From cotangent bundles to Liouville manifolds

Based on Section 7 of *Microlocal Morse Theory of Wrapped Fukaya Categories* by Ganatra, Pardon, and Shende

Microlocal theory of sheaves in symplectic geometry learning seminar
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Goal

Our goal is to prove

Theorem (GPS3, Theorem 1.4)

Suppose X is a real analytic Liouville manifold and $\Lambda \subset \partial_\infty X$ is a stop whose relative core $\mathfrak{c}_{X,\Lambda} := \mathfrak{c}_X \cup (\Lambda \times \mathbb{R}) \subset X$ is a subanalytic singular isotropic. For any stable polarization of X, there is a fully faithful functor

$$Perf\,\mathcal{W}(X,\Lambda)^{op} \hookrightarrow \mu sh_{\mathfrak{c}_{X,\Lambda}}(\mathfrak{c}_{X,\Lambda})^c$$

taking a homological cocore at a smooth Lagrangian point p of $\mathfrak{c}_{X,\Lambda}$ to a co-representative of the microstalk at p.

Note: If X is Weinstein (or more generally, admits all homological cocores), this embedding is an equivalence.

What we've seen

In Yash's talks, we saw

Theorem (GPS3, Theorem 1.1)

Suppose M is a real analytic and $\Lambda \subset S^*M$ be a subanalytic closed isotropic subset. There is a canonical equivalence

$$Perf\mathcal{W}(T^*M, \Lambda)^{op} \simeq Sh_\Lambda(M)^c$$

taking the linking disk at a smooth Lagrangian point $p \in \Lambda$ to a co-representative of the microstalk functor at p, and taking a cotangent fiber over $p \in M$ that does not meet Λ to a co-representative of the stalk functor.

Rough idea: Both sides have good functoriality properties with respect to stop removal and are generated by the objects mentioned above. Moreover, they can be described combinatorially when Λ is the conormal to a triangulation. This behavior is captured formally as a microlocal Morse theater uniquely determining both sides.

Strategy

We will produce a diagram

$$\begin{array}{ccc} \mathcal{W}(X,\Lambda)^{\mathrm{op}} & \hookrightarrow & \mathrm{Perf}\,\mathcal{W}(T^*M, D(\mathfrak{c}_{X,\Lambda}))^{\mathrm{op}} \\ & & \Big\| \text{Theorem 1.1} \\ \mu\mathrm{sh}_{\mathfrak{c}_{X,\Lambda}}(\mathfrak{c}_{X,\Lambda})^c & \hookrightarrow & \mathrm{Sh}_{D(\mathfrak{c}_{X,\Lambda})}(M)^c \end{array}$$

for a stop $D(\mathfrak{c}_{X,\lambda})$ called the double. The horizontal arrows land in categories generated by certain linking disks/co-representatives of micro-stalks.

Two ingredients:

- Find an embedding $X \hookrightarrow S^*M$ as a Liouville hypersurface
- Doubling trick to produce fully faithful functors.

Embedding

A *Liouville hypersurface embedding* of Liouville (domain) X into contact Y is a codimension 1 embedding such that there is contact form agreeing with the Liouville form on X. When $Y = \partial_\infty Z$, we call (Z, X) a *Liouville pair*.

Want: Liouville pair (T^*M, X) given any Liouville X satisfying some hypotheses.
$X \hookrightarrow S^*M$ would give:

- smooth map $f : X \to M$.
- splitting $f^*TM \simeq B \oplus \underline{\mathbb{R}}$.
- $TX \simeq B \otimes_{\mathbb{R}} \mathbb{C}$ as $\mathbb{C}$-vector bundles.

There is an h-principle that this formal data is enough.

Lemma (GPS3, Lemma 7.6 after Eliashberg-Mishachev)

If X is a Liouville manifold such that $\mathfrak{c}_X$ is contained in a finite union of locally closed submanifolds of dimension at most $\frac{1}{2}\dim X$, then every triple above arises from some $X \hookrightarrow S^*M$.

Embedding

We imposed that X has a stable polarization, i.e., $TX \oplus \underline{\mathbb{C}}^k \simeq B \otimes_{\mathbb{R}} \mathbb{C}$ for some $k < \infty$.

The h-principle lemma always applies to $X \times \mathbb{C}^k$ for for some $k < \infty$ and some further algebraic topology shows that a stable polarization produces the formal data needed.

