E_∞ Ring Spaces and E_∞ Ring Spectra

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Introduction

In topology, there is a dichotomy between two general classes of spaces and ways of thinking about their roles. On the one hand, there are concrete geometric spaces, most importantly the various types of manifolds. Typical problems one proposes to study about such spaces are their classification, at least up to cobordism, and the obstructions to the existence of an equivalence of a given space with a space with a richer type of structure. Bundles and fibrations over geometric spaces generally play a central role in the solution to such problems. On the other hand, there are the classifying spaces for bundles and fibrations theories and other cohomological invariants of spaces. These are thought of as tools for the analysis of geometric problems, and it is a familiar fact that theorems on the classifying space level often translate to yield intrinsic information on the bundle theory level. Thus, for example, Bott periodicity originated as a statement about the homotopy types of classifying spaces, but is most usefully interpreted as a statement about bundles and their tensor products.

The last decade has seen an intensive analysis of the homotopy types of the classifying spaces of geometric topology, with a view towards applications to the classification and obstruction theory of spaces less richly structured than differentiable manifolds. However, very much more structure than the mere homotopy type is relevant. The study of topological and PL manifolds and of Poincare duality spaces forces consideration of bundle and fibration theories the cohomology of whose classifying spaces is wholly inaccessible to the classical techniques and invariants of homotopy theory.

The appropriate framework for the study of these classifying spaces is infinite loop space theory and, in particular, its multiplicative elaboration which is the theme of this book. This is also the appropriate framework for the most structured development of algebraic K-theory, by which we understand the homotopy theory of discrete categories, and one of the main goals of this volume is a complete analysis of the relationship between the classifying spaces of geometric topology and the infinite loop spaces of algebraic K-theory. The results obtained have powerful calculational consequences, which are presented in [**26**]. For example, they make it possible to pass quite directly from representation theoretical computations of the homologies of appropriate finite groups to detailed analysis of characteristic classes for topological bundles (away from 2) and spherical fibrations.

From the point of view of classical algebraic topology, infinite loop space theory may be thought of as the use of unstable methods for the study of stable homotopy theory. Its starting point is a recognition principle for infinite loop spaces that allows one to pass back and forth between spectra and spaces with appropriate internal structure, namely E_∞ spaces. We shall enrich this additive theory with a multiplicative structure which allows one to pass back and forth between E_∞ ring spectra and E_∞ ring spaces.

Conceptually, the new theory is different in kind from the old: the appropriate multiplicative structure on spectra is itself unstable in that it appears not to admit an equivalent formulation expressible solely in terms of structure visible to the stable homotopy category. This is because the relevant structure requires very precise algebraic data on the point-set level on spaces which together comprise the spectrum. The application that concern us here center around space level exploitation of this algebraic data. It has very recently become possible to express a significant

portion of the structure in terms of maps in the stable category. This reformulation leads to applications in stable homotopy theory and will be the subject of a future volume.

The prototype of an E_∞ ring spectrum is the sphere spectrum $Q_\infty S^0$. Its zero$^{\text{th}}$ space QS^0 is the prototype of an E_∞ ring space. For purposes of both theory and calculation, the essential feature of QS^0 is the interrelationship between its additive infinite loop space structure and the multiplicative infinite loop space structure on the component SF of its identity element. It is such interrelationships which will be codified in our basic definitions.

If we ignore QS^0 and concentrate on SF, then the interest focuses on geometric topology and, in particular, on the relationships among F, Top, PL, the classical groups, and their homogeneous and classifying spaces. We give consistent infinite loop space structures to these spaces in Chapter I by use of the theory of E_∞ spaces developed in [**45**] (and recapitulated later in this volume). The basic results here were originally due to Boardman and Vogt [**19, 20**].

Our construction of an infinite loop space structure on SF depends on use not just of the $SF(n)$ but on the $SF(V)$ for all finite-dimensional real inner product spaces $V \subset \mathbb{R}^\infty$. If we are to see this structure in the spectrum $Q_\infty S^0$, then spectra too must be indexed on such V rather than just on the non-negative integers. We define such spectra in Chapter II, which is mainly an extract and summary from [**48**] of that slight amount of information about coordinate-free spectra and the stable homotopy category needed in this book.

We reach the definition of E_∞ ring spectra in Chapter IV, which is joint work with Frank Quinn and Nigel Ray. The fundamental idea of putting an E_∞ space structure on a spectrum in order to obtain the richest possible notion of a ring spectrum is entirely due to Frank Quinn. A manageable and technically correct way of doing this eluded us for some time. The essential insight leading to the correct definition came from Nigel Ray, who pointed out that good concrete constructions of Thom spectra gave naturally occurring examples of spectra with the right kind of structure and that these examples could be taken as models on which to pattern the general definition.

One key family of classifying spaces in geometric topology is missing from Chapter I, namely the classifying spaces for sphere bundles of a given type oriented with respect to a specified cohomology theory. In fact, no concrete constructions of such classifying spaces exist in the literature. We remedy this in Chapter III by use of the general classification theory for fibrations and bundles developed in [**47**]. In Section IV§3 we give these classifying spaces infinite loop space structures when the specified cohomology theory is represented by an E_∞ ring spectrum.

In Chapter V, we demonstrate that formal analysis on the classifying space level allows one to deduce sharpened versions of the results of Adams [**4, 5**] on $J(X)$ and of Sullivan [**72**] on topological bundle theory (away from the prime 2) from the kO-orientation of $Spin$-bundles and the $kO[1/2]$-orientation of $STop$-bundles together with the Adams operations, the cannibalistic classes derived from them by use of the specified orientations, and the Adams conjecture. In the last section, we combine various results from throughout the volume with results of Adams and Priddy [**8**], Madsen, Snaith, and Tornehave [**42**], and Ligaard [**38**] to analyze the infinite loop space structure of $BTop$ (away from 2) and of various other classifying spaces utilized in earlier parts of the chapter. This material completes most of the program envisioned in a preprint version of this chapter.

Aside from its last section, Chapter V is largely independent of infinite loop space theory and is reasonably self-contained. However, its earlier sections do make essential use of the fact that real connective K-theory is represented by an E_∞ ring spectrum kO. This fact implies in particular that $BO_\otimes$, the component of the identity element of the zero$^{\text{th}}$ space of kO, is an infinite loop space. Much more significantly, it encodes the interrelationship between kO and the spectrum determined by $BO_\otimes$. It is one easy consequence of this interrelationship which is exploited in Chapter V. In contrast to the case of Thom spectra, kO does not seem to occur in nature as an E_∞ ring spectrum and must therefore be manufactured.

