X real analytic, compact, spin

convention: all "subsets" are subanalytic subsets.

$$\mathrm{Sh}_{\text{naive}}(X) = \text{cxs } \mathcal{F}^\cdot \text{ of sheaves of } \mathbb{C}_X\text{-modules s.t. } \mathcal{H}^\cdot(\mathcal{F}^\cdot) \text{ constructible}$$

(+ bounded & finite rank)

dg cats

$$\mathrm{Sh}(X) := \mathrm{Sh}_{\text{naive}}(X)/_{q\text{-iso}}$$

$$\hom^\cdot_{\mathrm{sh}}(\mathcal{F}^\cdot, \mathcal{G}^\cdot) = \mathrm{RHom}(\mathcal{F}^\cdot, \mathcal{G}^\cdot)$$

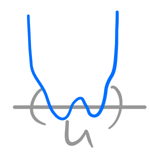
$$D\mathrm{Sh}(X) = H^0(\mathrm{Sh}(X)) \quad \text{"derived cat"}$$

$$\hom_D(\mathcal{F}^\cdot, \mathcal{G}^\cdot) = \mathrm{Ext}^0(\mathcal{F}^\cdot, \mathcal{G}^\cdot)$$

<u>Thm</u> (NZ) $\underset{\text{local topology, cones easy}}{D\mathrm{Sh}(X)} \overset{\text{equiv. of cats}}{\simeq} \underset{\text{global analysis, cones awful}}{D\overbrace{\mathrm{Fuk}}^{\text{tbd}}(T^*X)}$

local topology
cones easy

global analysis
cones awful

sheaf	brane
$D\,Sh_S(X)$ S Whitney strat	$D\,Fuk(X)_{\Lambda_S}$ (ass. conical Lagr $\subset \Lambda_S$)
$\mathbb{C}$	$X \subset T^*X$
$Ri_* \mathbb{C}_U$ $(U \underset{\text{open}}{\subset} X)$	$L_U = \Gamma_{df}$  $\left.\begin{array}{l} m : U \to \mathbb{R}_{\geq 0} \\ \partial U = X_{m=0} \end{array}\right\}$ defining function for ∂U $f = \log m : U \to \mathbb{R}$
<u>Verdier duality</u> $\mathbb{D}_X : DSh(X) \xrightarrow[\text{invol.}]{\simeq} DSh(X)^{opp}$ $A^\cdot \to RHom(A^\cdot, \omega_X)$ $\omega_X \cong \mathbb{C}[\underbrace{\dim X}_{\text{shift down}}]$	<u>Brane duality</u> $T^*X \xrightarrow[\sim]{-1} T^*X$ anti-sympl invol. $DFuk(T^*X) \xrightarrow{\simeq} DFuk(T^*X)^{opp}$ $(L, \mathcal{E}) \longrightarrow (-L, \mathcal{E}^\vee)$ local system on L

Application ($\pi_1 X = 0$ assume)

$L \hookrightarrow T^*X$ compact exact Lags
(spin , $\pi_1 L = 0$, Maslov zero
connected)

Cor (Nadler, Fukaya-Seidel-Smith)

$$H^\bullet(X, \mathbb{C}) \cong H^\bullet(L, \mathbb{C})$$

Proof $\mathcal{F}^\bullet \overset{NZ}{\rightsquigarrow} L$

- $\mathcal{F}^\bullet$ is constructible wrt trivial strat

  $\pi_1 L = 0 \Rightarrow \mathcal{H}^\cdot(\mathcal{F}^\bullet)$ is constant

- $Ext^\bullet(\mathcal{F}^\bullet, \mathcal{F}^\bullet) \cong H^\bullet(L) = 0$ if $\bullet > \dim X$

  $\Rightarrow \mathcal{F}^\bullet \cong \mathbb{C}_X^k$ (mod shifts) $k \geq 0$

  $\Rightarrow \mathcal{F}^\bullet \cong \mathbb{C}_X$ (bc. L connected)

Conclusion: $L \cong X$ in $\mathcal{D}Fuk$. ∎

Outline

$$
\begin{tikzcd}
 & \mathrm{Sh}(X) & & \mathrm{Fuk}(X) \\
RC_* \mathbb{C}_U \arrow[r, mapsto] & \mathrm{Open}(X) \arrow[u, hook, "\text{full subcat}"'] \arrow[rr, "\simeq"', "A_\infty \text{ q-equiv}"] & & \mathrm{Mor}(X) \arrow[u, hook, "A_\infty \text{ embedding}"']
\end{tikzcd}
$$

1   2   3

key: $\mathrm{Open}(X)$ generates $\mathrm{Sh}(X)$

$$\Rightarrow \quad \mu_X : \mathrm{Sh} \simeq \mathrm{Tw}\,\mathrm{Open} \xhookrightarrow[A_\infty \text{ q-emb}]{} \mathrm{Tw}\,\mathrm{Fuk}$$

(That μ_X is essentially surjective is proved in [Nadler 09] via a "resolution of the diagonal $\Delta \in \mathrm{Fuk}(X \times X)$" argument.)