Upshot: For any stable polarization of X, there is a Liouville hypersurface embedding $X \times \mathbb{C}^k \hookrightarrow S^*M$ compatible with stable polarization.

Interlude: homological cocores

Definition (GPS3, Definition 7.1)

Let (X, Λ) be a stopped Liouville manifold whose relative core $\mathfrak{c}_{X,\Lambda}$ with $\mathfrak{c}_{X,\Lambda}$ mostly Lagrangian. An object of Perf$\mathcal{W}(X, \Lambda)$ is a homological cocore at a smooth Lagrangian point p if its image under the Künneth embedding

$$\mathcal{W}(X, \Lambda) \hookrightarrow \mathcal{W}((X, \Lambda) \times (\mathbb{C}_{Re \geq 0}, \infty))$$

is isomorphic to the linking disk at $p \times \infty$.

Examples: Linking disks/cocores, any generalized cocore (exact, conical Lagrangian intersecting once at p).

Linking disks generate $\mathcal{W}((X, \Lambda) \times (\mathbb{C}_{\mathrm{Re} \geq 0}, \infty)) \implies$ existence of all homological cocores is equivalent to Künneth embedding being an equivalence.

Doubling

From a Liouville pair (Z, X) want to embed $\mathcal{W}(X)$ into some partially wrapped category on Z. Use doubling trick.

$(Z, X) \rightsquigarrow$ Liouville sector by:
In particular, have sector chart $X \times \mathbb{C}_{\mathrm{Re} \geq 0} \subset Z$ and thus functors

$$\mathcal{W}(X) \to \mathcal{W}(X \times \mathbb{C}_{\mathrm{Re}\geq 0}, \mathfrak{c}_X \times \{\infty\}) \to \mathcal{W}(Z, \mathfrak{c}_X)$$

However, wrapping may cause this not to be full and faithful.
Instead, double up. Set $D(\mathfrak{c}_X) := \mathfrak{c}_X \sqcup \mathfrak{c}_X^{\varepsilon}$. This stops the other end and becomes a full and faithful functor

$$\mathcal{W}(X) \hookrightarrow \mathcal{W}(Z, \mathfrak{c}_X \sqcup \mathfrak{c}_X^{\varepsilon}).$$

Doubling with stop

Really, wanted to work with $\mathcal{W}(X,\Lambda)$. In the sector chart $X \times \mathbb{C}_{\mathrm{Re}\geq 0} \subset Z$, look at

$$(X,\Lambda) \times (\mathbb{C}, \{\pm i\infty\}) = (X \times \mathbb{C}, (\mathfrak{c}_X \times \{\pm i\infty\}) \cup (\Lambda \times i\mathbb{R}))$$

and perturb slightly inwards. Result is defined to be $D(\mathfrak{c}_{X,\Lambda})$. Then, have

$$\begin{aligned} \mathcal{W}(X,\Lambda) &\to \mathcal{W}((X,\Lambda) \times (\mathbb{C}_{\mathrm{Re}\geq 0}, \infty)) \\ &\to \mathcal{W}(X \times \mathbb{C}, (\mathfrak{c}_X \times \{-\infty\}) \sqcup (\mathfrak{c}_X \times \{\pm i\infty\}) \sqcup (\Lambda \times i\mathbb{R})) \\ &\simeq \mathcal{W}(X \times \mathbb{C}_{\mathrm{Re}\geq 0}, D(\mathfrak{c}_{X,\Lambda})) \\ &\to \mathcal{W}(Z, D(\mathfrak{c}_{X,\Lambda})) \end{aligned}$$

This functor is also fully faithful.

The sheafy part

What we have done so far will give us a full and faithful embedding $\mathcal{W}(X,\Lambda)^{\mathrm{op}} \hookrightarrow \mathrm{Sh}_{D(\mathfrak{c}_{X,\Lambda})}(M)^c$. That is, we now "just" need to understand $\mu\mathrm{sh}_{\mathfrak{c}_{X,\Lambda}}(\mathfrak{c}_{X,\Lambda})^c$.

First, what is it?

In cotangent case, we recall μsh is the sheafification of

$$\Omega \mapsto \mathrm{Sh}(M)/\mathrm{Sh}_{T^*M\setminus\Omega}(M)$$

and have $\mu\mathrm{sh}_\Lambda$ consisting of objects with microsupport in Λ. Also, have microlocalization functions $\mathrm{Sh}_\Lambda(M) \to \mu\mathrm{sh}_\Lambda(\Lambda \cap \Omega)$.
In general, Shende has shown that $\mu\mathrm{sh}_\Lambda(\Lambda)$ is well-defined for any Λ embedded into a stably polarized contact manifold.

