Wrapped microlocal sheaves on pair-of-pants

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In this note we give an account of Nadler's computation of the dg category of wrapped microlocal sheaves on n-dimensional pair-of-pants, which in particular verifies the homological mirror symmetry conjecture in this case.

We will be working over an algebraically closed field $\mathbb{K}$ of characteristic 0.

1 Matrix factorizations

Consider a superpotential $W : \mathbb{A}^n \to \mathbb{A}^1$ such that $0 \in \mathbb{A}^1$ is the unique critical value. Denote by $X = W^{-1}(0)$ the singular fiber.

Let $Perf(X)$ and $Coh(X)$ be the dg enhancements of the triangulated category of perfect complexes and the bounded derived category of coherent sheaves, respectively. Let $D_{Sing}(X) = Coh(X)/Perf(X)$ be the 2-periodic dg quotient.

Let $MF(\mathbb{A}^n, W)$ be the differential $\mathbb{Z}/2$-graded category of matrix factorizations. Denote by $MF(\mathbb{A}^n, W)_{2\mathbb{Z}}$ the unfurling of $MF(\mathbb{A}^n, W)$, then there is a quasi-equivalence between 2-periodic dg categories

$$MF(\mathbb{A}^n, W)_{2\mathbb{Z}} \cong D_{Sing}(X). \tag{1}$$

We are interested in the superpotential $W_{n+1} : \mathbb{A}^{n+1} \to \mathbb{A}^1$ given by the product of coordinates $z_1 \cdots z_{n+1}$, so that the central fiber $X_n = W_{n+1}^{-1}(0)$ is the union of $n+1$ coordinate hyperplanes.

Consider the natural projection $\pi : X_n \to X_{n-1}$.

Proposition 1.1. *The pullback of coherent sheaves*

$$\pi^* : Coh(X_{n-1}) \to Coh(X_n) \tag{2}$$

induces an equivalence of differential $\mathbb{Z}/2$-graded categories

$$Coh(X_{n-1})_{\mathbb{Z}/2} \cong MF(\mathbb{A}^{n+1}, W_{n+1}). \tag{3}$$

Let $dgst_{\mathbb{K}}$ be the ∞-category of $\mathbb{K}$-linear small stable dg categories with exact functors. Let $dgSt_{\mathbb{K}}$ be the ∞-category of $\mathbb{K}$-linear cocomplete dg categories with continuous functors. Let $dgSt_{\mathbb{K}}^c \subset dgSt_{\mathbb{K}}$ be the (not full) ∞-subcategory of $\mathbb{K}$-linear cocomplete dg categories with functors preserving compact objects. Taking ind-categories provides an equivalence

$$Ind : dgst_{\mathbb{K}} \xrightarrow{\cong} dgSt_{\mathbb{K}}^c. \tag{4}$$

Taking compact objects provides an inverse equivalence

$$\kappa : dgSt_{\mathbb{K}}^c \xrightarrow{\cong} dgst_{\mathbb{K}}. \tag{5}$$

Let $\mathfrak{X}_{\mathbb{K}}$ be the category of affine locally complete intersection $\mathbb{K}$-schemes and closed embeddings. Passing to coherent sheaves and pushforwards yields a functor

$$Coh_* : \mathfrak{X}_{\mathbb{K}} \to dgst_{\mathbb{K}}. \tag{6}$$

Passing to perfect complexes and $*$-pullbacks yields a functor

$$Perf^* : \mathfrak{X}_{\mathbb{K}}^{op} \to dgst_{\mathbb{K}}, \tag{7}$$

and similarly for perfect complexes with proper support

$$Perf^*_{prop} : \mathcal{X}^{op}_{\mathbb{K}} \to dgst_{\mathbb{K}}. \tag{8}$$

Passing to ind-coherent sheaves and pushforwards provides a functor

$$IndCoh_* : \mathcal{X}_{\mathbb{K}} \to dgSt^c_{\mathbb{K}}. \tag{9}$$

Passing to quasi-coherent sheaves and $*$-pullbacks yields a functor

$$QCoh^* \simeq IndPerf^* : \mathcal{X}^{op}_{\mathbb{K}} \to dgSt^c_{\mathbb{K}}. \tag{10}$$

Passing to ind-coherent sheaves with $*$-pullbacks or !-pullbacks yield functors

$$IndCoh^* : \mathcal{X}^{op}_{\mathbb{K}} \to dgSt_{\mathbb{K}},\ IndCoh^! : \mathcal{X}^{op}_{\mathbb{K}} \to dgSt_{\mathbb{K}}. \tag{11}$$

Tensoring with the dualizing complex provides a natural intertwining equivalence

$$\otimes\omega : IndCoh^* \xrightarrow{\simeq} IndCoh^!. \tag{12}$$

Let us return to $(\mathbb{A}^n, W_n)$. Let $\mathcal{I}^\circ_n$ denote the poset of subsets $I \subset \{1, \cdots, n\}$ under inclusions. For $I \in \mathcal{I}^\circ_n$, consider the corresponding coordinate subspace

$$X_I = \mathrm{Spec}\mathbb{K}[z_1, \cdots, z_n]/(z_a | a \notin I). \tag{13}$$

We have a colimit diagram of closed embeddings

$$\mathrm{colim}_{I \in \mathcal{I}^\circ_n} X_I \xrightarrow{\cong} X_{n-1}. \tag{14}$$

Proposition 1.2. *The colimit diagram (14) is taken to a colimit diagram by Coh_* and $IndCoh_*$, and a limit diagram by $Perf^*$, $Perf^*_{prop}$, $IndCoh^!$, and $IndCoh^*$.*

2 Microlocal sheaves

Let Z be a real analytic manifold. We will often work with a closed conic Lagrangian subvariety $\Lambda \subset T^*Z$ and its Legendrian ideal boundary

$$\Lambda^\infty = (\Lambda \cap (T^*Z \backslash Z))/\mathbb{R}_{>0} \subset S^\infty Z. \tag{15}$$

Denote by Y the front projection $\pi^\infty(\Lambda^\infty)$, where $\pi^\infty : S^\infty Z \to Z$. In the generic situation, the projection $\pi^\infty|_{\Lambda^\infty} : \Lambda^\infty \to Y$ is finite, so the front projection is a hypersurface.

We will often fix a Whitney stratification $\mathcal{S} = \{Z_\alpha\}_{\alpha \in A}$ of Z so that $Y \subset Z$ is a union of strata. Hence we have inclusions

$$\Lambda \subset T^*_{\mathcal{S}} Z := \coprod_{\alpha \in A} T^*_{Z_\alpha} Z,\ \Lambda^\infty \subset S^\infty_{\mathcal{S}} Z := \coprod_{\alpha \in A} S^\infty_{Z_\alpha} Z. \tag{16}$$

Given a Whitney stratification $\mathcal{S}$, by a small open ball $B \subset Z$ around a point $z \in Z$ we will mean an open ball $B = B(r) \subset Z$ of some radius $r > 0$ such that the corresponding spheres $S(r') \subset Z$, for all $0 < r' < r$, are transverse to the strata of $\mathcal{S}$.

Let $Sh^\diamond(Z)$ denote the dg category of complexes of sheaves of $\mathbb{K}$-vector spaces on Z such that the total cohomology sheaf is locally constant with respect to some Whitney stratification $\mathcal{S}$. For a fixed Whitney stratification $\mathcal{S}$, denote by $Sh^\diamond_{\mathcal{S}}(Z) \subset Sh^\diamond(Z)$ the full subcategory which are cohomologically locally constant with respect to this specific $\mathcal{S}$. It follows that $Sh^\diamond(Z) = \bigcup_{\mathcal{S}} Sh^\diamond_{\mathcal{S}}(Z)$.

