$Sh(M)$ = (dg) category of sheaves of (dg)-modules on M.

For $f: N \to M$ cts $\exists$ adjoint pair (f_* right, f^* left)

$$f^* : Sh(M) \leftrightarrow Sh(N) : f_*$$

$$(f_* F)(U) = F(f^{-1}(U))$$

left adjoint of incl

$$(f^* G)(V) = G(f(V)) = \varinjlim_{f(V) \subseteq U} G(U) \quad \text{(sheafify)}$$

Rmk under GPS conventions, $\mathbb{Z}_M = pt^* \mathbb{Z}$ is $\mathbb{Z}_M(U)$ = chain cx computing $H^*(U;\mathbb{Z})$

If $j: U \overset{\text{open}}{\hookrightarrow} M$, then j^* has a left adjoint

$$j_! : V \mapsto \begin{cases} F(V) & \text{for } V \subseteq U \\ 0 & \text{otherwise} \end{cases} \quad \text{"ext. by 0"}$$

For f proper $f_* = f_!$, in general

$$f_! : Sh(N) \longrightarrow Sh(M)$$

$$F \longmapsto \varinjlim_{U \subset\subset N} f_* F_U \quad (F_U = j_! j^* F)$$

$f^!$ = right-adjoint of $f_!$ (b/c $f_!$ is co-cts)

M analytic mfd.

$SS(F) \subseteq T^*M$ is

$$SS(F) = \overline{\{ d\phi_m \mid \phi : M \to \mathbb{R} \text{ smooth}, \ m \in \phi^{-1}(t), \ (i^! F)_m \neq 0 \}}$$

for $i : \{x \mid \phi(x) \geq t\} \hookrightarrow M$

($m \notin SS(F)$ satisfy a propagation property)

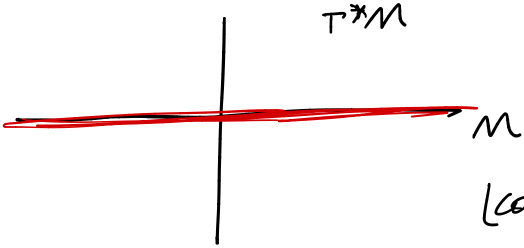
Let $j : \{\phi(x) < t\} \hookrightarrow M$. There is an exact triangle in $D(M)$, where $Z = \{x \mid \phi(x) \geq t\}$

$$R\Gamma_Z(F) \to F \to R\Gamma_{M \setminus Z}(F) \xrightarrow{+1}$$

Taking stalks at m shows that if $(m, d\phi_m) \notin SS(M)$

$\exists V \ni m$ on which $R\Gamma(V, F) \simeq R\Gamma(V \cap M \setminus Z, F)$

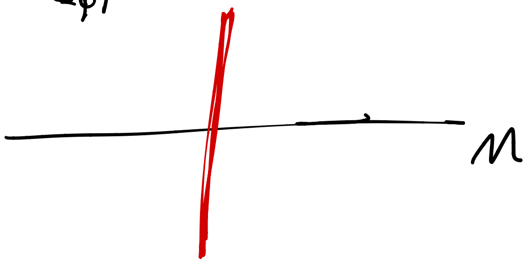
F constant sheaf $\implies ss(\mathcal{F}) = M$



(look at PP w/ const. function)

[cohomological critical points can only occur when $d\phi_x = 0$)

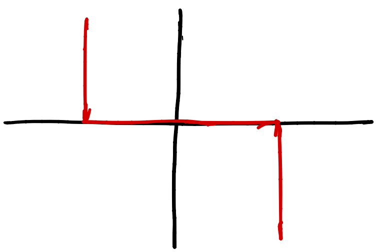
$F = \mathbb{C}_{pt}$



$j : U \hookrightarrow M$ open, ∂U smooth, $F = j_* \mathbb{Z}_U$.

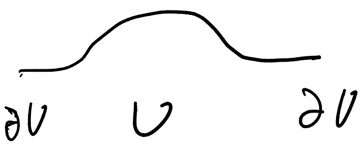
$\mathrm{supp}(F) = \overline{U}$.

Restriction maps are interesting for covectors pointing <u>into</u> U

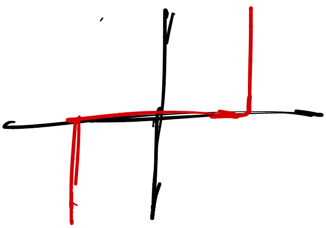


$F = j_! \mathbb{Z}_U$

Now sections stay away from ∂U



$\Rightarrow$ restriction maps are interesting pointing <u>outward</u>



<u>Prop:</u> (i) $ss(F)$ closed, conic

(ii) $ss(F) \cap M = \mathrm{Supp}(\mathcal{F})$

(iii) $ss(F) = ss(\mathcal{F}[j])$

(iv) $ss(F)$ is coisotropic

(v) Given $F_1 \to F_2 \to F_3 \xrightarrow{[1]}$ in $Sh(M)$,

$ss(F_i) \subseteq ss(F_j) \cup ss(F_k)$ for $\{i, j, k\} = \{1, 2, 3\}$

Thm (Morse thm)

$F \in Sh(M)$, $\psi : M \to \mathbb{R}$ proper on $Supp(F)$

If $d\psi_x \notin ss(F)$ for all $x \in \psi^{-1}([b,a))$

then $R\Gamma(M_b; F) \xrightarrow{\sim} R\Gamma(M_a; F)$ $(M_t = \psi^{-1}((-\infty, t)))$

Notation: S: $M = \coprod_{\alpha \in S} M_\alpha$ a stratif.

$Sh_S(M)$ = full subcat. of S-constructible sheaves
(for all $i : M_\alpha \hookrightarrow M$, $i^* \mathcal{F}$ loc const.)

For $X \subseteq T^*M$, $Sh_X(M)$ = full subcat spanned by $ss(\mathcal{F}) \subseteq X$.

Prop: S Whitney, $Sh_S(M) = Sh_{N^*S}(M)$

$(Sh_S(M) \subseteq Sh_{N^*S}(M))$ For $X = M_\alpha$, write

$X = \bigcup (X_0 \subseteq X_1 \subseteq \cdots)$, $N^* X_i \to N^* S$

hence $ss(\mathbb{Z}_X) = ss(\varinjlim \mathbb{Z}_{X_i}) \subseteq N^*S$

By same argument, any loc const F on X has $ss(i_! F) \subseteq N^*S$.

Any S-const. sheaf is locally a finite iterated extension of such sheaves $\Rightarrow Sh_S(M) \subseteq Sh_{N^*S}(M)$

$(Sh_{N^*S}(M) \subseteq Sh_S(M))$

If $ss(F) \subseteq N^*S$, X max'l stratum on which $F \neq 0$, then F is a local system in a nbhd of this stratum, so over X it agrees w/ $F_0 \in Sh_S(M) \subseteq Sh_{N^*S}(M)$, so it suffices to show that $\mathrm{Cone}(F_0 \to F)$ constructible

Iterating reduces to $F = 0$.