(infinitesimally wrapped)

① infinitesimal Fukaya cat

$T^*X \qquad \omega = dq \wedge dp \qquad \lambda = -p\,dq$

Liouville field $v = p\,\partial_p$, $\omega(v, \cdot) = \lambda$

(Weinstein, skel $= X$, construction below works $\forall$ Weinstein mflds)

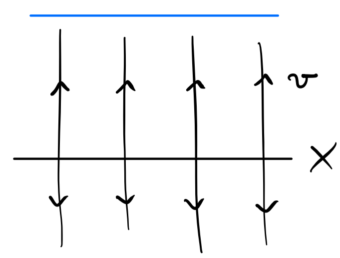
"Log" compactification $T^*X \subset \overline{T^*X}$

symplectic form has pole at the boundary

$$\overline{T^*X} = \frac{\overbrace{(0,\infty]}^{\text{transl}} \times \overbrace{T^*X}^{\text{Liouville flow}} \setminus \{\infty\} \times X}{\mathbb{R}_{\geq 0}}$$

compact with boundary

$$\partial\, \overline{T^*X} = S^*X$$



Choose ω-comp aC J st $\mathcal{L}_v J = 0$ @ ∞
($\neq$ Sasaki aC str)

$\rightsquigarrow$ (T^*X, ω, J) bounded geometry

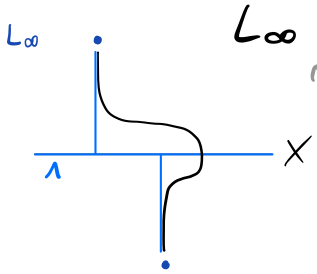
Objects of $Fuk(T^*X)$: $(L, \mathcal{E})$

L exact (maybe non-cpt) Maslov 0 & graded, spin connected

$\mathcal{E}$ local system on L

In addition :

ⓐ $\overline{L} \subset \overline{T^*X}$ subanalytic ($\approx$ conical)



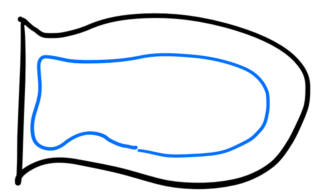
$L_\infty := \overline{L} \cap S^*X$ is an isotropic subset of S^*X (contact mfld)

"Legendrian at ∞"

$\Lambda := \text{cone}(L_\infty) \cup X$ "assoc. conical Lagrangian"

ⓑ L tame (bounded extrinsic geometry)

ex $L_U = \Gamma_{df}$ $f = \log m$



$\wr$ tame

$L_{U_\eta} = \Gamma_{df_\eta}$ $f_\eta = \log m_\eta$ $m_\eta = m - \eta$

$U_\eta = m_\eta > 0$

$\eta > 0$ small $\Rightarrow \partial U_\eta$ smooth

(conical Lagr $= T^*_{\partial U_\eta} X$)

hom's

"conical" Hamiltonians H : $\mathcal{L}_v H = H$ @ ∞

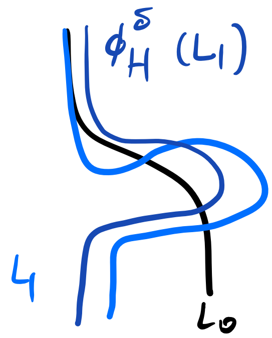
wlog $H(q,p) = |p|_g$ @ ∞

Aside: in wrapped Floer hly we have $\mathcal{L}_v H = 2 \cdot H$

<u>fact</u> Given L_0, L_1 $\exists\ \bar\delta > 0$ (small)

$\Uparrow$ real anal.

$0 < \delta \le \bar\delta \Rightarrow L_0 \cap \phi_H^\delta(L_1)$ is <u>compact</u>



key: ϕ_H^t extends as the <u>Reeb flow</u> on S^*X

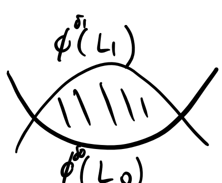
$\uparrow$ = geodesic flow

$*$ $\hom^\bullet_{Fuk}(L_0, L_1) := \bigoplus\limits_{p \in \phi_{H_0}^{\delta_0}(L_0) \cap \phi_{H_1}^{\delta_1}(L_1)} \mathrm{Hom}(\mathcal{E}_0, \mathcal{E}_1)\cdot p$

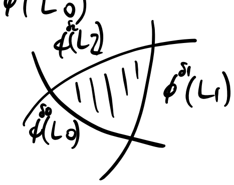
L_1 "further ahead" of L_0 i.e. $\delta_1 - \delta_0 > 0$ small

(Important: choices of η_i's, H_i's, ... form a <u>contractible</u> set!)