This brings us to the last four chapters which, aside from use of definitions contained in the first sections of Chapters I, II and IV, are largely independent of the first five. In Chapter IV, we

define E_∞ ring spaces and show that the classifying spaces of categories with appropriate internal structure, namely bipermutative categories, are examples of such spaces. In Chapter VII, we retool the machine constructed in [**45**, **46**] for the manufacture of spectra from E_∞ spaces so as to make it turn out coordinate-free spectra. We then show that if the machine is fed the additive structure of an E_∞ ring space, then it turns out an E_∞ ring spectrum. One immediate application is a multiplicatively enriched version of the Barratt–Quillen theorem [**16**, **68**, **46**] to the effect that QS^0, and thus the stable homotopy groups of spheres, can be constructed out of symmetric groups. Our version shows that the infinite loop space SF, and thus the classifying space BSF for stable spherical fibrations, can also be constructed out of symmetric groups.

In the last section of Chapter VII, we study the spectra turned out when the machine is fed the multiplicative structure of an E_∞ ring space. In particular, we obtain a purely multiplicative version of the relationship between SF and symmetric groups. In Chapter VIII, we give the promised analysis of the relationship between the classifying spaces of geometric topology and the infinite loop spaces of algebraic K-theory. This basic material is a mosaic of results due to Jørgen Tornehave and myself and includes new proofs and generalizations of the results originally given in his thesis [**75**] and in his unpublished preprints [**76**] and [**77**]; it is presented here under joint authorship. The connection between algebraic and topological K-theory was established in the work of Quillen [**58**, **59**], and we show that the maps given by Brauer lifting which he used to prove the Adams conjecture are infinite loop maps, both additively and multiplicatively. Via the Frobenius automorphism, this information yields a good understanding of the infinite loop space $B\operatorname{Coker} J$, which is the basic building block for BSF and for $BTop$ (away from 2) and turns out to be the classifying space for j-oriented stable spherical fibrations for a suitable E_∞ ring spectrum j. We also show that BSF splits as $B\operatorname{Im} J \times B\operatorname{Coker} J$ as an infinite loop space when localized at an odd prime p, and that, at 2, there is a (non-splittable) infinite loop fibration $B\operatorname{Coker} J \to BSF \to B\operatorname{Im} J$.

Chapter IX contains a theory of pairings in infinite loop space theory. This is used to compare our machine-built spectra of algebraic K-theory to the spectra constructed by Gersten and Wagoner [**30**, **79**].

Logically, this book is a sequel to [**45**, **46**]. However, I have tried to make it self-contained modulo proofs. Thus the definitions of operads and E_∞ spaces are recalled in Section VI§1, and the main results of the cited papers are stated without proof in Sections VII§1 to VII§3. Nevertheless, the reader may find a preliminary reading of the first three sections of [**45**] helpful, as they contain a leisurely explanation of the motivation behind the basic definitions. While a full understanding of the constructions used in Chapter VII requires preliminary reading of [**45**, §9 and §11], the pragmatic (and trusting) reader may regard the results of that chapter as existence statements derived by means of a black box, the internal intricacies of which can safely be ignored in the applications of the remaining chapters.

It is to be stressed, however, that all of our applications which go beyond the mere assertion that a given space is an infinite loop space depend on special features — the new multiplicative structure, various consistency statements, flexibility in the choice of raw materials — of our black box which allows us to fit together different parts of the theory.

We illustrate this point with a discussion of how BO appears in our theory. As explained in Chapter I, the ordinary classifying space of the infinite orthogonal group is an E_∞ space and thus an infinite loop space. As explained in Chapter VI, $\coprod_n BO(n)$ is an E_∞ ring space (and the relevant E_∞ operad is different from that used in Chapter I); the zero$^{\text{th}}$ space of the resulting E_∞ ring spectrum is equivalent to $BO \times \mathbb{Z}$. We thus have two infinite loop space structures on BO corresponding to two machine-built connective spectra. If we are to take these structures seriously, then we must prove that the machine-built spectra are equivalent to that obtained from the periodic Bott spectrum by killing its homotopy groups in negative degrees. (Other manufacturers of black boxes have not yet studied such consistency problems.) For our first model, the required proof

follows from a commutation relation between looping and delooping. For our second model, the required proof follows directly from the ring structure. In both cases, we rely on a characterization of the connective spectrum associated to a periodic space which only makes sense because of special features of our new construction of the stable homotopy category. That both models are necessary can be seen most clearly in the orientation sequence for kO-oriented stable spherical fibrations. This is a fibration sequence of infinite loop spaces

$$SF \longrightarrow BO_\otimes \longrightarrow B(SF; kO) \longrightarrow BSF$$

which is defined by use of the E_∞ structure on SF coming from Chapter I together with the E_∞ ring structure on kO given by the second model. The first model is essential to relate this sequence to the natural map $j\colon SO \to SF$ on the infinite loop level. Many of our applications of Chapters V and VIII center around this sequence, and its derivation really seems to require every bit of our general abstract machinery. That even much simpler consistency question about Bott periodicity is not altogether trivial is indicated by the fact that our easier second proof will apply equally well to prove the result about (additive) Brauer lifting cited above.

Beyond its new results, this book is intended to give a coherent account of the most important descriptive (as opposed to computational) applications of infinite loop space theory. A thorough study of the homology of E_∞ spaces and of E_∞ ring space appears in [**26**], where the theory and results of the present volume are applied to the study of characteristic classes. An informal summary of the material in both that volume and this one is given in [**49**], which gives an extended intuitive introduction to this general area of topology.

It is a pleasure to acknowledge my debt to the very many people who have helped me with this work. I owe details to Don Anderson, Zig Fiedorowicz, Dick Lashof, Arunas Liulevicius, Stewart Priddy, Vic Snaith, Mark Steinberger, Dick Swan, and Larry Taylor. I am particularly indebted to my coauthors Frank Quinn, Niel Ray, and Jørgen Tornehave for their ideas and insights, to Ib Madsen for key discussions of 2-primary phenomena and correspondence about various aspects of this work, and to Frank Adams for proofs of a number of technical lemmas and much other help. I owe a special debt to Jim Stasheff to his careful reading of the entire manuscript and his many suggestions for its improvement.

