LEGENDRIAN KNOTS AND CONSTRUCTIBLE SHEAVES

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The main reference of this talk is the paper [STZ17].

1. Definitions and context

Throughout this talk, let k be a commutative ring and M a real analytic manifold. We use the following definitions.

$$\begin{aligned}
\mathrm{Sh}_{\mathrm{naive}}(M) &:= \text{chain complexes of sheaves of } k\text{-modules on } M \\
&\quad\ \text{whose cohomology is constructible and has perfect stalks} \\
\mathrm{Sh}(M) &:= \mathrm{Sh}_{\mathrm{naive}}(M)/\text{acyclic complexes} \\
\mathrm{Sh}_{\mathcal{S}}(M) &:= \mathrm{Sh}(M) \text{ where cohomology of each object is} \\
&\quad\ \text{constructible wrt the stratification } \mathcal{S} \text{ of } M \\
\mathrm{Sh}_L(M) &= \mathrm{Sh}(M) \text{ where sheaves have singular support in} \\
&\quad\ \text{the closed conical subset } L \subset T^*M \\
\mathrm{Sh}_\Lambda(M) &= \mathrm{Sh}_{\mathbb{R}_{>0}\Lambda \cup 0_M}(M) \text{ where } \Lambda \subset ST^*M \text{ is Legendrian}
\end{aligned}$$

Last time Juan explained to us that the results in [NZ09, Nad09] give a quasi-equivalence

$$\mathrm{Sh}(M) \cong \mathrm{Fuk}(T^*M),$$

where $\mathrm{Fuk}(T^*M)$ is the "infinitesimally wrapped Fukaya category" of T^*M. In particular there is a quasi-equivalence $\mathrm{Sh}_\Lambda(M) \cong \mathrm{Fuk}_\Lambda(T^*M)$ where $\Lambda \subset ST^*M$ is a Legendrian, where $\mathrm{Fuk}_\Lambda(T^*M)$ has objects being exact conical Lagrangians asymptotic to the fixed Legendrian $\Lambda \subset ST^*M$ at infinity.

The goal of this talk is to give a combinatorial description $\mathrm{Sh}_\Lambda(M)$ from the point of view of the Legendrian $\Lambda \subset ST^*M$, in the cases $M = \mathbb{R}^2$ or $M = S^1 \times \mathbb{R}$.

$$\mathrm{Sh}_\Lambda(M)_0 := \mathrm{Sh}_\Lambda(M) \text{ where sheaves have acyclic stalks for } z \ll 0$$

More precisely, we will discuss the proof of some of the following theorems.

Theorem 1.1 ([STZ17])**.** *A contactomorphism inducing a Legendrian isotopy $\Lambda \simeq \Lambda'$ induces a quasi-equivalence $\mathrm{Sh}_\Lambda(M) \xrightarrow{\sim} \mathrm{Sh}_{\Lambda'}(M)$. This quasi-equivalence preserves the subcategory $\mathrm{Sh}_\Lambda(M)_0$.*

Theorem 1.2 ([STZ17])**.** *If Λ is a stabilized Legendrian knot (see Definition 2.4 for the definition), then every element of $\mathrm{Sh}_\Lambda(M)$ is locally constant. In particular $\mathrm{Sh}_\Lambda(M)_0 = 0$.*

Theorem 1.3 ([STZ17])**.** *Every element of $\mathrm{Sh}_\Lambda(M)$ is periodic with period $2\,\mathrm{rot}(\Lambda)$; in particular, if $\mathrm{rot}(\Lambda) \neq 0$, then there are no bounded complexes of sheaves in $\mathrm{Sh}_\Lambda(M)$.*

Let $\mathcal{C}_1(\Lambda) \subset \mathrm{Sh}_\Lambda(M)_0$ be the subcategory of objects with "microlocal rank 1" (see Definition 4.6). Associated to the Legendrian Λ is a certain A_∞-category called the *augmentation category* which is denoted by $\mathrm{Aug}_+(\Lambda)$. Its objects are dg-algebra maps $\varepsilon \colon CE^*(\Lambda) \longrightarrow \mathbb{Z}$, where $CE^*(\Lambda)$ is the Chekanov–Eliashberg dg-algebra.