$*$ μ_1 :



$*$ μ_2 :



etc ...

② From sheaves to Morse theory

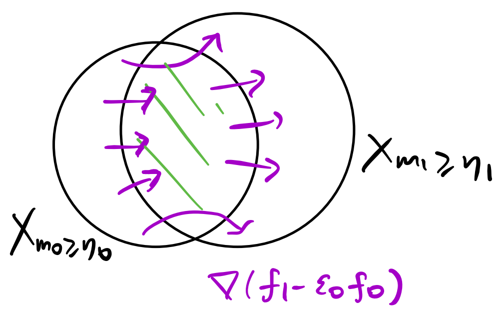
<u>Morse A_∞ - cat</u> (à la Fukaya) $\mathrm{Mor}(X)$

* Ob: (U, m) ⇝ abbrev U

* $\hom^{\cdot}_{\mathrm{Mor}}(U_0, U_1) = CM^{\cdot}(X_{m_0 \geq \eta_0,\, m_0 \geq \eta_1},\ f_1 - \varepsilon_0 f_0)$

$\eta_1 - \eta_0 > 0$ small $\Rightarrow \partial X_{m_j \geq \eta_j}$ smooth & $\partial X_{m_0 \geq \eta_0} \pitchfork \partial X_{m_1 \geq \eta_1}$

$\varepsilon_0 > 0$ <u>small</u> $\Rightarrow$

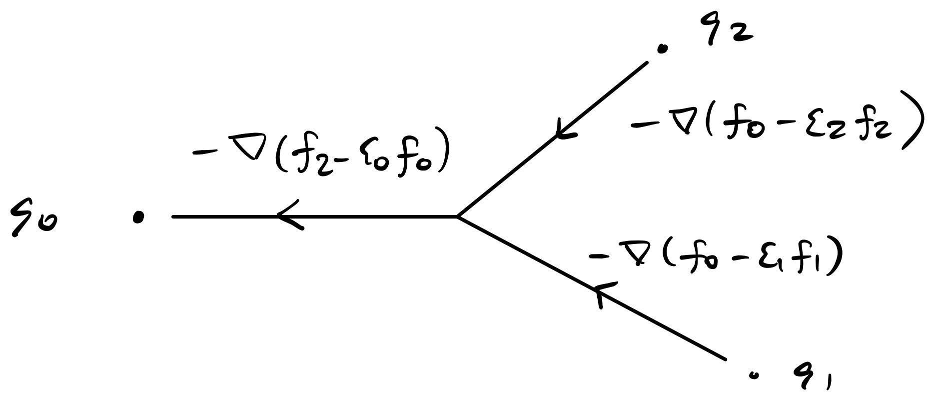


This guarantees Morse cx is defined

(<u>Important</u>: can organize choices into a <u>contractible</u> set)

* μ_1 = Morse differential $\qquad q_0 \xleftarrow{-\nabla(f_1 - \varepsilon_0 f_0)} q_0'$

* μ_2 counts "flow triangles"



* higher μ_k count "flow trees"

<u>Prop</u> $\exists$ canonical q-equiv $\mathrm{Open}(X) \xrightarrow{\simeq} \mathrm{Mor}(X)$

<u>Idea</u> $\exists$ canonical q-equivalences of chain cxs

$$B\Gamma(\mathbb{C}) \xrightarrow{\simeq} \Omega_{dR}(X) \xrightarrow{\simeq} CM^{\cdot}(X).$$

Poincaré Lemma $\qquad$ Harvey-Lawson

③ From Morse to Floer

Prop $\mathrm{Mor}(X) \hookrightarrow \mathrm{Fuk}(T^*X)$ (A_∞ emb)

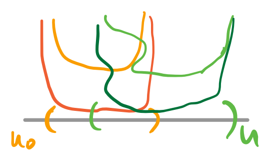
$$u \longrightarrow L_{u_\eta}$$

Idea Want to apply Floer/Fukaya-Oh Thm

One can hamiltonian deform

$$\phi_{H_0}^{\delta_0}(L_{u_0}) \ \& \ \phi_{H_1}^{\delta_1}(L_{u_1}) \rightsquigarrow \tilde{L}_0 \ \& \ \tilde{L}_1$$

issue: not graphs, not close to zero section



so that ($\sigma > 0$ very small)

$$* \quad \tilde{L}_0 \cap \tilde{L}_1 = \sigma \cdot \Gamma_{\varepsilon_0 df_0} \cap \Gamma_{df_1}$$

Then Floer's proof identifies (for $\sigma > 0$ small) gradient lies & holo disks.

$$\Rightarrow \mu_1^{\mathrm{Mor}} = \mu_1^{\mathrm{Fuk}}$$

For $\mu_2, \mu_3, \ldots$ use Fukaya-Oh's argument to identify flow trees = holo polygons.