Let $Sh(Z) \subset Sh^\diamond(Z)$ be the full dg subcategory of constructible complexes of sheaves of $\mathbb{K}$-vector spaces on Z. In other words, it consists of objects of $Sh^\diamond(Z)$ whose total cohomology sheaf, when restricted to each Z_α, has finite rank. We can introduce the notation $Sh_{\mathcal{S}}(Z)$ as above, and it follows that $Sh(Z) = \bigcup_{\mathcal{S}} Sh_{\mathcal{S}}(Z)$. The objects of $Sh^{\diamond}(Z)$ will be referred to as *large constructible sheaves*, and the objects of $Sh(Z)$ as *constructible sheaves*.

All functors between dg categories of sheaves will be derived in the dg sense. When dealing with large constructible sheaves, since we are working with co-complete dg categories, the functors should also be co-continuous (preserves colimits). For example, for a closed embedding $i : Y \to Z$, by the !-restriction $i^! : Sh^{\diamond}(Z) \to Sh^{\diamond}(Y)$, we will mean the shifted cone $i^! \simeq Cone(\mathcal{F} \to j_* j^* \mathcal{F})[-1]$, where $j : U \to Z$ is the inclusion of the open complement $U = Z \backslash i(Y)$. For a smooth map $f : Y \to Z$, by the !-pullback $f^! : Sh^{\diamond}(Z) \to Sh^{\diamond}(Y)$, we will mean the twist of the $*$-pullback $f^! \mathcal{F} \cong f^* \mathcal{F} \otimes \omega_f$, where $\omega_f \cong or_f[\dim Y/Z]$ is the relative dualizing complex.

Fix a point $(z,\xi) \in T^*Z$. Let $B \subset Z$ be an open ball around $z \in Z$, and $f : B \to \mathbb{R}$ a smooth function such that $f(z) = 0$ and $df|_z = \xi$. We will refer to f as a *compatible test function*.

Define the *vanishing cycles functor*

$$\phi_f : Sh^{\diamond}(Z) \to Mod_{\mathbb{K}}, \tag{17}$$

$$\phi_f(\mathcal{F}) = \Gamma_{\{f \geqslant 0\}}(B, \mathcal{F}|_B) \cong Cone\left(\Gamma(B, \mathcal{F}|_B) \to \Gamma(\{f < 0\}, \mathcal{F}|_{\{f<0\}})\right)[-1], \tag{18}$$

where we take $B \subset Z$ to be sufficiently small.

To any object $\mathcal{F}$ of $Sh^{\diamond}(Z)$, define its *singular support* $ss(\mathcal{F}) \subset T^*Z$ to be the largest closed subset such that $\phi_f(\mathcal{F}) \cong 0$ for any $(z,\xi) \in T^*Z \backslash ss(\mathcal{F})$, and any compatible test function f.

For a conic Lagrangian subvariety $\Lambda \subset T^*Z$, write $Sh^{\diamond}_{\Lambda}(Z) \subset Sh^{\diamond}(Z)$, resp. $Sh_{\Lambda}(Z) \subset Sh(Z)$ for the full dg subcategory with singular support $ss(\mathcal{F}) \subset \Lambda$.

Given a Whitney stratification $\mathcal{S}$, an inclusion $\Lambda \subset T^*_{\mathcal{S}}Z$ induces the fully faithful embeddings $Sh^{\diamond}_{\Lambda}(Z) \subset Sh^{\diamond}_{\mathcal{S}}(Z)$, $Sh_{\Lambda}(Z) \subset Sh_{\mathcal{S}}(Z)$. More generally, an inclusion $\Lambda \subset \Lambda'$ induces the fully faithful embeddings $Sh^{\diamond}_{\Lambda}(Z) \subset Sh^{\diamond}_{\Lambda'}(Z)$, $Sh_{\Lambda}(Z) \subset Sh_{\Lambda'}(Z)$.

When $U \subset Z$ is an open subset, we will abuse notations and write $Sh^{\diamond}_{\Lambda}(U) \subset Sh^{\diamond}(U)$, resp. $Sh_{\Lambda}(U) \subset Sh(U)$ for the full dg subcategory with objects satisfying $ss(\mathcal{F}) \subset \Lambda \cap \pi^{-1}(U)$. $\pi : T^*Z \to Z$ is the natural projection.

Remark 2.1. *Let $\omega_Z \cong or_Z[\dim Z] \cong p^! \mathbb{K}_{pt}$, for $p : Z \to pt$, be the Verdier dualizing complex. For a conic Lagrangian subvariety $\Lambda \subset T^*Z$ and the antipodal conic Lagrangian subvariety $-\Lambda \subset T^*Z$. Verdier duality provides an involutive equivalence*

$$\mathbf{D}_Z : Sh_{\Lambda}(Z)^{op} \xrightarrow{\cong} Sh_{-\Lambda}(Z), \mathbf{D}_Z(\mathcal{F}) = \mathcal{H}om(\mathcal{F}, \omega_Z). \tag{19}$$

The above discussions can be generalized to the slightly more general setting. To a conic open subspace $\Omega \subset T^*Z$, we associate the dg category $\mu Sh^{\diamond}_{\Lambda}(\Omega)$ of large microlocal sheaves on Ω supported along Λ. Before describing its construction, we first mention some of its formal properties.

- Given an inclusion of conic open subspaces $\Omega' \subset \Omega$, there is a natural restriction functor $\mu Sh^{\diamond}_{\Lambda}(\Omega) \to \mu Sh^{\diamond}_{\Lambda}(\Omega')$. These assignments assemble into a sheaf $\mu Sh^{\diamond}_{\Lambda}$ of dg categories supported along Λ.

- There exists a Whitney stratification of Λ such that the restriction of $\mu Sh^{\diamond}_{\Lambda}$ to each stratum is locally constant. Thus we can reconstruct $\mu Sh^{\diamond}_{\Lambda}$ from the assignments $\mu Sh^{\diamond}_{\Lambda}(\Omega)$ for small conic open neighborhoods Ω of $(z,\xi) \in \Lambda$.

- Given a closed embedding of conic Lagrangian subvarieties $\Lambda' \subset \Lambda$, there is a natural full embedding $\mu Sh^{\diamond}_{\Lambda'} \subset \mu Sh^{\diamond}_{\Lambda}$ of sheaves of dg categories.

All of the above facts follows from the local description of $\mu Sh^{\diamond}_{\Lambda}(\Omega)$ which we now recall. Note that for a point $(z,\xi) \in \Lambda$ there are two local cases to consider: either $\xi = 0$

so locally Ω is the cotangent bundle T^*B of a small open ball $B \subset Z$, or $\xi \neq 0$ so that locally Ω is the cone over a small open ball $\Omega^\infty \subset S^\infty Z$.

- For $B = \pi(\Omega)$, there is always a canonical functor $Sh^\diamond_\Lambda(B) \to \mu Sh^\diamond_\Lambda(\Omega)$. When $\Omega = T^*B$, this functor is in fact an equivalence

$$Sh^\diamond_\Lambda(B) \xrightarrow{\cong} \mu Sh^\diamond_\Lambda(T^*B). \tag{20}$$

- Suppose Ω is the cone over small open ball $\Omega^\infty \subset S^\infty Z$. Set $B = \pi(\Omega)$, and let $Sh^\diamond_\Lambda(B, \Omega) \subset Sh^\diamond(B)$ denote the full dg subcategory of objects with $ss(\mathcal{F}) \cap \Omega \subset \Lambda$. Then there is a natural equivalence

$$Sh^\diamond_\Lambda(B, \Omega)/K^\diamond(B, \Omega) \xrightarrow{\cong} \mu Sh^\diamond_\Lambda(\Omega), \tag{21}$$

  where $K^\diamond(B, \Omega) \subset Sh^\diamond_\Lambda(B, \Omega)$ denote the full dg subcategory of objects with $ss(\mathcal{F}) \cap \Omega = \varnothing$.

We similarly introduce the full dg subcategory $\mu Sh_\Lambda(\Omega) \subset \mu Sh^\diamond_\Lambda(\Omega)$ of microlocal sheaves on Ω supported along Λ. It is constructed as above by working with constructible sheaves instead of large constructible sheaves.