Sheaf doubling

There is a generalization of doubling for subanalytic $\Lambda \subset S^*M$ with some nice coordinates near its boundary when not closed. The doubling trick for sheaves is addressed in *Sheaf quantization in Weinstein symplectic manifolds* by Nadler and Shende.

Theorem (NS, Theorem 6.30)

Let $\Lambda \subset S^*M$ be sufficiently nice. Then, $Sh_{D(\Lambda\times(0,1))}(M \times \mathbb{R})$ is an orthogonal direct sum of $Sh_{\emptyset}(M \times \mathbb{R})$ and $Sh_{D(\Lambda\times(0,1))}(M \times \mathbb{R})_0$ and microlocalization gives an equivalence

$$Sh_{D(\Lambda\times(0,1))}(M \times \mathbb{R})_0 \simeq \mu sh_{\Lambda\times(0,1)}(\Lambda \times (0,1)) = \mu sh_{\Lambda}(\Lambda).$$

Finishing the proof

We will show that microlocalization $\mu : \mathrm{Sh}_{D(\mathfrak{c}_{X,\Lambda})}(M) \to \mu\mathrm{sh}_{\mathfrak{c}_{X,\Lambda}}(\mathfrak{c}_{X,\Lambda})$ has a fully faithful left adjoint
Consider

$$\begin{array}{ccccc}
\mathrm{Sh}_{D(\mathfrak{c}_{X,\Lambda})}(M) & \underset{\sim}{\longleftarrow} & \mathrm{Sh}_{D(\mathfrak{c}_{X,\Lambda})\times(0,1)}(M \times (0,1)) & \longleftarrow & \mathrm{Sh}_{D(\mathfrak{c}_{X,\Lambda}\times(0,1))}(M \times \mathbb{R}) \\
\downarrow & & \downarrow & \swarrow & \\
\mu\mathrm{sh}_{\mathfrak{c}_{X,\Lambda}}(\mathfrak{c}_{X,\Lambda}) & \underset{\sim}{\longleftarrow} & \mu\mathrm{sh}_{\mathfrak{c}_{X,\Lambda}\times(0,1)}(\mathfrak{c}_{X,\Lambda} \times (0,1)) & &
\end{array}$$

where vertical arrows are microlocalization μ and horizontal arrows are restriction r.

Sheaf doubling theorem implies that furthest right μ is a projection onto an orthogonal direct summand. Thus, μ^* is inclusion, and, in particular, fully faithful.

Thus, reduced to showing
$r^* : \mathrm{Sh}_{D(\mathfrak{c}_{X,\Lambda})\times(0,1)}(M \times (0,1)) \to \mathrm{Sh}_{D(\mathfrak{c}_{X,\Lambda}\times(0,1))}(M \times \mathbb{R})$ is fully faithful on co-representatives of microstalks on the first copy of $\mathfrak{c}_{X,\Lambda} \times (0,1)$.

Fukaya categories strike back

This is in a cotangent bundle, and we can translate back to Fukaya categories. Want to show

$$\mathcal{W}(T^*(M \times (0,1)), D(\mathfrak{c}_{X,\Lambda}) \times (0,1)) \to \mathcal{W}(T^*(M \times \mathbb{R})), D(\mathfrak{c}_{X,\Lambda} \times (0,1)))$$

is fully faithful on linking disks of the first copy of $\mathfrak{c}_{X,\Lambda} \times (0,1)$. This comes from the fact that

$$(X,\Lambda) \times (\mathbb{C}, \pm\infty) \times (\mathbb{C}_{\mathrm{Re}\geq 0}, \infty) \hookrightarrow (T^*(M \times \mathbb{R}), D(\mathfrak{c}_{X,\Lambda} \times (0,1)))$$

induces a fully faithful functor on linking disks of the first copy of the stop by the doubling theorem. Moreover, this map factors through $(T^*(M \times (0,1)), D(\mathfrak{c}_{X,\Lambda}) \times (0,1))$ and full-faithfulness of the first map is proved by repeating the argument of the doubling theorem with a factor of $T^*(0,1)$.

The end

Thank you!