Theorem 1.4 ([NRS[+]20])**.** *There exists an equivalence of A_∞-categories $\mathrm{Aug}_+(\Lambda) \cong \mathcal{C}_1(\Lambda)$.*

2. Legendrian knots

We now focus on the case $M = \mathbb{R}^2_{x,z}$, and Legendrian knots $\Lambda \subset ST^*\mathbb{R}^2_{x,z}$. In fact, since $ST^*\mathbb{R}^2_{x,z} \cong\cong \mathbb{R}^2_{x,z} \times S^1$ we will furthermore assume that Λ is null-homologous. Without loss of generality we may thus assume $\Lambda \subset \mathbb{R}^2_{x,z} \times S^1_{\text{lower}} \cong \mathbb{R}^3_{x,y,z}$ where $\mathbb{R}^3_{x,y,z}$ is equipped with the standard contact form $\alpha = dz - ydx$.

We give a short introduction to Legendrian knot theory in $\mathbb{R}^3$. The front projection is defined by $(x, y, z) \mapsto (x, z)$. Pick a parametrization $t \mapsto (x(t), y(t), z(t))$ of Λ and note that Λ is Legendrian (by definition) iff $T\Lambda \subset \xi := \ker(dz - ydx)$ iff $y(t) = \frac{\dot{z}(t)}{\dot{x}(t)}$. Thus given a front projection of a Legendrian knot we can always lift it to a Legendrian by $(x, z) \mapsto \left(x, \frac{dz}{dx}, z\right)$. Note that this excludes front diagrams with vertical tangencies. Instead of vertical tangencies, front diagrams contains cusps.

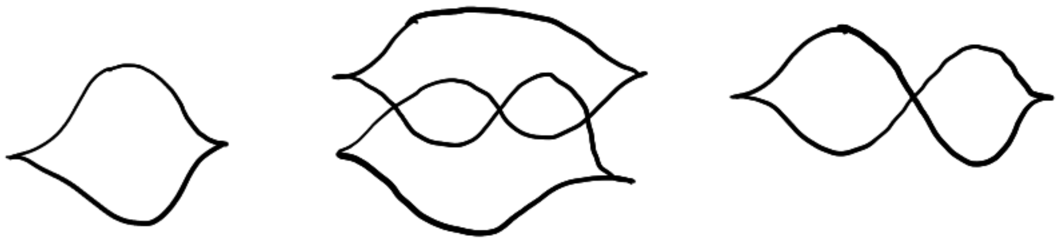
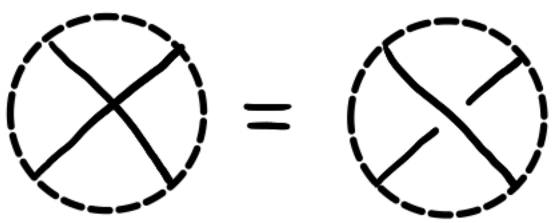


Figure 1. Left to right: The unknot, the trefoil and a stabilized unknot.

Remark 2.1. Whenever we draw a front diagram, we never have to indicate over and under crossings. The strand with lower slope (= lower y value in the lift) is always the over strand.



Since the tangent vectors $(\dot{x}(t), \dot{z}(t))$ are never vertical it follows that the downward normal vectors $(\dot{z}(t), -\dot{x}(t))$ are never horizontal. Thus a front diagram lifts directly to a Legendrian in $\mathbb{R}^2_{x,z} \times S^1_{\text{lower}}$ by defining the component in the S^1_{lower}-factor to be the unit downward conormal at the point.