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CHAPTER I

$\mathscr{I}$ functors

In [**19**], Boardman and Vogt introduced the concept of $\mathscr{I}$-functor. Their purpose was to show how certain collections of spaces, such as BFV or $BTopV$, indexed on inner product spaces V produce E_∞ spaces and therefore infinite loop spaces by passing to limits over $V \subset \mathbb{R}^\infty$. In Section I§1 we give a detailed exposition of this part of their theory, reformulated in terms of operads and operad actions as defined in [**45**, §1]. In particular, we give a systematic discussion of the classical groups of their homogeneous spaces as $\mathscr{I}$-functors and display the Bott maps as morphisms of $\mathscr{I}$-functors. In Section I§2, we related the two-sided geometric bar construction to $\mathscr{I}$-functors and rederive the theorems of Boardman and Vogt to the effect that F, Top, PL and the related classifying spaces and homogeneous spaces are infinite loop spaces. (We shall use brief ad hoc arguments based on the triangulation theorem to handle PL.) Most of the material of these sections dates from 1971 and has been circulating in preprint form since 1972. Boardman and Vogt's own account of their theory has since appeared [**20**]. Their language and choice of details are quite different from ours, and there is very little overlap. The present language and results will be needed in the rest of this book.

1. Linear isometries and classical groups

Let $\mathscr{T}$ denote the category of compactly generated and nondegenerately based weak Hausdorff spaces. For an operad $\mathscr{C}$, let $\mathscr{C}[\mathscr{T}]$ denote the category of $\mathscr{C}$-spaces [**45**, §1] (or Section VI§1).

To obtain an action of an operad on the infinite classical groups and related spaces, it is conceptually and notationally simplest to pass from a functor defined on a certain category $\mathscr{I}$ to an action by a related operad $\mathscr{L}$. The definition and properties of $\mathscr{I}$ and $\mathscr{L}$ are due to Boardman and Vogt [**19, 20**].

Definition I.1.1. Define the category $\mathscr{I}$ of linear isometries as follows. The objects of $\mathscr{I}$ are finite or countably infinite dimensional real inner product spaces, topologized as the limits of their finite dimensional subspaces. The morphisms $\mathscr{I}(V, W)$ from V to W are linear isometries $V \to W$, and $\mathscr{I}(V, W)$ is given the (compactly generated) compact-open topology. Note that the direct sum $\oplus \colon \mathscr{I} \times \mathscr{I} \to \mathscr{I}$ is a continuous functor and is commutative, associative, and unital (with unit $\{0\}$) up to coherent natural isomorphism.

Definition I.1.2. Define the linear isometries operad $\mathscr{L}$ by $\mathscr{L}(j) = \mathscr{I}((\mathbb{R}^\infty)^j, \mathbb{R}^\infty)$, where $(\mathbb{R}^\infty)^j$ is the direct sum of j copies of $\mathbb{R}^\infty$ with its standard inner product; the requisite data are specified by

    (a) $\gamma(f; g_1, \ldots, g_k) = f \circ (g_1 \oplus \cdots \oplus g_k)$, $f \in \mathscr{L}(k)$ and $g_i \in \mathscr{L}(j_i)$.
    (b) $1 \in \mathscr{L}(i)$ is the identity map.
    (c) $(f\sigma)(y) = f(\sigma y)$ for $f \in \mathscr{L}(j)$, $\sigma \in \Sigma_j$, and $y \in (\mathbb{R}^\infty)^j$.

In other words, $\mathscr{L}$ is required to be a sub-operad of the endomorphism operad of $\mathbb{R}^\infty$ (where $\mathbb{R}^\infty$ has basepoint zero).

It is trivial to verify that Σ_j acts freely on $\mathscr{L}(j)$. The following lemma therefore implies that $\mathscr{L}$ is an E_∞ operad. Recall that isometries need not be isomorphisms.

Lemma I.1.3. $\mathscr{I}(V, \mathbb{R}^\infty)$ *is contractible for all inner product spaces V.*

PROOF. Let $\{e_i \mid i \geq 1\}$, $\{e_i', e_i'' \mid i \geq 1\}$, $\{f_j\}$, $\{f_j', f_j''\}$ be orthonormal bases for $\mathbb{R}^\infty$, $\mathbb{R}^\infty \oplus \mathbb{R}^\infty$, V, and $V \oplus V$ respectively. Define $\alpha \colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ by $\alpha(e_i) = e_{2i}$ and define $\beta \colon \mathbb{R}^\infty \to \mathbb{R}^\infty \oplus \mathbb{R}^\infty$ by $\beta(e_{2i-1}) = e_i''$ and $\beta(e_{2i}) = e_i'$. Then β is an isomorphism such that $\beta\alpha = i'$, the injection of the first summand. Define a path $H_1 \colon I \to \mathscr{I}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ from the identity to α and define a map $H_2 \colon I \to \mathscr{I}(V, V \oplus V)$ from i' to i'' by normalizing the obvious linear paths

$$G_1(t)(e_i)(1-t)e_i + te_{2i} \quad \text{and} \quad G_2(t)(f_j) = (1-t)f_j' + tf_j''.$$

Fix $\gamma \in \mathscr{I}(V, \mathbb{R}^\infty)$ and define $H \colon I \times \mathscr{V}(V, \mathbb{R}^\infty) \to \mathscr{I}(V, \mathbb{R}^\infty)$ by

$$H(t,k) = \begin{cases} H_1(2t) \circ k, & \text{if } 0 \leq t \leq 1/2 \\ \beta^{-1} \circ (k \oplus \gamma) \circ H_2(2t-1), & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

Then $H(0,k) = k$ and $H(1,k) = \beta^{-1}i''\gamma$, which is independent of k. $\square$

We now define $\mathscr{I}$-functors and a functor from $\mathscr{I}$-functors to $\mathscr{L}$-spaces.

