such that $\widehat{\pi}|_{\widehat{C}}\colon \widehat{C} \to \operatorname{Core} X_0$ is null-homotopic. Such extension exists because the critical Weinstein handle attached to X_0 is along a Legendrian that belongs to a Darboux ball in $\partial_\infty X_0$. Therefore we conclude

$$\widehat{\pi}_*[\widehat{L}] = \widehat{\pi}_*[\widehat{C}] = 0 \text{ in } \widetilde{H}_n(\operatorname{Core} X_0; MO\langle k+2\rangle) \tag{2.1}$$

3. Sketch of the proof of the main theorem

We consider the composition

$$\widehat{L} \xrightarrow{\widehat{\pi}|_L} \operatorname{Core} X_0 \xrightarrow{q} \frac{(\operatorname{Core} X_0)_{n-k}}{(\operatorname{Core} X_0)_{n-k-1}} \simeq \bigvee_j S^{n-k}.$$

Let $q_i\colon \bigvee_j S^{n-k} \to S^{n-k}$ be the components of the wedge sum.

Since $n \geq 2k+2$, $q \circ \widehat{\pi}|_L$ being null-homotopic is in fact equivalent to $f^i := q_i \circ q \circ \widehat{\pi}|_L$ being null-homotopic for each i. Now, (2.1) implies $f^i_*[\widehat{L}] = 0$ in $\widetilde{H}_n(S^{n-k}; MO\langle k+2\rangle)$. We have a commutative diagram

$$\begin{array}{ccc} \widetilde{H}_n(S^n; MO\langle k+2\rangle) & \xrightarrow{f^i_*} & \widetilde{H}_n(S^{n-k}; MO\langle k+2\rangle) \\ \downarrow\cong & & \downarrow\cong \\ \pi_0(MO\langle k+2\rangle) & \xrightarrow{f^i_*} & \pi_k(MO\langle k+2\rangle) \\ \uparrow\cong & & \uparrow\cong \\ \pi_0^{\mathrm{st}} & \xrightarrow{f^i_*} & \pi_k^{\mathrm{st}} \end{array}$$

We now have $f^i_* = 0$ since the fundamental class generates $\widetilde{H}_n(S^n; MO\langle k+2\rangle)$. Now, we are in the stable range because $n \geq 2k+2$, so we conclude that f^i is null-homotopic for each i, and hence $q \circ \widehat{\pi}|_L$ is null-homotopic. □

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