Theorem 2.2. *Two front diagrams represent the same Legendrian knot iff they are related by regular homotopy and a finite sequence of Reidemeister moves as shown in Figure 2.*

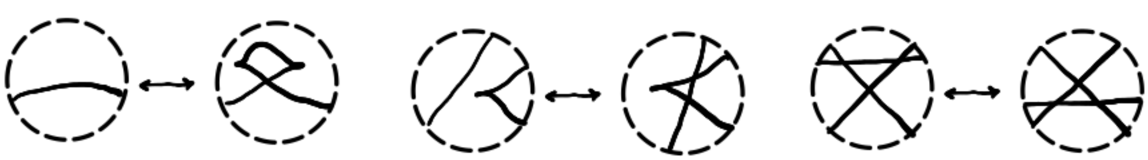
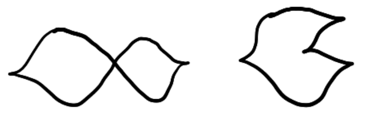


Figure 2. The three Reidemeister moves in the front projection.

Exercise 2.3. Show that the following two front diagrams represent the same knot.

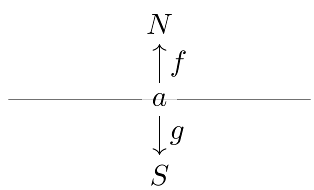


Definition 2.4. *A front diagram is called* stabilized *if it contains a zig-zag:* 

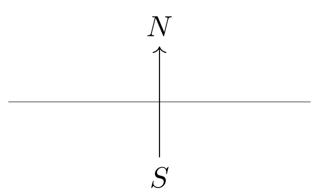
3. Constructible sheaves

We have seen from Sebastian's talk that for a Whitney stratification $\mathcal{S}$ of a manifold M that sheaves on M with singular support in $N^*\mathcal{S}$ are exactly constructible sheaves on M wrt $\mathcal{S}$ (see [GPS18, Proposition 4.8]). In our case, the category at hand is $\mathrm{Sh}_\Lambda(\mathbb{R}^2)$, but in our case $\mathbb{R}_{>0}\Lambda$ is just half of the conormal. However, we can still describe $\mathrm{Sh}_\Lambda(\mathbb{R}^2)$ in terms of constructible sheaves, but with certain conditions.

Arc. Near arcs we have the following local picture.



Since the singular support is contained in the downward normal directions it means that g is a quasi-isomorphism. This sheaf is thus determined up to quasi-isomorphism by the following data near arcs.



Cusp. Near cusps we have the following local picture

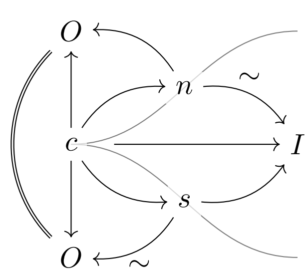
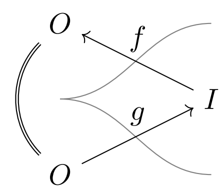


Figure 3.

From the definition of singular support one can work out that the map $c \to s$ is also a quasi-isomorphism. The conclusion is that the sheaf near cusps is determined up to quasi-isomorphism by the following commutative diagram near cusps.



Crossing. Near crossings we have the following local picture

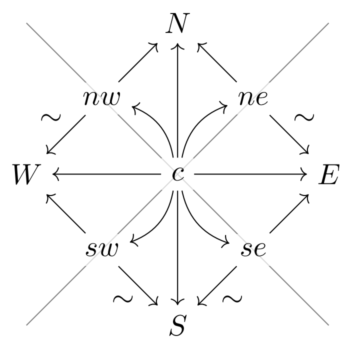
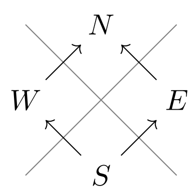


FIGURE 4.

Again studying the definition of singular suppot we find that the maps $c \to se$ and $c \to sw$ are quasi-isomorphisms. There is an additional condition which we do not write out in terms of these maps. We summarize by saying that a sheaf near a crossing is determined by the following data

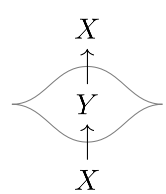


with the extra condition that the total complex

$$S \longrightarrow W \oplus E \longrightarrow N$$

should be acyclic.

Example 3.1. Consider the front diagram of the unknot. Elements in $\mathrm{Sh}_{\mathrm{unknot}}(\mathbb{R}^2)$ are specified by two complexes X and Y of k-modules with maps $X \xrightarrow{f} Y \xrightarrow{g} X$ whose composition is the identity.



