X topological space.

A stratification S of X is a decomposition

$$X = \coprod_{\alpha \in S} X_\alpha \text{ of locally closed subsets}$$

s.t. $\overline{X_\alpha} \setminus X_\alpha$ is a union of strata

rmk S is a poset: $\beta \to \alpha$ iff $X_\alpha \subseteq \overline{X_\beta}$

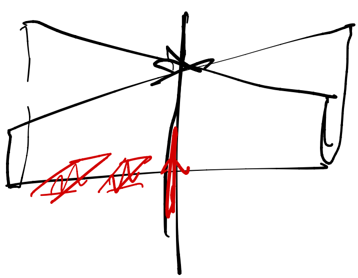
If M is a C^1-mfd a stratification by C^1-submfds is a Whitney stratification if it satisfies (X, Y strata)

(A) For any $(x_i) \to y$, $x_i \in X$, $y \in Y$ s.t. $T_{x_i} X$ converges to $V \subseteq T_y M$, we have $T_y Y \subseteq V$

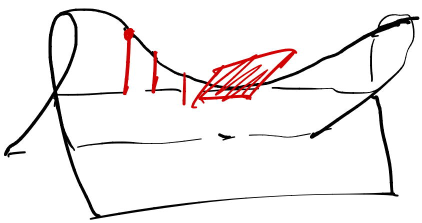
(B) For (x_i) in X, (y_i) in Y both converging to $y \in Y$ s.t. $T_{x_i} X$ converges to $V \subseteq T_y M$ and secant directions from y_i to x_i converge to $L \subseteq T_y M$, we have $L \subseteq V$

Non-examples:

(Whitney Umbrella)



(Whitney cusp)



(A) is a microlocal condition:

$$N^*S := \bigcup_\alpha N^*M_\alpha \subseteq T^*M \text{ is closed} \Leftrightarrow (A)$$

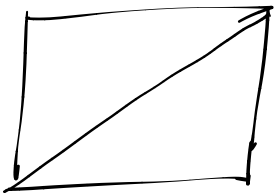
Say that $Y \subseteq M$ is <u>constructible</u> if it is a union of strata.

(<u>Slogan</u>: Compact, constructible sets have nbhds microlocally close nbhds $U \supseteq X$; N^*U converges to N^*X.)

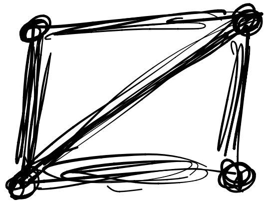
<u>Prop</u>: Fix $1 \leq p \leq \infty$, S a C^p-Whitney strat, $X \subseteq M$ an S-constructible compact,

$$N^*X := \bigcup_{M_\alpha \subseteq X} N^*M_\alpha.$$

Then $\exists$ a decreasing family X^η of nbhds of X w/ C^p-cornered boundary s.t. as $\eta \to 0$, (1) N^*X^η becomes contained in arbitrarily small cornered nbhds of N^*X, (2) N^*X^η remains disjoint from N^*S



X



X^η

Desired nbhds are

$$X^{\vec{\varepsilon}} := \bigcup_{M_\alpha \subseteq X} T_\alpha^{\varepsilon_\alpha} \quad \left(\begin{array}{c}\text{auxilliary norm} \\ \rho_\alpha : T_\alpha \to \mathbb{R}_{\geq 0}\end{array}\right)$$

vectors of norm $< \varepsilon_\alpha$ in the tube around M_α

Can write X as $\bigcup X_i$ for $X_0 \subseteq X_1 \subseteq \cdots$ loc. closed submflds w/ boundary whose conormals approach N^*S

(apply Prop to $\partial X = \overline{X} \setminus X$, $X_i := X \setminus U_i$ where U_i = nbhd of boundary)

$$SS(\mathbb{Z}_X) = ss(\varinjlim \mathbb{Z}_{X_i}) \subseteq N^*S$$