Definition I.1.4. An $\mathscr{I}$-functor (T, ω) is a continuous functor $T \colon \mathscr{I} \to \mathscr{T}$ together with a commutative, associative, and continuous natural transformation $\omega \colon T \times T \to T \circ \oplus$ (of functors $\mathscr{I} \times \mathscr{I} \to \mathscr{T}$) such that

(a) if $x \in TV$ and if $1 \in T\{0\}$ is the basepoint, then

$$\omega(x, 1) = x \in T(V \oplus \{0\}) = TV$$

(b) if $V = V' \oplus V''$, $\dim V' < \infty$, and if $i \colon V' \to V$ is the inclusion, then $Ti \colon TV' \to TV$ (which , by (a), is given by $Ti(x) = \omega(x,1)$, where 1 is the basepoint of TV'') is a homeomorphism onto a closed subset, and

(c) $TV = \varinjlim TV'$ as a space, where V' runs over the finite dimensional sub inner product spaces of V.

We call ω the Whitney sum; for $x_i \in TV_i$, $1 \leq i \leq j$, we write $\omega(x_1, \ldots, x_j) = x_1 \oplus \cdots \oplus x_j$. A morphism $\Phi \colon (T, \omega) \to (T', \omega')$ of $\mathscr{I}$-functors is a continuous natural transformation $\Phi \colon T \to T'$ which commutes with the Whitney sums. $\mathscr{I}[\mathscr{T}]$ denotes the category of $\mathscr{I}$-functors.

Remark I.1.5.   (i) The category $\mathscr{I}[\mathscr{T}]$ has finite products; if $t \colon T' \times T \to T \times T'$ is the interchange natural transformation, then

$$(T, \omega) \times (T', \omega') = (T \times T', (\omega \times \omega')(1 \times t \times 1)).$$

Similarly, the category $\mathscr{I}[\mathscr{T}]$ has fibred products.

(ii) For $C \in \mathscr{T}$ and an $\mathscr{I}$-functor (T, ω), define the function space $\mathscr{I}$-functor $F(C, T)$ by $F(C, T)(V) = F(C, TV)$, with Whitney sum the composite

$$F(C, TV) \times F(C, TW) \longrightarrow F(C \times C, TV \times TW) \xrightarrow{\Phi(\Delta, \omega)} F(C, T(V \oplus W)).$$

(iii) If U is the universal covering space functor and if (T, ω) is an $\mathscr{I}$-functor, then $(UT, U\omega)$ is an $\mathscr{I}$-functor, where $U\omega$ is induced from $P\omega$ on $PT = F(I, T)$ by passage to quotient spaces (compare [**26**, I.4.8]).

Definition I.1.6. Define a functor $\Theta \colon \mathscr{I}[\mathscr{T}] \to \mathscr{L}[\mathscr{T}]$ by letting $\Theta(T, \omega) = (T\mathbb{R}^\infty, \theta)$ on objects and $\Theta(\Phi) = \Phi \colon T\mathbb{R}^\infty \to T'\mathbb{R}^\infty$ on morphisms, where $\theta_j \colon \mathscr{L}(j) \times (T\mathbb{R}^\infty)^j \to T\mathbb{R}^\infty$ is defined by

$$\theta_j(f, x_1, \ldots, x_j) = (Tf)(x_1 \oplus \cdots \oplus x_j), \ f \in \mathscr{L}(j) \text{ and } x_i \in T\mathbb{R}^\infty.$$

θ_j is continuous by the continuity of T and ω. Observe that Θ commutes with the various constructions specified in the previous remarks, where these constructions are defined on $\mathscr{L}[\mathscr{T}]$ by [**45**, 1.5–1.7] and [**26**, I.4.8]. As is customary, we shall often write T both for $\mathscr{I}$-functors and for the derived $\mathscr{L}$-spaces $T\mathbb{R}^\infty$.

Remark I.1.7. Let $\alpha\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ be a linear isometric isomorphism. Then α determines an automorphism $\mathscr{L}\alpha$ of the operad $\mathscr{L}$ by $(\mathscr{L}\alpha)(f) = \alpha f(\alpha^{-1})^j\colon (\mathbb{R}^\infty)^j \to \mathbb{R}^\infty$ for $f \in \mathscr{L}(j)$. If (T, ω) is an $\mathscr{I}$-functor, then $T\alpha\colon T\mathbb{R}^\infty \to T\mathbb{R}^\infty$ is an $\mathscr{L}\alpha$-equivariant homeomorphism, in the sense that $T\alpha \circ \theta_j = \theta_j \circ (\mathscr{L}\alpha \times (T\alpha)^j)$.

To construct $\mathscr{L}$-space, we need only construct $\mathscr{I}$-functors. We next show that to construct $\mathscr{I}$-functors we need only study finite-dimensional inner product spaces and their linear isometric isomorphisms.

Definition I.1.8. Let $\mathscr{I}_n$, $n < \infty$, be the full subcategory of $\mathscr{I}$ whose objects are n-dimensional, and let $\mathscr{I}_*$ be the graded subcategory of $\mathscr{I}$ consisting of the union of the $\mathscr{I}_n$. Note that the functor $\oplus\colon \mathscr{I}_m \times \mathscr{I}_n \to \mathscr{I}_{m+n}$ is a continuous functor $T\colon \mathscr{I}_* \to \mathscr{T}$ together define a graded functor $\oplus\colon \mathscr{I}_* \times \mathscr{I}_* \to \mathscr{I}_*$. An $\mathscr{I}_*$-functor (T, ω) is a continuous functor $T\colon \mathscr{I}_* \to \mathscr{T}$ together with a commutative, associative, and continuous natural transformation $\omega\colon T \times T \to T \circ \oplus$ such that

(a) if $x \in TV$ and if $1 \in T\{0\}$ is the basepoint, then
$$\omega(x, 1) = x \in T(V \oplus \{0\}) = TV.$$

(b) if $V = V' \oplus V''$, $\dim V < \infty$, then the map $TV' \to TV$ given by $x \to \omega(x, 1)$ is a homeomorphism onto a closed subset.

Morphisms of $\mathscr{I}_*$-functors are defined in the evident way, and $\mathscr{I}_*[\mathscr{T}]$ denotes the category of $\mathscr{I}_*$-functors.