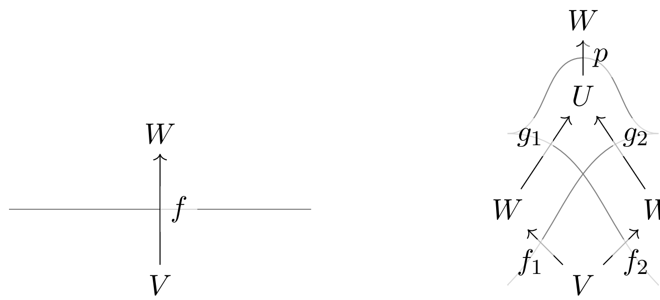
If we consider elements in $\mathrm{Sh}_{\mathrm{unknot}}(\mathbb{R}^2)_0$, that is to say that X is acyclic, then any such sheaf is in fact determined by the choice of Y. So $\mathrm{Sh}_{\mathrm{unknot}}(\mathbb{R}^2)_0$ is quasi-equivalent to the derived category of complexes of k-modules.

Let us now prove the invariance theorem.

Theorem 3.2 ([STZ17])**.** *A contactomorphism inducing a Legendrian isotopy $\Lambda \simeq \Lambda'$ induces a quasi-equivalence $\mathrm{Sh}_\Lambda(M) \xrightarrow{\sim} \mathrm{Sh}_{\Lambda'}(M)$. This quasi-equivalence preserves the subcategory $\mathrm{Sh}_\Lambda(M)_0$.*

Proof. By Theorem 2.2 it is enough to study the three Reidemeister moves.

Reidemeister 1: Consider the first Reidemeister move:



We first go from right to left. By the cusp conditions we have $pg_1 = pg_2 = 1$. By commutativity we obtain $f_1 = f_2$. Thus we obtain $V \xrightarrow{f} W$ by letting $f := f_1 = f_2$.

From left to right, we first note that without loss of generality f is injective on the chain level (if not, replacing W with the mapping cylinder of f gives a diagram representing the same sheaf). Then define $f_1 = f_2 = f$,

$$U = \mathrm{coker}(V \xrightarrow{(f,-f)} W \oplus W)$$

and $p\colon U \longrightarrow W$ is induced by $W \oplus W \longrightarrow W$.

Checking invariance under the other two Reidemeister moves is left as an exercise. (See [STZ17, Sections 4.4.2 and 4.4.3].) □

4. Microlocal monodromy

An n-periodic Maslov potential of a front diagram Φ is a map $\mu\colon\ \mathrm{strands}(\Phi) \longrightarrow \mathbb{Z}/n\mathbb{Z}$ such that when two strands meet at a cusp we have

$$\mu(\text{upper strand}) = \mu(\text{lower strand}) + 1.$$

The existence of such a potential is equivalent to $n \mid 2\,\mathrm{rot}(\Lambda)$.

Given a Maslov potential, we now define a functor

$$\mu\mathrm{mon}\colon\ \mathrm{Sh}_\Lambda(M) \longrightarrow \mathrm{Loc}(\Lambda)$$

to local systems of complexes of k-modules up to quasi-isomorphism on Λ. To define this functor, we will pull back the given stratification $\mathcal{S}$ of a front diagram to a stratification of $\Lambda \subset \mathbb{R}^3$ via the front projection.

- Arcs in $\mathcal{S}$ have unique preimages in Λ.
- The preimage of a crossing c is two points in Λ which we denote by $c_{\diagup}$ and $c_{\diagdown}$ respectively, see Figure 5.
- The preimage of a cusp is a closed interval in Λ, i.e. one 1-dimensional stratum that we denote by $c_{\prec}$, and two 0-dimensional strata which we denote by $c_{\precsim}$ and $c_{\preccurlyeq}$, respectively. All together we denote the preimages and maps relating them in the stratification of Λ as $c_{\precsim} \to c_{\prec} \leftarrow c_{\preccurlyeq}$, see Figure 5.

