Proof of GPS3, Main theorem.

§ Thm (Thm 1.1, GPS3)

M be a real analytic manifold and $\Lambda \subset S^*M$ a subanalytic closed isotropic subset. There is an equivalence

$$\mathrm{Perf}\, \mathcal{W}(T^*M, \Lambda)^{op} \simeq Sh_\Lambda(M)^c$$

which takes

① linking disc at smooth Leg pt $p \in \Lambda$ to corepresentative of microstalk at p, and

② cotangent fiber at $p \in M$, not in img of Λ, to corepresentative of stalk functor.

Strategy: Any subanalytic clsd isotropic Λ contained insider some $N^*_\infty \mathcal{S}$, conormal to some Whitney triangulation $\mathcal{S}$.

① Check the equivalence explicitly for the case $\Lambda = N^*_\infty \mathcal{S}$

② Check that both sides transform in the same way when Λ gets smaller.

$\rightsquigarrow$ ① sheaf side : quotient by corep. of μsth's

② Fuk side : quotient by linking disks.

§ Microlocal Morse categories:

Axiomatic setup for characterizing wFuk and sheaf categories wrt properties ① & ② above.

Defn ① A morse-character is a functor

$$\mathcal{C} : \left\{ \begin{matrix} \text{subanalytic singular} \\ \text{clsd isotropics} \\ \subset S^*M \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{dg-categories} \\ \text{over } \mathbb{Z} \end{matrix} \right\}$$

with an isomorphism of functors

$$\left(\mathcal{S} \to \mathcal{C}(N^*_\infty \mathcal{S})\right) \simeq \left(\mathcal{S} \mapsto \mathrm{Perf}\,\mathcal{S}\right)$$

on Whitney triangulations.

② For (a) a smooth Leg pt $p \in \Lambda$

(b) $\epsilon > 0$

(c) Analytic function with Morse Λ-critical pt at p w/ critical value 0 and no other critical values in $[-\epsilon, \epsilon]$

(d) $\mathcal{S}$ a Whiteny triangulation st. $\Lambda \subset N^*_\infty \mathcal{S}$ and $f^{-1}(-\infty, -\epsilon)$, $f^{-1}(-\infty, \epsilon)$ are $\mathcal{S}$-constructible.

recall Morse Λ-critical pt. is a transverse intersect^n pt between Γ_{df} and $\Omega_M \cup (\mathbb{R}_{>0} \times \Lambda)$.

Define Morse character $\mathcal{X}_{\Lambda,p}(f, \epsilon, \mathcal{S})$ to be image under $\underset{\mathrm{Perf}(\mathcal{S})}{\overset{}{\mathcal{C}(\mathcal{S})}} \to \mathcal{C}(\Lambda)$ of the cone

$$\mathrm{Hom}\left(1_{f^{-1}(-\infty,-\epsilon)} \to 1_{f^{-1}(-\infty,\epsilon)}\right) \in \mathrm{Perf}(\mathcal{S}) = \mathcal{C}(N^*_\infty \mathcal{S})$$

③ Microlocal Mors theater is a pre-theater $\mathcal{C}$ st. For $\Lambda \subset \Lambda'$ any any collection of Morse characters $\mathcal{X}_{\Lambda',p}(f, \epsilon, \mathcal{S}) \in \mathcal{C}(\Lambda')$, at least one on each smooth leg locus of Λ,

$\mathcal{C}(\Lambda') \to \mathcal{C}(\Lambda)$ is given by quotient by Morse-characters.

Prop^n: Any two Morse characters are isomorphic.

Pf: $\quad e(\Lambda) = e(N^*_\infty S) / \mathcal{X}_{N^*_S S \setminus \Lambda}$

all morse char. at all sm. Leg pts. of $N^*S \setminus \Lambda$.

§ Microlocal Morse theatre: Sheaf categories:

Thm: The functor $\Lambda \longrightarrow Sh_\Lambda(M)^c$ is a microlocal Morse functor, with co-representatives of microstalk functor at sm. pts of Λ as Morse characters.

① For $\Lambda \subset \Lambda'$ have functor $Sh_\Lambda(M) \longrightarrow Sh_{\Lambda'}(M)$ maps are continuous, co-continuous so has left left adjoint which preserves compact obj. this gives the functor

$$Sh_{\Lambda'}(M)^c \longrightarrow Sh_\Lambda(M)^c.$$

$$Sh_S(M) \simeq Sh_{\mathcal{S}}(M) \simeq Mod(CS) = Fun(S^{op}, Mod\,\mathbb{Z})$$

$\mathcal{S}$-constructible sheaves

strata poset of $\mathcal{S}$

$$Sh_S(M)^c \simeq Sh_S(M)^c \simeq Perf(\mathcal{S})$$

② $\Lambda \subset \Lambda'$,

$$(Sh_{\Lambda'}(M)^c / \mathcal{D}) \xrightarrow{\sim} Sh_\Lambda(M)$$

$\mathcal{D}$ = corepresentative of microstalk at smooth points

§ Microlocal Fukaya Category:

$\Lambda \longmapsto \text{Perf } \mathcal{W}(T^*M, \Lambda)^{op}$ is a microlocal Morse theater with Morse character linking disks at sm. pts of Λ.

Things to check

① For $\mathcal{S}$ Whitney stratification

$$\text{Perf}(\mathcal{S}) \xrightarrow{\sim} \mathcal{W}(T^*M, N^*\mathcal{S})$$

② Stop Removal formula:

$\Lambda \subset \Lambda'$ then

$$\mathcal{W}(X, \Lambda')/\mathcal{D} \xrightarrow{\sim} \mathcal{W}(X, \Lambda)$$

For $\mathcal{D}$ = linking disks at sm. pt of $\Lambda' \setminus \Lambda$

∴ It remains to check Morse characters correspond to linking disks.