The dg category $\mu Sh_\Lambda(\Omega)$ is the sections of a subsheaf $\mu Sh_\Lambda \subset \mu Sh^\diamond_\Lambda$ of full dg subcategories supported along Λ. Given a Whitney stratification of Λ such that the restriction of $\mu Sh^\diamond_\Lambda$ to each stratum is locally constant, the restriction of μSh_Λ to each stratum will also be locally constant. Finally, given a closed embedding of conic Lagrangian subvarieties $\Lambda' \subset \Lambda$, the full embedding $\mu Sh^\diamond_{\Lambda'} \subset \mu Sh^\diamond_\Lambda$ restricts to a full embedding $\mu Sh_{\Lambda'} \subset \mu Sh_\Lambda$.

Remark 2.2. *For a conic Lagrangian subvariety $\Lambda \subset T^*Z$, with antipodal subvariety $-\Lambda \subset T^*Z$, and conic open subspace $\Omega \subset T^*Z$, with antipodal subspace $-\Omega \subset T^*Z$. Verdier duality induces an involutive equivalence*

$$\mathbf{D}_Z : \mu Sh_\Lambda(\Omega)^{op} \xrightarrow{\cong} \mu Sh_{-\Lambda}(-\Omega). \tag{22}$$

Fix a Whitney stratification $\mathcal{S} = \{Z_\alpha\}_{\alpha \in A}$ of Z such that $\Lambda \subset T^*_\mathcal{S} Z := \coprod_{\alpha \in A} T^*_{Z_\alpha} Z$. To each stratum $Z_\alpha \subset Z$, introduce the frontier $\partial(T^*_{Z_\alpha} Z) := \overline{T^*_{Z_\alpha} Z} \setminus T^*_{Z_\alpha} Z$ of its conormal bundle, and the dense, open, smooth locus of their complement

$$(T^*_\mathcal{S} Z)^\circ := T^*_\mathcal{S} Z \setminus \bigcup_{\alpha \in A} \partial(T^*_{Z_\alpha} Z). \tag{23}$$

Introduce the corresponding dense, open, smooth locus

$$\Lambda^\circ := \Lambda \cap (T^*_\mathcal{S} Z)^\circ \subset \Lambda. \tag{24}$$

Note that Λ° depends on $\mathcal{S}$, refining $\mathcal{S}$ leads to smaller Λ°.

Fix a point $(z, \xi) \in \Lambda^\circ$. Let $B \subset Z$ be a small open ball around $z \in Z$, and $f : B \to \mathbb{R}$ a compatible test function. Let $L \subset T^*Z$ be the graph of df, and assume that L intersects Λ° transversely at the single point $(z, \xi) \in \Lambda^\circ$.

Definition 2.1. *Let $\Omega \subset T^*Z$ be a conic open subspace containing $(z, \xi) \in \Lambda^\circ$. Define the microstalk along $L \subset T^*Z$ to be the vanishing cycles*

$$\phi_L : \mu Sh^\diamond_\Lambda(\Omega) \to Mod_\mathbb{K}, \ , \phi_L(\mathcal{F}) := \Gamma_{\{f \geqslant 0\}}(B, \widetilde{\mathcal{F}}|_B), \tag{25}$$

where $\widetilde{\mathcal{F}} \in Sh^\diamond_\Lambda(B, \Omega_B)$ represents the restriction of $\mathcal{F} \in \mu Sh^\diamond_\Lambda(\Omega)$ to a small open neighborhood $\Omega_B \subset \Omega$ of the point $(z, \xi) \in T^*Z$.

Remark 2.3. *The microstalk is well-defined since by construction it vanishes on the kernel of the localization $Sh^\diamond_\Lambda(B, \Omega_B) \to \mu Sh^\diamond_\Lambda(\Omega_B)$ with respect to the singular support.*

Lemma 2.1. *An object $\mathcal{F} \in \mu Sh^\diamond_\Lambda(\Omega)$ is trivial if and only if all of its microstalks are trivial. An object $\mathcal{F} \in \mu Sh^\diamond_\Lambda(\Omega)$ lies in the full subcategory $\mu Sh_\Lambda(\Omega) \subset \mu Sh^\diamond_\Lambda(\Omega)$ if and only if all its microstalks are perfect (i.e. proper) $\mathbb{K}$-modules.*

3 Wrapped microlocal sheaves

Definition 3.1. *Define the small dg category $\mu Sh^w_\Lambda(\Omega)$ of wrapped microlocal sheaves on Ω supported along Λ to be the full dg subcategory of compact objects within the dg category $\mu Sh^\diamond_\Lambda(\Omega)$ of large microlocal sheaves.*

The dg categories $\mu Sh_\Lambda(\Omega)$ and $\mu Sh^w_\Lambda(\Omega)$ can now be defined for any Liouville manifold instead of conic open subsets in the cotangent bundle. See the work of Nadler-Shende.

Remark 3.1. *Given the full dg subcategory $\mathcal{C}_c \subset \mathcal{C}$ of compact objects in a stable cocomplete dg category, the canonical functor $Ind\mathcal{C}_c \to \mathcal{C}$ is an equivalence. Thus we have*

$$Ind\mu Sh^w_\Lambda(\Omega) \cong \mu Sh^\diamond_\Lambda(\Omega). \tag{26}$$

Geometrically, the partially wrapped Fukaya category $\mathcal{W}(X, \mathfrak{f})$ is generated by cocores and linking discs. Denote their endomorphism A_∞-algebra by $\mathcal{W}_{(X,\mathfrak{f})}$. Then the definition above just says that the derived category $D^{perf}\mathcal{W}(X, \mathfrak{f})$ can be defined as the category $D^{perf}(\mathcal{W}_{(X,\mathfrak{f})})$ of perfect modules over the A_∞-algebra $\mathcal{W}_{(X,\mathfrak{f})}$. Note that $D^{perf}(\mathcal{W}_{(X,\mathfrak{f})}) \subset D^{mod}(\mathcal{W}_{(X,\mathfrak{f})})$ is the subcategory of compact objects.

There is a more concrete geometric characterization of wrapped microlocal sheaves. Recall the microstalk functors $\phi_L : \mu Sh^\diamond_\Lambda(\Omega) \to Mod_{\mathbb{K}}$. Note that ϕ_L preserves products, hence admits a left adjoint $\phi_L^\ell : Mod_{\mathbb{K}} \to \mu Sh^\diamond_\Lambda(\Omega)$, and also preserves coproducts, hence ϕ_L^ℓ preserves compact objects.

Definition 3.2. *Define the microlocal skyscraper $\mathcal{F}_L = \phi_L^\ell(\mathbb{K}) \in \mu Sh^w_\Lambda(\Omega)$ to be the object corepresenting the microstalk*

$$\phi_L(\mathcal{F}) \cong \hom(\mathcal{F}_L, \mathcal{F}), \ \mathcal{F} \in \mu Sh^\diamond_\Lambda(\Omega). \tag{27}$$

Lemma 3.1. *$\mu Sh^w_\Lambda(\Omega)$ is split-generated by the microlocal skyscrapers $\mathcal{F}_L \in \mu Sh^w_\Lambda(\Omega)$.*

Proof. By Lemma 2.1, the microlocal skyscrapers $\mathcal{F}_L$ compactly generate $\mu Sh^\diamond_\Lambda(\Omega) \cong Ind\mu Sh^w_\Lambda(\Omega)$. (For any non-trivial object $\mathcal{F}$ of $\mu Sh^\diamond_\Lambda(\Omega)$, there must be some L so that $\hom(\mathcal{F}_L, \mathcal{F}) \neq 0$.) Thus we may invoke the general fact that if a collection of objects of a small stable dg category $\mathcal{C}_c$ generates the ind-category $\mathcal{C} \cong Ind\mathcal{C}_c$, then it split-generates $\mathcal{C}_c$. □