Proposition I.1.9. *The forgetful functor $\mathscr{I}[\mathscr{T}] \to \mathscr{I}_*[\mathscr{T}]$ is an isomorphism of categories.*

PROOF. We must verify that an $\mathscr{I}_*$-functor (T, ω) admits a unique extension to an $\mathscr{I}$-functor. If $\dim V = \infty$, we can and must define TV by Definition I.1.4(c); we shall write $x' \oplus 1$ for the image in TV of $x' \in TV'$ (since $V = V' \oplus (V')^\perp$ when $\dim V' < \infty$). Similarly, we can and must define $\omega\colon TV \times TW \to T(V \oplus W)$ by $x \oplus y = (x' \oplus y') \oplus 1$ if $x = x' \oplus 1$ and $y = y' \oplus 1$ with $x' \in TV'$ and $y' \in TW'$ for finite dimensional subspaces V' of V and W' of W. Finally, if $f\colon V \to W$ is a linear isometry and if $x = x' \oplus 1 \in TV$, with $x' \in TV'$ where $\dim V' < \infty$, we define

(I.1)
$$(Tf)(x) = (Tf')(x') \oplus 1, \text{ where } f' = f|_{V'}\colon V' \to f(V').$$

This definition is forced since the image of $f'' = f|_{(V')^\perp}$ is contained in $f(V')^\perp$, hence $f = f' \oplus f''$, and we therefore must have
$$(Tf)(x) = T(f' \oplus f'')(x' \oplus 1) = (Tf')(x') \oplus (Tf'')(1) = (Tf')(x') \oplus 1,$$
the last equality holding since Tf'' must preserve basepoints. It is straightforward to verify that (T, ω), so constructed, is indeed a well-defined $\mathscr{I}$-functor. Similarly, morphisms of $\mathscr{I}_*$-functors extend uniquely to morphisms of $\mathscr{I}$-functors by passage to limits. $\square$

Henceforward, we shall identify the categories $\mathscr{I}_*[\mathscr{T}]$ and $\mathscr{I}[\mathscr{T}]$. We shall speak of $\mathscr{I}$-functors but shall only construct the underlying $\mathscr{I}_*$-functors. The following remarks will be basic to the applications.

Remark I.1.10. For many of the $\mathscr{I}$-functors (T, ω) of interest, the points $x \in TV$ will be (or will be derived from) maps $tV \to tV$ for some space tV depending functorially on V; the basepoint of TV will be the identity map of tV. Moreover, when $\dim V = \dim W < \infty$, a point $f \in \mathscr{I}(V, W)$ will determine a homeomorphism $tf\colon tV \to tW$ and we will have

(I.2)
$$(Tf)(x) = tf \circ x \circ (tf)^{-1} \text{ for } x \in TV.$$

Henceforward, we shall generally replace formulas (I.1) and (I.2) by the notationally simpler expression
$$(Tf)(x) = fxf^{-1} \text{ for } x \in TV \text{ and } f \in \mathscr{I}(V, W).$$

We thus suppress from the notation both the passage from f to tf and the required restriction to finite dimensional subspaces. It will often be the case that TV is a sub topological monoid of the monoid (under composition) of maps $tV \to tV$. It will follow from that the composition product $c\colon TV \times TV \to TV$ defines a morphism of $\mathscr{I}$-functors. Indeed, the commutativity of the diagrams

$$
\begin{array}{ccc}
TV \times TV & \xrightarrow{\ c\ } & TV \\
{\scriptstyle Tf \times Tf}\downarrow & & \downarrow{\scriptstyle Tf} \\
TW \times TW & \xrightarrow{\ c\ } & TW
\end{array}
\quad\text{and}\quad
\begin{array}{ccc}
TV \times TV \times TW \times TW & \xrightarrow{\ c \times c\ } & TV \times TW \\
{\scriptstyle \omega}\downarrow & & \downarrow{\scriptstyle \omega} \\
T(V \oplus W) \times T(V \oplus W) & \xrightarrow{\ c\ } & T(V \oplus W)
\end{array}
$$

simply amounts to the validity of the formulas

$$
fgg'f^{-1} = (fgf^{-1})(fg'f^{-1}) \quad\text{and}\quad gg' \oplus hh' = (g \oplus h)(g' \oplus h')
$$

for $f \in \mathscr{I}(V, W)$, $g, g' \in TV$, and $h, h' \in TW$. On $T\mathbb{R}^\infty$, the composition product will be homotopic to the (internal) Whitney sum induced from the action of $\mathscr{L}$ (see [**45**, 8.7] or [**46**, 3.4]). When, further, TV is a sub topological group of the group of homeomorphisms $tV \to tV$, the inverse map $i\colon TV \to TV$ will define a morphism of $\mathscr{I}$-functors by virtue of the formulas

$$
(fgf^{-1})^{-1} = fg^{-1}f^{-1} \quad\text{and}\quad (g \oplus h)^{-1} = g^{-1} \oplus h^{-1}.
$$

We now define the classical groups and their homogeneous spaces systematically as $\mathscr{I}$-functors. Let $\mathbb{K}$ denote one of the normed division rings $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (real numbers, complex numbers, or quaternions). For a real inner product space V, let $V_{\mathbb{K}}$ denote $\mathbb{K} \otimes_{\mathbb{R}} V$ regarded as a (left) inner product space over $\mathbb{K}$ and, for $\mathbb{J} \subset \mathbb{K}$, let $V_{\mathbb{K}}^{\mathbb{J}}$ denote $V_{\mathbb{K}}$ regarded as an inner product space over $\mathbb{J}$. By a classical group G, we understand any one of the following functors from $\mathscr{I}$ to the category of topological groups:

$$
\begin{array}{ccccccccc}
e & \subset & SO(V) & \subset & O(V) \\
& & \cap & & \cap \\
SU(V_{\mathbb{C}}) & \subset & U(V_{\mathbb{C}}) & \subset & SO(V_{\mathbb{C}}^{\mathbb{R}}) & \subset & O(V_{\mathbb{C}}^{\mathbb{R}}) \\
& & \cap & & \cap & & \cap \\
Sp(V_{\mathbb{H}}) & \subset & SU(V_{\mathbb{H}}^{\mathbb{C}}) & \subset & U(V_{\mathbb{H}}^{\mathbb{C}}) & \subset & SO(V_{\mathbb{H}}^{\mathbb{R}}) & \subset & O(V_{\mathbb{H}}^{\mathbb{R}}).
\end{array}
$$