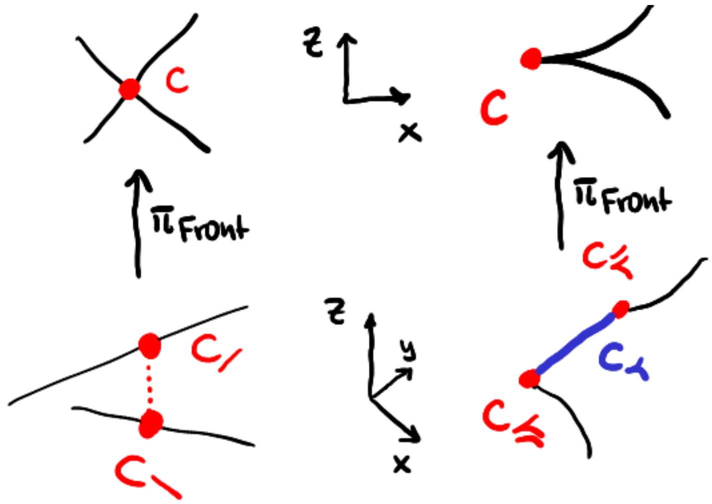


FIGURE 5.

Let us now first define the unnormalized microlocal monodromy.

Definition 4.1 (Unnormalized microlocal monodromy)**.** *Given a stratification $\mathcal{S}$ of a front diagram and the corresponding stratification Δ of Λ we define the unnormalized microlocal monodromy functor $\mu\mathrm{mon}'$ as follows.*

- *If $a \in \mathcal{S}$ is an arc, we denote its preimage by $a \in \Delta$. Denote the region above a in the front diagram by N, and define*

$$\mu\mathrm{mon}'(a) := \mathrm{Cone}(a \to N).$$

- *If $c \in \mathcal{S}$ is a crossing we have a diagram as in Figure 4. We define*

$$\mu\mathrm{mon}'(c_{\diagup}) := \mathrm{Cone}(c \to nw)$$
$$\mu\mathrm{mon}'(c_{\diagdown}) := \mathrm{Cone}(c \to ne)$$

There are furthermore maps in Δ as follows: $nw \leftarrow c_{\diagdown} \to se$ and $ne \leftarrow c_{\diagup} \to sw$, and the corresponding maps after applying $\mu\mathrm{mon}'$

$$\mu\mathrm{mon}'(nw) \leftarrow \mu\mathrm{mon}'(c_{\diagdown}) \to \mu\mathrm{mon}'(se)$$
$$\mu\mathrm{mon}'(ne) \leftarrow \mu\mathrm{mon}'(c_{\diagup}) \to \mu\mathrm{mon}'(sw)$$

are defined via functoriality of cones

$$\begin{array}{ccccc} c & \longrightarrow & nw & \longrightarrow & \mu\mathrm{mon}'(c_{\diagup}) \\ \downarrow & & \downarrow & & \downarrow \\ ne & \longrightarrow & N & \longrightarrow & \mu\mathrm{mon}'(ne) \end{array} \qquad \begin{array}{ccccc} c & \longrightarrow & nw & \longrightarrow & \mu\mathrm{mon}'(c_{\diagup}) \\ \downarrow & & \downarrow & & \downarrow \\ sw & \longrightarrow & W & \longrightarrow & \mu\mathrm{mon}'(sw) \end{array}$$

and the corresponding diagrams for the maps $\mu\mathrm{mon}'(ne) \leftarrow \mu\mathrm{mon}'(c_{\diagup}) \to \mu\mathrm{mon}'(sw)$.