Remark 3.2. *One should think of $\mu Sh^w_\Lambda(\Omega)$ as the (derived) partially wrapped Fukaya category associated to the stopped Liouville manifold $(\Omega, \Lambda \cap \partial_\infty\Omega)$, and $\mu Sh_\Lambda(\Omega)$ the (derived) infinitesimal Fukaya category. The microlocal skyscrapers correspond to cocores and linking discs which intersect the smooth part of $\Lambda \cap \Omega$ transversely at a single point. Geometrically, they are given by the Lagrangian disc $L \subset T^*B$.*

Recall that for conic open subspaces $\Omega \subset T^*Z$, the dg category $\mu Sh^\diamond_\Lambda(\Omega)$ of large microlocal sheaves is the sections of a sheaf $\mu Sh^\diamond_\Lambda$ of dg categories supported along Λ. For an inclusion $\Omega' \subset \Omega$ of conic open subspaces, the restriction functor $\rho : \mu Sh^\diamond_\Lambda(\Omega) \to \mu Sh^\diamond_\Lambda(\Omega')$ preserves products, hence admits a left adjoint $\rho^\ell : \mu Sh^\diamond_\Lambda(\Omega') \to \mu Sh^\diamond_\Lambda(\Omega)$, and also preserves coproducts, hence ρ^ℓ preserves compact objects. Thus its restriction to the subcategory of compact objects defines a natural corestriction functor

$$\rho^w : \mu Sh^w_\Lambda(\Omega') \to \mu Sh^w_\Lambda(\Omega). \tag{28}$$

Proposition 3.1. *The dg categories $\mu Sh^w_\Lambda(\Omega)$ for conic open subspaces $\Omega \subset T^*Z$ and corestriction functors $\rho^w : \mu Sh^w_\Lambda(\Omega') \to \mu Sh^w_\Lambda(\Omega)$ for inclusions $\Omega' \subset \Omega$, assemble into a cosheaf μSh^w_Λ of dg categories supported along Λ. Furthermore, there exists a Whitney stratification of Λ such that the restriction of μSh^w_Λ to each stratum is locally constant.*

Given a closed embedding of conic Lagrangian subvarieties $\Lambda' \subset \Lambda$, there is a natural full embedding $i : \mu Sh^{\diamond}_{\Lambda'} \to \mu Sh^{\diamond}_{\Lambda}$ of sheaves of dg categories. Observe that i preserves products, hence admits a left adjoint $i^{\ell} : \mu Sh^{\diamond}_{\Lambda} \to \mu Sh^{\diamond}_{\Lambda'}$, and also preserves coproducts, so i^{ℓ} preserves compact objects. Thus its restriction to compact objects defines an essentially surjective functor (i.e. surjective on objects up to isomorphism)

$$i^w : \mu Sh^w_{\Lambda} \to \mu Sh^w_{\Lambda'}. \tag{29}$$

The evaluation of i^w on a microlocal skyscraper $\mathcal{F}_L \in \mu Sh^w_{\Lambda}(\Omega)$ is straightforward. If the small Lagrangian ball $L \subset T^*Z$ is centered at a point $(z, \xi) \in \Lambda^{\circ}$ that is not contained in $\Lambda' \subset \Lambda$, then $i^w(\mathcal{F}_L) \cong 0$. If the small Lagrangian ball L is centered at a point $(z, \xi) \in \Lambda^{\circ}$ contained in $\Lambda' \subset \Lambda$, then $i^w(\mathcal{F}_L)$ simply represents the restriction of the microstalk functor to sections of $\mu Sh^{\diamond}_{\Lambda'} \subset \mu Sh^{\diamond}_{\Lambda}$. (Geometrically, what the functor i^w does is sending linking discs of Λ to linking discs of Λ'. Since Λ has more linking discs, i^w is essentially surjective.)

Theorem 3.1. *The natural* hom*-pairing provides an equivalence*

$$\mu Sh_{\Lambda}(\Omega) \cong Fun^{ex}\left(\mu Sh^w_{\Lambda}(\Omega)^{op}, Perf_{\mathbb{K}}\right), \tag{30}$$

where Fun^{ex} denotes the dg category of exact functors, and $Perf_{\mathbb{K}}$ that of perfect $\mathbb{K}$-modules.

Remark 3.3. *While objects of $\mu Sh^w_{\Lambda}(\Omega)$ similarly give functionals on $\mu Sh_{\Lambda}(\Omega)$, it is not in general true that they produce all possible functionals. One could think about the specific example where $\Lambda = S^1 \subset T^*S^1$ is the zero section, and $\Omega = T^*S^1$ is the entire cotangent bundle. Then we have $\mu Sh_{S^1}(T^*S^1) \cong Perf_{prop}(\mathbb{G}_m)$ and $\mu Sh^w_{S^1}(T^*S^1) \cong Coh(\mathbb{G}_m)$. The* hom*-pairing gives an equivalence*

$$Perf_{prop}(\mathbb{G}_m) \cong Fun^{ex}\left(Coh(\mathbb{G}_m)^{op}, Perf_{\mathbb{K}}\right). \tag{31}$$

Clearly there are more functionals on $Perf_{prop}(\mathbb{G}_m)$ then those coming from $Coh(\mathbb{G}_m)$. For example, one could take the hom*-pairing with a direct sum of skyscraper sheaves at infinitely many points.*

Remark 3.4. *In terms of symplectic topology, Theorem 3.1 is a version of the Eilenberg-Moore equivalence between the partially wrapped Fukaya category and the infinitesimal Fukaya category associated to the stopped Liouville manifold $(\Omega, \Lambda \cap \partial_{\infty}\Omega)$. Choosing Λ to be the zero section, $\Lambda \cap \partial_{\infty}\Omega = \varnothing$, the infinitesimal Fukaya category becomes the compact Fukaya category, and the partially wrapped Fukaya category becomes the fully wrapped Fukaya category. In general, this Eilenberg-Moore equivalence does not give the Koszul duality between Fukaya categories, a typical example is $\Omega = T^*S^1$.*

On the mirror side, Koszul duality between $Coh(X)$ and $Perf_{prop}(X)$ holds for proper schemes X.

Proof of Theorem 3.1. First, let us observe that it suffices to prove the assertion locally. If one choose a cover $\{\Omega_i\}_{i \in I}$ of Ω by conic open subspaces, since $\mu Sh_{\Lambda}(\Omega)$ is a sheaf and $\mu Sh^w_{\Lambda}(\Omega)$ is a cosheaf we have

$$\mu Sh_{\Lambda}(\Omega) \cong \lim_{i \in I} \mu Sh_{\Lambda}(\Omega_i), \;\; \mu Sh^w_{\Lambda}(\Omega) \cong \mathrm{colim}_{i \in I} \mu Sh^w_{\Lambda}(\Omega_i). \tag{32}$$

Thus if we have the assertion locally, i.e.

$$\mu Sh_{\Lambda}(\Omega_i) \xrightarrow{\cong} Fun^{ex}\left(\mu Sh^w_{\Lambda}(\Omega_i)^{op}, Perf_{\mathbb{K}}\right), \tag{33}$$

then we have it globally

$$\begin{aligned}\mu Sh_\Lambda(\Omega) &\cong \lim_{i\in I}\mu Sh_\Lambda(\Omega_i) \cong \lim_{i\in I} Fun^{ex}\left(\mu Sh^w_\Lambda(\Omega_i)^{op}, Perf_{\mathbb{K}}\right)\\ &\cong Fun^{ex}\left(\mathrm{colim}_{i\in I}\mu Sh^w_\Lambda(\Omega_i)^{op}, Perf_{\mathbb{K}}\right) \cong Fun^{ex}\left(\mu Sh^w_\Lambda(\Omega)^{op}, Perf_{\mathbb{K}}\right).\end{aligned} \tag{34}$$

We may assume that $\Omega \subset T^*Z$ is the cone over a small open ball $\Omega^\infty \subset S^\infty Z$ centered at a point of $\Lambda^\infty \subset S^\infty Z$.