(The vertical inclusions are all of the form $g \to 1 \otimes g$.) If G is a classical group and if G_i, $1 \le i \le n$, are classical subgroups of G (in the sense of the lattice above), then the homogeneous space $G / \prod_{i=1}^{n} G_i$ is defined to be the functor $\mathscr{I} \to \mathscr{T}$ which is given on objects by

$$
\left(G \Big/ \prod_{i=1}^{n} G_i \right)(V) = G(V^n) \Big/ \prod_{i=1}^{n} G_i(V).
$$

For $f \in \mathscr{I}(V, W)$, $\dim V = \dim W < \infty$, $(G / \prod_{i=1}^{n} G_i)(f)$ is induced by passage to orbit spaces from $g \to f^n g (f^{-1})^n$ for $g \in G(V^n)$, where $f^n\colon V^n \to W^n$ is the direct sum of f with itself n times. With Whitney sum induced by passage to orbit spaces from the composite

$$
G(V^n) \times G(W^n) \xrightarrow{\ \oplus\ } G(V^n \oplus W^n) \xrightarrow{\ G(\nu)\ } G((V \oplus W)^n),
$$

where ν is the evident shuffle isometry, each $(G / \prod_{i=1}^{n} G_i, \omega)$ is an $\mathscr{I}$-functor. If $H \subset G$ and $H_i \subset G_i$, then the evident maps of orbit spaces define a morphism of $\mathscr{I}$-functors $H / \prod_{i=1}^{n} H_i \to G / \prod_{i=1}^{n} G_i$; when $H = G$, the inclusion of the fibre $\prod_{i=1}^{n} G_i / H_i$ in $G / \prod_{i=1}^{n} H_i$ is also a morphism of $\mathscr{I}$-functors. By the universal bundle of a classical group G, we understand the morphism of $\mathscr{I}$-functors

$$
\pi\colon EG = G/(e \times G) \longrightarrow G/(G \times G) = BG
$$

obtained by setting $n = 2$, $G_1 = e$ or G, and $G_2 = G$ in the framework above. Remark I.1.10 show that the product and inverse map on each classical group G are morphisms of $\mathscr{I}$-functors.

We can define $Spin$ by letting $Spin(V)$ be the universal cover of $SO(V)$. Alternatively, and preferably, $Spin$, and also $Spin^c$, Pin, and Pin^c can be explicitly described as $\mathscr{I}$-functors by means of their standard descriptions in terms of the Clifford algebras of inner product spaces [12]. The product and inverse maps on these groups and the usual maps between these groups and classical groups are then morphisms of $\mathscr{I}$-functors.

We observe next that each of the Bott maps may be regarded as a morphism of $\mathscr{I}$-functors of the form
$$\beta\colon H/(H_1 \times H_2) \longrightarrow \Omega(G/(G_1 \times G_2)).$$
In fact, each β is induced by passage to orbit spaces from a map $\beta\colon H(V^2) \to \Omega G(V^2)$ of the form
$$\beta(g)(t) = g\alpha(t)g^{-1}\alpha^{-1}(t), \ g \in H(V^2) \text{ and } 0 \le t \le 1.$$
Here $\alpha(t)\colon V_{\mathbb{K}}^2 \to V_{\mathbb{K}}^2$ is a linear isometric isomorphism of the general form
$$\alpha(t)(v, v') = (v\alpha_1(t), v'\alpha_2(t)) \text{ for } v, v' \in V_{\mathbb{K}}$$
where the $\alpha_i(t)$ are elements of norm one in the relevant ground field $\mathbb{K}$. For example,
$$\beta\colon BU = U/(U \times U) \longrightarrow \Omega(SU/(e \times e)) = \Omega SU$$
is so determined by $\alpha_1(t) = e^{\pi i t}$ and $\alpha_2(t) = e^{-\pi i t}$. Explicit definitions of the $\alpha_i(t)$ required in the real case may be found in [21, 25]. The verification that each β is a morphism of $\mathscr{I}$-functors is an easy calculation from the form of the maps (and explicit expressions for the $\alpha_i(t)$ are not needed). The point is that if $f \in \mathscr{I}(V, W)$, then $f_{\mathbb{K}}^2$ commutes with $\alpha_i(t)$.

In order to iterate the Bott maps, it is necessary to use the natural maps $\zeta\colon G \to \Omega BG$ for $G = O$, U, and Sp. Propositions I.2.3 and I.2.4 below show that, after passage to $\mathscr{L}$-spaces, these maps become composites of $\mathscr{L}$-maps and of homotopy inverses of $\mathscr{L}$-maps which are homotopy equivalences. As our definition of the stable category and our arguments in Section II§3 will make clear, these inverses will not complicate the following discussion.

Definition I.1.6 yields a natural structure of $\mathscr{L}$-space on the homogeneous spaces $G/\prod_{i=1}^n G_i = (G/\prod_{i=1}^n G_i)(\mathbb{R}^\infty)$, on $Spin$, etc. Each of these spaces is grouplike (π_0 is a group) and is thus an infinite loop space by [46, 2.3] (or Proposition VII.3.2 below). Certain of these spaces are also infinite loop spaces by Bott periodicity. To show the consistency of these structures, let X be one of the spaces entering into Bott periodicity of period d, $d = 2$ or $d = 8$, such as $\Omega^2 BU \simeq BU \times \mathbb{Z}$ or $\Omega^8 BO \simeq BO \times \mathbb{Z}$. The $\mathscr{L}$-space structure on X determines a connective coordinatized spectrum (or infinite loop sequence) $B_\infty X = \{B_i X\}$ such that $B_0 X$ is naturally equivalent to X. The iterated Bott map $\beta\colon X \to \Omega^d X$ induces a weak homotopy equivalence $B_\infty \beta\colon B_\infty X \to B_\infty \Omega^d X$ of spectra. As will be explained in Proposition VII.3.4, [46, 3.1] gives a map $B_\infty \Omega^d X \to \Omega^d B_\infty X$ in the stable category, the zero$^{\text{th}}$ level of which is equivalent to the identity map of $\Omega^d X$. By composition, we therefore have a map $B_\infty X \to \Omega^d B_\infty X$, the zero$^{\text{th}}$ level of which is equivalent to $\beta\colon X \to \Omega^d X$. It is intuitively obvious and will be rigorously proven in Section II§3, that the connective spectrum associated to the periodic spectrum with zero$^{\text{th}}$ space X determined by β is characterized, up to isomorphism in the stable category, by precisely these conditions.