- *If $c \in \mathcal{S}$ is a cusp we have a diagram as in Figure 3. The preimage of c is the diagram $c_{\precneqq} \to c_{\prec} \leftarrow c_{\preccurlyeq}$ and we define*

$$\mu\mathrm{mon}'(c_{\precneqq}) = \mu\mathrm{mon}'(c_{\prec}) := \mathrm{Cone}(c \to n)$$
$$\mu\mathrm{mon}'(c_{\preccurlyeq}) := \mu\mathrm{mon}'(n) = \mathrm{Cone}(n \to O).$$

Since we have maps $s \leftarrow c \to n$ in the cusp diagram Figure 3 we need to provide maps

$$\mu\mathrm{mon}'(s) \leftarrow \mu\mathrm{mon}'(c_{\precapprox}) \xrightarrow{\mathrm{id}} \mu\mathrm{mon}'(c_{\prec}) \leftarrow \mu\mathrm{mon}'(c_{\preccurlyeq}) \xrightarrow{\mathrm{id}} \mu\mathrm{mon}'(n).$$

where $\mu\mathrm{mon}'(c_{\precapprox}) \to \mu\mathrm{mon}'(s)$ is defined by functoriality of cones via the diagram

$$\begin{array}{ccccc} c & \longrightarrow & a & \longrightarrow & \mu\mathrm{mon}'(c_{\precapprox}) \\ \downarrow & & \downarrow & & \downarrow \\ b & \longrightarrow & I & \longrightarrow & \mu\mathrm{mon}'(s) \end{array}$$

the map $\mu\mathrm{mon}'(c_{\preccurlyeq}) \to \mu\mathrm{mon}'(c_{\prec})$ is defined as follow. Applying the octahedral axiom to the sequence $c \to n \to O$ gives the triangle

$$\mathrm{Cone}(c \to n) \longrightarrow \mathrm{Cone}(c \to O) \longrightarrow \mathrm{Cone}(n \to O) \xrightarrow{[1]}$$

which gives a map $\mu\mathrm{mon}'(c_{\preccurlyeq}) \to \mu\mathrm{mon}'(c_{\prec})[1]$.

Proposition 4.2. *After applying $\mu\mathrm{mon}'$, all arrows are quasi-isomorphisms (or a shifted quasi-isomorphism in the case of cusps).*

Proof. The precise statements are that all maps defined above

$$\mu\mathrm{mon}'(nw) \leftarrow \mu\mathrm{mon}'(c_{\diagdown}) \to \mu\mathrm{mon}'(se)$$
$$\mu\mathrm{mon}'(ne) \leftarrow \mu\mathrm{mon}'(c_{\diagup}) \to \mu\mathrm{mon}'(sw)$$
$$\mu\mathrm{mon}'(c_{\precapprox})' \to \mu\mathrm{mon}'(s)$$
$$\mu\mathrm{mon}'(c_{\preccurlyeq}) \to \mu\mathrm{mon}'(c_{\prec})$$

are quasi-isomorphisms. First, the crossing condition that the complex $c \to ne \oplus nw \to N$ is acyclic is equivalent to the maps $\mu\mathrm{mon}'(c_{\diagup}) \longrightarrow \mu\mathrm{mon}'(ne)$ and $\mu\mathrm{mon}'(c_{\diagdown}) \longrightarrow \mu\mathrm{mon}'(nw)$ being quasi-isomorphisms. Secondly, by studying Figure 4 we have immediately that the maps $\mu\mathrm{mon}'(c_{\diagup}) \to \mu\mathrm{mon}'(sw)$ and $\mu\mathrm{mon}'(c_{\diagdown}) \to \mu\mathrm{mon}'(se)$ are quasi-isomorphisms.

By studying Figure 3 we have that $\mu\mathrm{mon}'(c_{\precapprox})' \to \mu\mathrm{mon}'(s)$ is a quasi-isomorphism, and we have that the cusp condition gives that $\mu\mathrm{mon}'(c_{\preccurlyeq}) \to \mu\mathrm{mon}'(c_{\prec})$ is a quasi-isomorphism. □

Proposition 4.3. *If a is an arc on one component of a front diagram, then traveling around the component gives a sequence of quasi-isomorphisms*

$$\mu\mathrm{mon}'(a) \xleftarrow{\sim} \cdots \xrightarrow{\sim} \mu\mathrm{mon}'(a)[\#\textit{down cusps - }\#\textit{up cusps}] = \mu\mathrm{mon}'(a)[-2\,\mathrm{rot}(\Lambda)].$$

In particular if $\mathrm{rot}(\Lambda) \neq 0$, $\mu\mathrm{mon}'(a)$ must be either unbounded in both directions or acyclic.