We may deform $\Lambda^\infty \subset S^\infty Z$ to a Legendrian subvariety $\Lambda^\infty_{arb} \subset S^\infty Z$ with arboreal singularities. Taking Λ_{arb} to be the cone over Λ^∞_{arb}, we have an equivalence $\mu Sh^\diamond_\Lambda(\Omega) \cong \mu Sh^\diamond_{\Lambda_{arb}}(\Omega)$, restricting to an equivalence $\mu Sh_\Lambda(\Omega) \cong \mu Sh_{\Lambda_{arb}}(\Omega)$. Passing to the compact objects in the first equivalence, we have $\mu Sh^w_\Lambda(\Omega) \cong \mu Sh^w_{\Lambda_{arb}}(\Omega)$.

Thus we may assume that the conic Lagrangian subvariety $\Lambda \subset T^*Z$ has arboreal singularities. Moreover, we may further assume that $\Omega \subset T^*Z$ is the cone over a small open ball $\Omega^\infty \subset S^\infty Z$ centered at a point of $\Lambda^\infty \subset S^\infty Z$ that is an arboreal singularity. In this situation, $\mu Sh^\diamond_\Lambda(\Omega)$ is equivalent to the dg category $Mod_{\mathbb{K}}(T)$ of modules over a directed tree T, and $\mu Sh_\Lambda(\Omega)$ is equivalent to the dg category $Perf_{\mathbb{K}}(T)$ of perfect modules. Passing to compact objects under the first equivalence, we have that $\mu Sh^w_\Lambda(\Omega)$ is also equivalent to $Perf_{\mathbb{K}}(T)$.

Finally, for perfect modules over a directed tree, it is straightforward to check that the hom-pairing provides an equivalence

$$Perf_{\mathbb{K}}(T) \cong Fun^{ex}\left(Perf_{\mathbb{K}}(T)^{op}, Perf_{\mathbb{K}}\right). \tag{35}$$

□

Remark 3.5. *Locally, the dg categories $\mu Sh_\Lambda(\Omega)$ and $\mu Sh^w_\Lambda(\Omega)$, which correspond respectively to the infinitesimal Fukaya category and the partially wrapped Fukaya category, are Koszul dual. Moreover, these two categories are smooth and proper, therefore "self-dual" in the derived sense. This explains the fact that their derived categories are both equivalent to $Perf_{\mathbb{K}}(T)$ in the above argument.*

4 Lagrangian skeleton

By the n-dimensional *pair-of-pants*, we mean the Liouville manifold (P_n, α_{P_n}) given by the generic hyperplane

$$P_n = \{1 + z_1 + \cdots + z_{n+1} = 0\} \subset T^{n+1}_{\mathbb{C}}, \tag{36}$$

equipped with the restriction of the Liouville form α_n on T^*T^{n+1}. The Lagrangian skeleton of P_n can be described by the combinatorics of the permutahedron.

To get a more convenient Lagrangian skeleton, we need to break the symmetry and work with the pair-of-pants in a slightly modified form where we alter its embedding near infinity. This is called *tailored pair-of-pants*, and we denote it by (Q_n, α_{Q_n}).

To provide the tailored pair-of-pants a particularly simple skeleton, it will be useful to break the symmetry and apply a natural isotopy to its Liouville structure. For $x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}$, consider the family of Liouville structures on T^*T^{n+1} given by

$$\alpha^x_n = \sum_{a=1}^{n+1}(\xi_a - x_a)d\theta_a, \omega^x_n = d\alpha^x_n = \sum_{a=1}^{n+1}(\xi_a - x_a)d\theta_a. \tag{37}$$

The restriction of α^x_n and ω^x_n to the pair-of-pants P_n provide a family of Liouville structures. We may construct the tailored pair-of-pants $Q_n \subset T^{n+1}_{\mathbb{C}}$ so that the restricted Liouville form $\alpha^x_{Q_n} = \alpha^x_n|_{Q_n}$ provide a family of Liouville structures as well. Choose

$\ell \gg 0$, and let $x_\ell = (-\ell, \cdots, -\ell) \in \mathbb{R}^{n+1}$. Let us focus on the Liouville structure on Q_n given by

$$\beta_{Q_n} := \alpha_{Q_n}^{x_\ell} = \left(\sum_{a=1}^{n+1} (\xi_a + \ell) d\theta_a\right)|_{Q_n}. \tag{38}$$

Write $L_n \subset Q_n$ for the resulting skeleton, we will describe its geometry.

Let $S^1_\Delta \subset T^{n+1}$ be the diagonal circle. The translation S^1_Δ-action on T^{n+1} induces a Hamiltonian S^1-action on T^*T^{n+1}, with moment map

$$\mu_\Delta : T^*T^{n+1} \to \mathbb{R},\ \mu_\Delta(\theta_1, \xi_1, \cdots, \theta_{n+1}, \xi_{n+1}) = \sum_{a=1}^{n+1} \xi_a. \tag{39}$$

Consider the quotient $\mathbb{T}^n = T^{n+1}/S^1_\Delta$ consisting of $(n+1)$-tuples $[\theta_1, \cdots, \theta_{n+1}]$ taken up to simultaneous translation. If we distinguish the last entry, then we obtain an identification $\mathbb{T}^n \cong T^n$ via the coordinates $\theta_a - \theta_{n+1}$, where $1 \leqslant a \leqslant n$.

Let $\mathfrak{t}_n^* = \left\{\sum_{a=1}^{n+1} \xi_a = 0\right\} \subset \mathbb{R}^{n+1}$ be the dual of the Lie algebra of $\mathbb{T}^n$. We have the identification $T^*\mathbb{T}^n \cong \mathbb{T}^n \times \mathfrak{t}_n^*$. In terms of coordinates, a point of $T^*\mathbb{T}^n$ can be represented by $([\theta_1, \cdots, \theta_{n+1}], (\xi_1, \cdots, \xi_{n+1}))$, where $\sum_{a=1}^{n+1} \xi_a = 0$.

For $\chi \in \mathbb{R}$, we have a twisted Hamiltonian reduction correspondence

$$T^*T^{n+1} \xleftarrow{q_\chi} \mu_\Delta^{-1}(\chi) \xrightarrow{p_\chi} T^*\mathbb{T}^n, \tag{40}$$

where q_χ is the inclusion of level set, while p_χ is the translation projection

$$p_\chi((\theta_1, \cdots, \theta_{n+1}), (\xi_1, \cdots, \xi_{n+1})) := ([\theta_1, \cdots, \theta_{n+1}], (\xi_1 - \hat{\chi}, \cdots, \xi_{n+1} - \hat{\chi})), \tag{41}$$

where $\hat{\chi} = \chi/(n+1)$. In particular, when $\chi = 0$, we recover the usual Hamiltonian reduction correspondence

$$T^*T^{n+1} \xleftarrow{q_0} T^*_{S^1_\Delta} T^{n+1} \xrightarrow{p_0} T^*\mathbb{T}^n, \tag{42}$$

where $T^*_{S^1_\Delta} T^{n+1} \subset T^*T^{n+1}$ is the conormal bundle.