2. The bar construction; F, Top, and PL

The two-sided geometric bar construction will play a central role in our theory, and the following notations will be used throughout the later chapters. Let G be a topological monoid the identity element of which is a nondegenerate basepoint and let X and Y be left and right G-spaces. Then there is a simplicial topological space $B_*(Y, G, X)$, the n-th space of which is $Y \times G^n \times X$ [45, §10]. Its geometric realization [45, 11.1] will be denoted $B(Y, G, X)$. We shall always write
$$p\colon B(Y, G, X) \longrightarrow B(Y, G, *) \quad \text{and} \quad q\colon B(Y, G, X) \longrightarrow B(*, G, X)$$

for the maps induced from the trivial G-maps $X \to *$ and $Y \to *$, where $*$ is the one point G-space. p and q are quasi-fibrations with fibres X and Y when G is grouplike and are G-bundles when G is a topological group [**47**, 7.6 and 8.2]. We shall write

$$\tau = \tau(\rho)\colon Z \longrightarrow B(Y, G, X) \quad \text{and} \quad \varepsilon = \varepsilon(\lambda)\colon B(Y, G, X) \longrightarrow Z$$

for the maps induced by a map $\rho\colon Z \to Y \times X$ and by a map $\lambda\colon Y \times X \to Z$ such that $\lambda(yg, x) = \lambda(y, gx)$ [**45**, 9.2]; the intended choice of ρ and λ should be clear from the context. By [**45**, 9.8, 9.9, and 11.10], the G-maps

$$\varepsilon\colon B(Y, G, G) \longrightarrow Y \quad \text{and} \quad \varepsilon\colon B(G, G, X) \longrightarrow X$$

are deformation retractions with right inverses the evident maps τ. We agree to abbreviate $B(\alpha, \gamma, \beta) = B\alpha$ if γ and β are identity maps, and similarly in the other variables.

$BG = B(*, G, *)$ is the standard classifying space of G. If G is group-like, the natural inclusion $\zeta\colon G \to \Omega BG$ is a weak homotopy equivalence. If, further, $p'\colon E' \to B'$ is a principal quasi G-fibration with E' aspherical, then the maps

$$B' \xleftarrow{\varepsilon(p')} B(E', G, *) \xrightarrow{q} BG$$

display a weak homotopy equivalence between B' and BG [**47**, 8.7 and 7.7].

For a morphism $j\colon H \to G$ of grouplike topological monoids, define the left and right "homogeneous spaces" by

$$G/H = B(G, H, *) \quad \text{and} \quad H\backslash G = B(*, H, G).$$

There is a weak homotopy equivalence $\theta\colon G/H \to FBj$, where FBj denotes the homotopy theoretic fibre, and a quasifibration sequence

$$H \xrightarrow{j} G \xrightarrow{\tau} G/H \xrightarrow{q} BH \xrightarrow{Bj} BG$$

[**47**, 8.8]. By symmetry, the same statements hold for $H\backslash G$.

Now let $\mathscr{C}$ be any operad. As explained in [**46**, §3], $B(Y, G, X)$ is a $\mathscr{C}$-space when Y, G, and X are $\mathscr{C}$-spaces and the product and unit of G and its action on Y and on X are morphisms of $\mathscr{C}$-spaces. Such a G is said to be a monoid in $\mathscr{C}[\mathscr{T}]$; its monoid product is homotopic to the product given by the action of $\mathscr{C}$ [**46**, 3.4]. When all given maps in any of the constructions of the previous paragraphs are morphisms of $\mathscr{C}$-spaces, then so are all derived maps (where homotopies $X \times I \to Y$ are interpreted as maps $X \to F(I^+, Y)$ and where the fibre of a $\mathscr{C}$-map is a $\mathscr{C}$-space via [**45**, 1.8]). For most of the maps above, this statement follows from the fact that geometric realization defines a functor from simplicial $\mathscr{C}$-spaces to $\mathscr{C}$-spaces [**45**, 12.2]. For the remaining maps, easy direct calculations as in [**46**, 3.6], where $\zeta\colon G \to \Omega BG$ is handled, are required.

We next show that precisely similar statements apply to $\mathscr{I}$-functors.

Definition I.2.1. A monoid-valued $\mathscr{I}$-functor, or monoid in $\mathscr{I}[\mathscr{T}]$, is an $\mathscr{I}$-functor G such that each GV is a topological monoid, the identity element of GV is its basepoint, and the products $GV \times GV \to GV$ define a morphism of $\mathscr{I}$-functors. G is said to be group-valued if each GV is a topological group and the inverse maps $GV \to GV$ define a morphism of $\mathscr{I}$-functors. G is said to be grouplike if each $\pi_0 GV$ is a group. A left action of a monoid-valued $\mathscr{I}$-functor G on an $\mathscr{I}$-functor X is a morphism of $\mathscr{I}$-functors $G \times X \to X$ such that the map $GV \times XV \to XV$ is an action of GV on XV for each V.

Definition I.2.2. Let G be a monoid-valued $\mathscr{I}$-functor which acts from the left and right on $\mathscr{I}$-functors X and Y. Define an $\mathscr{I}$-functor $B(Y, G, X)$ as follows. For $V \in \mathscr{I}$ and for a morphism $f\colon V \to V'$ in $\mathscr{I}$, define

$$B(Y, G, X)(V) = B(YV, GV, XV) \quad \text{and} \quad B(Y, G, X)(f) = B(Yf, Gf, Xf).$$

The Whitney sum on $B(Y, G, X)$ is defined by the composite maps

$$B(YV, GV, XV) \times B(YW, GW, XW),$$

$$\downarrow \xi$$

$$B(YV \times YW, GV \times GW, XV \times XW)$$

$$\downarrow B(\omega, \omega, \omega)$$

$$B(Y(V \oplus W), G(V \oplus W), X(V \oplus W))$$

where ξ is the commutative and associative natural homeomorphism given by [**45**, 10.1 and 11.5].

It is easily verified that $B(Y, G, X)$ is well-defined. The only point worth mentioning is that the functor $B(Y, G, X)$ is continuous because geometric realization is continuous if the set of maps of simplicial spaces from C to D is topologized as a subspace of the product over n of the spaces of maps from C_n to D_n. Clearly the $\mathscr{L}$-space obtained by first applying the functor B to (Y, G, X) and then applying the functor Θ of Definition I.1.6 coincides with the $\mathscr{L}$-space obtained by first applying Θ to Y, G, and X and then applying B.