Definition 4.4 (Normalized microlocal monodromy)**.** *Fix a Maslov potential $p\colon \mathrm{strands}(\Phi) \longrightarrow \mathbb{Z}/n\mathbb{Z}$. We define the functor $\mu\mathrm{mon}\colon \mathrm{Sh}_\Lambda(M) \longrightarrow \mathrm{Loc}(\Lambda)$ as follows. If x is the preimage of an arc or crossing, then*

$$\mu\mathrm{mon}(x) := \mu\mathrm{mon}'(x)[-p(x)]\,.$$

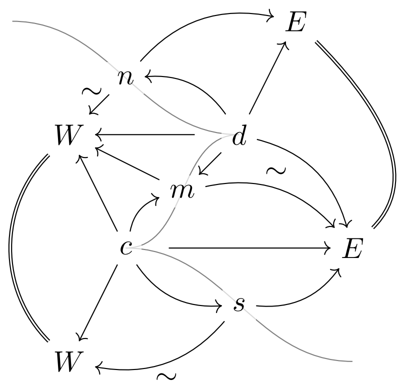
For the preimage $c_{\precapprox} \leftarrow c_{\prec} \to c_{\preccurlyeq}$ of a cusp c, we define

$$\mu\mathrm{mon}(c_{\preccurlyeq}) := \mu\mathrm{mon}(n) = \mathrm{Cone}(n \to O)[-p(n)]$$
$$\mu\mathrm{mon}(c_{\precapprox}) = \mu\mathrm{mon}(c_{\prec}) := \mathrm{Cone}(c \to n)[-p(n) + 1]$$

We now finish with the proof of the following theorem.

Theorem 4.5 ([STZ17])**.** *If Λ is a stabilized Legendrian knot (see Definition 2.4 for the definition), then its microlocal monodromy vanishes.*

Proof. Assuming Λ is stabilized, there is some zig-zag in the diagram. Near such a zig-zag we have the following diagram



Furthermore by definition of singular support (also see Figure 3 and surrounding discussion) we have that $c \to s$ and $d \to m$ are quasi-isomorphisms. By commutativity it implies that $c \to W$ and $d \to E$ are quasi-isomorphisms too. So we have the following diagram

By passing to cohomology and utilizing a trick (see [STZ17, Corollary 3.18]) we obtain

$$c \to d \to n \to E$$

where both compositions of consecutive arrows equals the identity map, and $c = n$ and $d = E$. It follows that $d \to n$ is an isomorphism, from which it follows that the microlocal monodromy vanishes. □

Definition 4.6 (Microlocal rank)**.** *A sheaf is said to have* microlocal rank r *wrt to a fixed Maslov potential if* $\mu\mathrm{mon}(x)$ *is quasi-isomorphic to a locally free* k*-module of rank* r *placed in degree* 0*. We write* $\mathcal{C}_r(\Lambda) \subset \mathrm{Sh}_\Lambda(\mathbb{R}^2)_0$ *for the full subcategory of microlocal rank* r *objects.*

References

[GPS18] Sheel Ganatra, John Pardon, and Vivek Shende. Microlocal morse theory of wrapped Fukaya categories. *arXiv:1809.08807*, 2018.

[Nad09] David Nadler. Microlocal branes are constructible sheaves. *Selecta Math. (N.S.)*, 15(4):563–619, 2009.

[NRS+20] Lenhard Ng, Dan Rutherford, Vivek Shende, Steven Sivek, and Eric Zaslow. Augmentations are sheaves. *Geom. Topol.*, 24(5):2149–2286, 2020.

[NZ09] David Nadler and Eric Zaslow. Constructible sheaves and the Fukaya category. *J. Amer. Math. Soc.*, 22(1):233–286, 2009.

[STZ17] Vivek Shende, David Treumann, and Eric Zaslow. Legendrian knots and constructible sheaves. *Invent. Math.*, 207(3):1031–1133, 2017.