Introduce the conic Lagrangian subvariety

$$\Lambda_1 := \{(\theta, 0) | \theta \in S^1\} \cup \{(0, \xi) | \xi \in \mathbb{R}_{\geqslant 0}\} \subset T^*S^1, \tag{43}$$

and the product conic Lagrangian subvariety

$$\Lambda_{n+1} := (\Lambda_1)^{n+1} \subset T^*T^{n+1}. \tag{44}$$

Note that $\Lambda_{n+1} \subset \mu_\Delta^{-1}(\mathbb{R}_{\geqslant 0})$, and that Λ_{n+1} and $\mu_\Delta^{-1}(\chi)$ are transverse for $\chi > 0$. Fix some $\chi > 0$, define the Lagrangian subvariety

$$\mathfrak{L}_n := p_\chi\left(q_\chi^{-1}(\Lambda_{n+1})\right) \subset T^*\mathbb{T}^n. \tag{45}$$

Remark 4.1. *We do not include χ in the notation for $\mathfrak{L}_n$ as we will eventually specialize to the case $\chi = n + 1$.*

To describe $\mathfrak{L}_n \subset T^*\mathbb{T}^n$, consider the moment map $\mu_{n+1} : T^*T^{n+1} \to \mathbb{R}^{n+1}$ of the Hamiltonian T^{n+1}-action and restrict it to $\Lambda_{n+1} \subset T^*T^{n+1}$. Note that $\mu_{n+1}(\Lambda_{n+1}) = \mathbb{R}^{n+1}_{\geqslant 0}$. For $I \subset \{1, \cdots, n+1\}$, consider the relatively open coordinate cone

$$\sigma_I = \{\xi_a = 0, \xi_b > 0 | a \in I, b \notin I\} \subset \mathbb{R}^{n+1}_{\geqslant 0}. \tag{46}$$

For $x \in \sigma_I$, $\mu_{n+1}^{-1}(x) \cap \Lambda_{n+1}$ is the orthogonal coordinate subtorus

$$T^I = \{\theta_a = 0 | a \notin I\} \subset T^{n+1}. \tag{47}$$

Consider the closed simplex

$$\widetilde{\Xi}(\chi) = \left\{ (\xi_1, \cdots, \xi_{n+1}) | \xi_a \geqslant 0 \text{ for } 1 \leqslant a \leqslant n+1, \sum_{a=1}^{n+1} \xi_a = \chi \right\} \subset \mathbb{R}^{n+1}_{\geqslant 0}. \tag{48}$$

Note that the projection p_χ restricts to an isomorphism

$$\mu_\Delta^{-1}(\chi) \cap \Lambda_{n+1} = \mu_{n+1}^{-1}\left(\widetilde{\Xi}(\chi)\right) \xrightarrow{\cong} \mathfrak{L}_n \tag{49}$$

since for any point of $\mu_\Delta^{-1}(\chi) \cap \Lambda_{n+1}$, we must have $\xi_a > 0$ and hence $\theta_a = 0$ for some $a \in \{1, \cdots, n+1\}$, so that no points are identified by the S^1_Δ-translations.

For a proper subset $I \subset \{1, \cdots, n+1\}$, consider the relatively open subsimplex

$$\widetilde{\Xi}_I(\chi) = \widetilde{\Xi}_n(\chi) \cap \sigma_I. \tag{50}$$

Then p_χ restricts to an isomorphism

$$\bigcup_I T^I \times \widetilde{\Xi}_I(\chi) \xrightarrow{\cong} \mathfrak{L}_n, \tag{51}$$

where we take the union over non-empty $I \subset \{1, \cdots, n+1\}$. Note that when $n = 2$, $\mathfrak{L}_n$ is the union of two circles and an open interval.

Theorem 4.1. *There is an open neighborhood $U_n \subset Q_n$ of the Lagrangian skeleton $L_n \subset Q_n$ and an open symplectic embedding*

$$j : U_n \to T^*\mathbb{T}^n \tag{52}$$

which restricts to an isomorphism

$$j|_{L_n} : L_n \xrightarrow{\cong} \mathfrak{L}_n. \tag{53}$$

5 Contactification and symplectization

By Theorem 4.1, the symplectic geometry of a neighborhood $U_n \subset Q_n$ of $L_n \subset Q_n$ is equivalent to that of a neighborhood $\mathfrak{U}_n \subset T^*\mathbb{T}^n$ of $\mathfrak{L}_n \subset T^*\mathbb{T}^n$.

We introduce the Liouville form β_n on the neighborhood $\mathfrak{U}_n \subset T^*\mathbb{T}^n$ obtained by transporting the Liouville form β_{Q_n} restricted to the neighborhood $U_n \subset Q_n$. Thus β_n provides a primitive to the restriction of the canonical symplectic form $\omega_{T^*\mathbb{T}^n}|_{\mathfrak{U}_n} = d\beta_n$. The Lagrangian subvariety $\mathfrak{L}_n \subset T^*\mathbb{T}^n$ is conic with respect to its associated Liouville vector field.

In general, let M be a Liouville manifold with Liouville form α_M. The *circular contactification* of M is the contact manifold $N = M \times S^1$, with contact form $\lambda_N = dt + \alpha_M$, and contact structure $\xi_N = \ker(\lambda_N)$. The *contactification* of M is the contact manifold $\widetilde{N} = M \times \mathbb{R}$, with contact form $\lambda_{\widetilde{N}} = dt + \alpha_M$, and contact structure $\xi_{\widetilde{N}} = \ker(\lambda_{\widetilde{N}})$. Note that there is a natural contact $\mathbb{Z}$-cover $\widetilde{N} \to N$.

Definition 5.1. *A Lagrangian subvariety $L \subset M$ is integral if there is a continuous function $f : L \to S^1$ such that the restriction of f to any submanifold of L is differentiable and a primitive for the restriction of α_M.*

A Lagrangian subvariety $L \subset M$ is exact if in addition there exists a lift of $f : L \to S^1$ to a continuous function $\tilde{f} : L \to \mathbb{R}$.

Remark 5.1. *A Lagrangian subvariety $L \subset M$ is integral if and only if it admits a Legendrian lift $\mathcal{L} \subset N$. Similarly, $L \subset M$ is exact if and only if it admits a Legendrian lift $\widetilde{\mathcal{L}} \subset \widetilde{N}$.*

Return to the neighborhood $\mathfrak{U}_n \subset T^*\mathbb{T}^n$. It admits two Liouville forms: β_n and the canonical Liouville form $\alpha_{T^*\mathbb{T}^n}$. The Lagrangian subvariety $\mathfrak{L}_n \subset \mathfrak{U}_n$ is conic with respect to the Liouville vector field associated to β_n, and thus exact with respect to β_n. On the other hand, if we construct $\mathfrak{L}_n$ using $\chi > 0$ with $\hat{\chi} = \chi/(n+1)$ integral, then the function

$$\tilde{f} : \mu_\Delta^{-1}(\chi) \cap \Lambda_{n+1} \to S^1, \tilde{f} = \sum_{a=1}^{n+1} (\xi_a - \hat{\chi})\theta_a \tag{54}$$

is invariant under S^1_Δ-translations, hence descends to a function $f : \mathfrak{L}_n \to S^1$. A straightforward computation shows that f provides an integral structure of $\mathfrak{L}_n$ for $\alpha_{T^*\mathbb{T}^n}$.

Consider the circular contactification (N_n, λ_n) of $\mathfrak{U}_n$. Denote by $\mathcal{L}_n$ the Legendrian lift of $\mathfrak{L}_n$ to (N_n, λ_n).

From now on we further specialize to $\chi = n+1$ so that $\hat{\chi} = 1$. Introduce the conic open subspace and its spherical projectivization

$$\Omega_{n+1} = \mu_\Delta^{-1}(\mathbb{R}_{>0}) \subset T^*T^{n+1}, \Omega_{n+1}^\infty = \Omega_{n+1}/\mathbb{R}_{>0} \subset S^\infty T^{n+1}. \tag{55}$$

The natural projection gives an isomorphism of contact manifolds $\mu_\Delta^{-1}(\chi) \cong \Omega_{n+1}^\infty$.

Lemma 5.1. *We have a finite contact cover*

$$\mathfrak{p}_\chi : \Omega_{n+1}^\infty \cong \mu_\Delta^{-1}(\chi) \to T^*\mathbb{T}^n \times S^1 \tag{56}$$

given by $\mathfrak{p}_\chi = p_\chi \times \delta$, where $\delta : T^{n+1} \to S^1$ is the diagonal character.

The cover is trivializable over the neighborhood $N_n \subset T^*\mathbb{T}^n \times S^1$ of the Legendrian $\mathcal{L}_n \subset N_n$ with a canonical section $s : N_n \to \Omega_{n+1}^\infty$ such that $s(\mathcal{L}_n) = \Lambda_{n+1}^\infty$.

It follows that the contact geometry of the circular contactification $N_n \subset T^*\mathbb{T}^n \times S^1$ near the Legendrian lift $\mathcal{L}_n$ is equivalent to that of the open subspace $\Omega_{n+1}^\infty \subset S^\infty T^{n+1}$ near the Legendrian subvariety Λ_{n+1}^∞.

Introduce the circular contactification $Q_n \times S^1$, and its symplectization $\widetilde{Q}_n = Q_n \times S^1 \times \mathbb{R}$, with their natural projections

$$\widetilde{Q}_n = Q_n \times S^1 \times \mathbb{R} \xrightarrow{s} Q_n \times S^1 \xrightarrow{c} Q_n. \tag{57}$$

The Lagrangian skeleton $L_n \subset Q_n$ lifts under c to the Legendrian subvariety $L_n \times \{0\} \subset Q_n \times S^1$, and we can take its inverse image under s to obtain a conic Lagrangian subvariety

$$\widetilde{L}_n = s^{-1}(L_n \times \{0\}) \subset \widetilde{Q}_n. \tag{58}$$

The following is a consequence of Theorem 4.1.

Theorem 5.1. *Fix $\chi = n+1$. There is a conic open neighborhood $\widetilde{U}_n \subset \widetilde{Q}_n$ of the Lagrangian subvariety $\widetilde{L}_n \subset \widetilde{Q}_n$, a conic open neighborhood $\Upsilon_{n+1} \subset \Omega_{n+1}$ of the intersection $\Lambda_{n+1} \cap \Omega_{n+1}$, and an exact symplectomorphism*

$$\tilde{j} : \widetilde{U}_n \xrightarrow{\cong} \Upsilon_{n+1} \tag{59}$$

restricting to an isomorphism

$$\tilde{j}|_{\widetilde{L}_n} : \widetilde{L}_n \xrightarrow{\cong} \Lambda_{n+1} \cap \Omega_{n+1}. \tag{60}$$

6 Mirror symmetry

We calculate (what is supposed to be) the dg category of wrapped microlocal sheaves on the pair-of-pants. Note that Theorem 5.1 allows us to define the dg category $\mu Sh_{L_n}(Q_n)$ of wrapped microlocal sheaves on Q_n supported along L_n to be the dg category of wrapped microlocal sheaves on Ω_{n+1} supported along Λ_{n+1}.

Lemma 6.1. *There are mirror equivalences*

$$Sh^{\diamond}_{\Lambda_{n+1}}(T^{n+1}) \cong QCoh(\mathbb{A}^{n+1}), \tag{61}$$

$$Sh_{\Lambda_{n+1}}(T^{n+1}) \cong Perf_{prop}(\mathbb{A}^{n+1}), \tag{62}$$

$$Sh^{w}_{\Lambda_{n+1}}(T^{n+1}) \cong Coh(\mathbb{A}^{n+1}). \tag{63}$$

Remark 6.1. *Geometrically, $Sh_{\Lambda_1}(S^1)$ and $Sh^w_{\Lambda_1}(S^1)$ correspond respectively to the infinitesimal and partially wrapped Fukaya categories associated to the Landau-Ginzburg model $(\mathbb{C}^*, z)$, which is the mirror of $\mathbb{A}^1$.*

Fix a subset $I \subset \{1, \cdots, n+1\}$, with complement $I^c = \{1, \cdots, n+1\} \backslash I$. Let $T^I \subset T^{n+1}$ be the subtorus defined by $\theta_a = 0$ for $a \in I^c$. Let $\Lambda_I = (\Lambda_1)^I \subset T^*T^I$ be the product conic Lagrangian subvariety.

Consider the hyperbolic restriction

$$\eta_I : Sh^{\diamond}_{\Lambda_{n+1}}(T^n) \to Sh^{\diamond}_{\Lambda_I}(T^I), \ \eta_I(\mathcal{F}) := p_* q^! \mathcal{F} \tag{64}$$

built from the correspondence

$$T^I \xleftarrow{p} T^I \times [0, 1/2)^{I^c} \xrightarrow{q} T^{n+1}, \tag{65}$$

where p is the projection and q is the inclusion.

Let $f : \mathbb{A}^I = \mathrm{Spec}\mathbb{K}[t_a | a \in I] \to \mathbb{A}^n = \mathrm{Spec}\mathbb{K}[t_1, \cdots, t_n]$ be the affine subspace defined by $t_a = 0$ for $a \in I^c$.

Lemma 6.2. *The equivalences (61) and (62) fit into commutative diagrams*

$$\begin{array}{ccc} Sh^{\diamond}_{\Lambda_{n+1}}(T^{n+1}) & \xrightarrow{\cong} & QCoh(\mathbb{A}^{n+1}) \\ {\scriptstyle \eta_I}\downarrow & & \downarrow{\scriptstyle f^*} \\ Sh^{\diamond}_{\Lambda_I}(T^I) & \xrightarrow{\cong} & QCoh(\mathbb{A}^I) \end{array} \tag{66}$$

$$\begin{array}{ccc} Sh_{\Lambda_{n+1}}(T^{n+1}) & \xrightarrow{\cong} & Perf_{prop}(\mathbb{A}^{n+1}) \\ {\scriptstyle \eta_I}\downarrow & & \downarrow{\scriptstyle f^*} \\ Sh_{\Lambda_I}(T^I) & \xrightarrow{\cong} & Perf_{prop}(\mathbb{A}^I) \end{array} \tag{67}$$

Theorem 6.1. *There are mirror equivalences*

$$\mu Sh^{\diamond}_{\Lambda_{n+1}}(\Omega_{n+1}) \cong IndCoh(X_n), \tag{68}$$

$$\mu Sh_{\Lambda_{n+1}}(\Omega_{n+1}) \cong Perf_{prop}(X_n), \tag{69}$$

$$\mu Sh^{w}_{\Lambda_{n+1}}(\Omega_{n+1}) \cong Coh(X_n). \tag{70}$$

Proof. Let $\mathcal{I}_{n+1}$ be the category whose objects are subsets $I \subset \{1, \cdots, n+1\}$, and morphisms $I \to I'$ are inclusions $I \subset I'$. Let $\mathcal{I}^{\circ}_{n+1} \subset \mathcal{I}_{n+1}$ denote the full subcategory whose objects are proper subsets of $\{1, \cdots, n+1\}$.

Define a functor $A : \mathcal{I}^{\circ}_{n+1} \to \mathcal{X}_{\mathbb{K}}$ as follows. For an object I of $\mathcal{I}^{\circ}_{n+1}$, take the affine space $A(I) = \mathbb{A}^I$, and for a morphism $I \subset I'$, take the inclusion $A(I, I') : \mathbb{A}^I \to \mathbb{A}^{I'}$, given by setting $t_a = 0$ for each $a \in I' \backslash I$.

Recall the functor $IndCoh^* : \mathcal{X}^{op}_{\mathbb{K}} \to dgSt_{\mathbb{K}}$ that assigns a scheme its ind-coherent sheaves and a proper morphism of schemes its $*$-pullback. Recall also the full subfunctor $Perf^*_{prop} : \mathcal{X}^{op}_{\mathbb{K}} \to dgst_{\mathbb{K}}$ of perfect complexes with proper support.

Consider the composite functor $IndCoh^* \circ A : (\mathcal{I}^\circ_{n+1})^{op} \to dgSt_{\mathbb{K}}$, and $Perf^*_{prop} \circ A(\mathcal{I}^\circ_{n+1})^{op} \to dgst_{\mathbb{K}}$. By Proposition 1.2, the canonical maps are equivalences

$$IndCoh(X_n) \xrightarrow{\cong} \lim_{(\mathcal{I}^\circ_{n+1})^{op}} IndCoh(\mathbb{A}^I), \tag{71}$$

$$Perf_{prop}(X_{n+1}) \xrightarrow{\cong} \lim_{(\mathcal{I}^\circ_{n+1})^{op}} Perf_{prop}(\mathbb{A}^I), \tag{72}$$

$$Coh(X_n) \xleftarrow{\cong} \operatorname{colim}_{\mathcal{I}^\circ_{n+1}} Coh(\mathbb{A}^I). \tag{73}$$

To prove the theorem, we will similarly identify $\mu Sh^\diamond_{\Lambda_{n+1}}(\Omega_{n+1})$ as the limit of a functor

$$\mu Sh^\diamond : (\mathcal{I}^\circ_{n+1})^{op} \to dgSt_{\mathbb{K}}, \tag{74}$$

and then provide an equivalence of functors $\mu Sh^\diamond \simeq IndCoh^* \circ A$. This will immediately prove the first and the third equivalences. For the second one, we observe that $\mu Sh_{\Lambda_{n+1}}(\Omega_{n+1})$ is the limit of a full subfunctor $\mu Sh \subset \mu Sh^\diamond$, which is equivalent to the subfunctor $Perf^*_{prop} \circ A \subset IndCoh^* \circ A$.

For each $I \in \mathcal{I}^\circ_{n+1}$, introduce the conic open subspace $\Omega_I \subset \Omega_{n+1}$, cut out by the additional requirement $\xi_a \neq 0$ for $a \notin I$. Thus for $I \subset I'$, we have the open inclusion $\Omega_I \subset \Omega_{I'}$, and for $I = \{1, \cdots, n+1\}$, we have $\Omega_I = \Omega_{n+1}$. Note that the collection $\{\Omega_I\}_{I \in \mathcal{I}^\circ_{n+1}}$ forms a conic open cover of Ω_{n+1} with the property $\Omega_{I \cap I'} = \Omega_I \cap \Omega_{I'}$. Define the functor $\mu Sh^\diamond$ by

$$\mu Sh^\diamond(I) = \mu Sh^\diamond_{\Lambda_{n+1}}(\Omega_I), \tag{75}$$

with inclusions $I \subset I'$ taken to the restriction maps along the inclusions $\Omega_I \subset \Omega_{I'}$. Define the full subfunctor $\mu Sh \subset \mu Sh^\diamond$ by $\mu Sh(I) = \mu Sh_{\Lambda_{n+1}}(\Omega_I)$.

Since $\mu Sh^\diamond_{\Lambda_{n+1}}$ forms a sheaf, $\mu Sh_{\Lambda_{n+1}} \subset \mu Sh^\diamond_{\Lambda_{n+1}}$ is a full subsheaf, and $\{\Omega_I\}_{I \in \mathcal{I}^\circ_{n+1}}$ is an open conic cover of Ω_{n+1}, the canonical functors are equivalences

$$\mu Sh^\diamond_{\Lambda_{n+1}}(\Omega_{n+1}) \xrightarrow{\cong} \lim_{(\mathcal{I}^\circ_{n+1})^{op}} \mu Sh^\diamond_{\Lambda_{n+1}}(\Omega_I), \tag{76}$$

$$\mu Sh_{\Lambda_{n+1}}(\Omega_{n+1}) \xrightarrow{\cong} \lim_{(\mathcal{I}^\circ_{n+1})^{op}} \mu Sh_{\Lambda_{n+1}}(\Omega_I). \tag{77}$$

Next let us define an additional functor to interpolate between $IndCoh^* \circ A$ and $\mu Sh^\diamond$.

For $I \in \mathcal{I}^\circ_{n+1}$, define the functor

$$Sh^\diamond : (\mathcal{I}^\circ_{n+1})^{op} \to dgSt_{\mathbb{K}}, \; Sh^\diamond(I) = Sh^\diamond_{\Lambda_I}(T^I) \tag{78}$$

with inclusions $I \subset I'$ taken to the hyperbolic restrictions

$$\eta_{I \subset I'} : Sh_{\Lambda'_I} \diamond (T^{I'}) \to Sh^\diamond_{\Lambda_I}(T^I), \; \eta_{I \subset I'}(\mathcal{F}) = p_* q^! \mathcal{F} \tag{79}$$

built from the correspondence

$$T^I \xleftarrow{p} T^I \times [0, 1/2)^{I' \setminus I} \xrightarrow{q} T^{I'}, \tag{80}$$

where p is the projection and q is the inclusion. Define the full subfunctor $Sh \subset Sh^\diamond$ by $Sh(I) = Sh_{\Lambda_I}(T^I)$.

Lemmas 6.1 and 6.2 imply that we have equivalences

$$Sh^\diamond \simeq IndCoh^* \circ A, \; Sh \simeq Perf^*_{prop} \circ A. \tag{81}$$

It remains to establish equivalences of functors

$$Sh^\diamond \simeq \mu Sh^\diamond, \; Sh \simeq \mu Sh. \tag{82}$$

For any $I \in \mathcal{I}^\circ_{n+1}$, let us return to the hyperbolic restriction

$$\eta_I : Sh^\diamond_{\Lambda_{n+1}}(T^{n+1}) \to Sh_{\Lambda_I}(T^I), \; \eta_I(\mathcal{F}) = p_* q^! \mathcal{F}. \tag{83}$$

First, η_I factors through the microlocalization

$$Sh^{\diamond}_{\Lambda_{n+1}}(T^{n+1}) \to \mu Sh^{\diamond}_{\Lambda_I}(\Omega_I) \xrightarrow{\tilde{\eta}_I} Sh_{\Lambda_I}(T^I) \tag{84}$$

since the hyperbolic restriction in the coordinate direction indexed by $a \in T^c$ vanishes on sheaves whose singular support does not intersect the locus $\{\xi_a > 0\} \subset T^*T^{n+1}$.

Next, for $I \subset I'$, the induced functors extend to natural commutative diagrams

$$\begin{array}{ccc} \mu Sh^{\diamond}_{\Lambda'_I}(\Omega_{I'}) & \xrightarrow{\tilde{\eta}_{I'}} & Sh^{\diamond}_{\Lambda'_I}(T^{I'}) \\ {\scriptstyle \rho_{I\subset I'}}\downarrow & & \downarrow{\scriptstyle \eta_{I\subset I'}} \\ \mu Sh^{\diamond}_{\Lambda_I}(\Omega_I) & \xrightarrow{\tilde{\eta}_I} & Sh^{\diamond}_{\Lambda_I}(T^I) \end{array} \tag{85}$$

Thus we have a map of functors $\tilde{\eta} : \mu Sh^{\diamond} \to Sh^{\diamond}$, restricting to a map of subfunctors $\mu Sh \to Sh$. It remains to show that $\tilde{\eta}$ is an equivalence. It suffices to show that

$$\tilde{\eta}_I : \mu Sh^{\diamond}_{\Lambda_I}(\Omega_I) \to Sh^{\diamond}_{\Lambda_I}(T^I) \tag{86}$$

is an equivalence for any $I \in \mathcal{I}^{\circ}_{n+1}$. Note that it admits a inverse induced by the pushforward

$$j_{I*} : Sh^{\diamond}_{\Lambda_I}(T^I) \to Sh^{\diamond}_{\Lambda_{n+1}}(T^{n+1}) \tag{87}$$

along the natural inclusion $j_I : T^I \to T^{n+1}$. To see this, note that j_I is simply the product of inclusions in the coordinate directions indexed by I^c, and the identity in the coordinate directions indexed by I. □

Corollary 6.1. *There is a quasi-equivalence of differential $\mathbb{Z}/2$-graded categories*

$$\mu Sh^{w}_{\Lambda_{n+1}}(\Omega_{n+1})_{\mathbb{Z}/2} \cong MF(\mathbb{A}^{n+2}, W_{n+2}). \tag{88}$$

For a $\mathbb{Z}$-graded version of the above equivalence proved for the actual wrapped Fukaya category $\mathcal{W}(P_n)$, see Lekili-Polischuk.