In view of Remark I.1.5, all of the material of the first few paragraphs of this section can be rephrased in terms of $\mathscr{I}$-functors (where a homotopy between maps $T \to T*$ of $\mathscr{I}$-functors means a morphism of $\mathscr{I}$-functors $T \to F(I^+, T')$ and where, just as for spaces, the fibre of a map of $\mathscr{I}$-functors is defined as the evident fibred product). It is again straightforward to verify that all constructed maps are morphisms of $\mathscr{I}$-functors when all given maps are so. We summarize results in the following statements.

Proposition I.2.3. *If $j\colon H \to G$ is a morphism of grouplike monoid-valued $\mathscr{I}$-functors, then the following is a quasi-fibration sequence of $\mathscr{I}$-functors*

$$H \xrightarrow{\ j\ } G \xrightarrow{\ \tau\ } G/H \xrightarrow{\ q\ } BH \xrightarrow{\ Bj\ } BG.$$

$\zeta\colon G \to \Omega BG$ *and* $\theta\colon G/H \to FBj$ *are weak equivalences of $\mathscr{I}$-functors.*

The classical groups define group-valued $\mathscr{I}$-functors by Remark I.1.10

Proposition I.2.4. *When G is a classical group $\mathscr{I}$-functor, the maps*

$$G/(G \times G) \xleftarrow{\ \varepsilon(\pi)\ } B(G/(E \times G), G, *) \xrightarrow{\ q\ } BG$$

of $\mathscr{I}$-functors determine a weak equivalence of $\mathscr{L}$-spaces between the two natural classifying spaces of G.

Henceforward, we shall work relative to a fixed continuous sphere-valued functor $t\colon \mathscr{I}_* \to \mathscr{T}$ such that $tV \wedge tW = t(V \oplus W)$. By the continuity, commutativity, and associativity of the smash product, t is itself an $\mathscr{I}_*$-functor with the natural projection $tV \times tW \to t(V \oplus W)$ as Whitney sum. We have the following list of monoid-valued $\mathscr{I}_*$-functors which act from left on t.

Example I.2.5. Let $\widetilde{F}$ denote the $\mathscr{I}_*$-functor specified by

$$\widetilde{F}V = F(tV, tV) \quad \text{and} \quad \widetilde{F}f = F(tf^{-1}, tf) \text{ for } f\colon V \longrightarrow V',$$

with the Whitney sum $\omega\colon \widetilde{F}V \times \widetilde{F}W \to \widetilde{F}(V \oplus W)$ given by the smash product of maps. Define sub $\mathscr{I}_*$-functors F, SF, Top and $STop$ of $\widetilde{F}$ by restricting attention to based homotopy equivalences, degree one homotopy equivalences, homeomorphisms and degree one homeomorphisms of tV. Define e to be the trivial sub $\mathscr{I}_*$-functor of $\widetilde{F}$, $eV = \{1\}$. When tV is the one-point compactification V^* of V, O and SO become sub $\mathscr{I}_*$-functors of Top and $STop$ via one-point compactifications of maps. Moreover, the twisted adjoint representation of [**12**, p. 7] defines morphisms of $\mathscr{I}_*$-functors from Pin, $Spin$, Γ, Pin^c, $Spin^c$, and Γ^c to O. Then $tV = V_{\mathbb{C}}^*$, U and SU become sub $\mathscr{I}_*$-functors

of $STop$ via one-point compactification, and similarly for Sp when $tV = V_{\mathbb{H}}^*$. By Remark I.1.10 and [**12**], all of these $\mathscr{I}_*$-functors are monoid-valued and all but SF, F, and $\widetilde{F}$ are group-valued.

We pass from $\mathscr{I}_*$-functors to $\mathscr{I}$-functors by Proposition I.1.9.

Write $\widetilde{F}\mathbb{R}^n = \widetilde{F}(n)$, $\widetilde{F}\mathbb{R}^\infty = \widetilde{F}$, and similarly for other $\mathscr{I}$-functors. It is usual to define $G(n)$ to be the space of homotopy equivalences of S^{n-1} and to regard $F(n-1)$ as the subspace of based homotopy equivalences. It is clear that $G = \varinjlim G(n)$ is then homotopy equivalent to F. G is the more natural space to consider in some geometric situations, but we shall work with F since it is this space which occurs naturally in our theory.

We pass from $\mathscr{I}$-functors to $\mathscr{L}$-spaces via Definition I.1.6 and from $\mathscr{L}$-spaces to infinite loop spaces via [**45, 46**] (or Section VII§3). Thus F, SF, Top, $STop$, the classical groups, $Spin$, $Spin^c$, and related groups and all of their classifying spaces and homogeneous spaces are infinite loop spaces, and all of the natural maps between these spaces are infinite loop maps.

It remains to consider PL and related spaces. One could perhaps develop a complete geometric theory by introducing categories with the same objects as $\mathscr{I}$ but with simplicial sets of morphisms from V to W, the k-simplices of which would be appropriate piecewise linear and piecewise differential fibrewise homeomorphisms $\Delta_k \times V \to \Delta_k \times W$; here PD would be required in order to relate PL to O. I have not attempted to go through the details.

A second approach would be to consider the $\mathscr{I}$-functor "PL" such that "PLV" is the subspace of FV consisting of based piecewise linear homeomorphisms of V^*. Unfortunately, "PL" has the wrong homotopy type; according to Rourke (private communication), the inclusion of "PL" in Top is a weak homotopy equivalence. It is at least conceivable that a larger topology on "PL" exists which does have the right homotopy type.

Our approach is to ignore these difficulties, to recall that PL was largely introduced in order to study the triangulation problem, and to observe that the homotopy types of PL, BPL, PL/O, etc. are completely determined as infinite loop spaces by the solution to this problem and by the infinite loop space structures already derived on TOP, $BTop$, Top/O, etc.

In detail, we note that $Top/PL = K(\mathbb{Z}_2, 3)$, Top/O is a 2-connected space, $\pi_3(Top/O) = \mathbb{Z}_2$, and the natural map $Top/O \to Top/PL$ induces an isomorphism on π_3. Recall that there is only one $(n-1)$-connected spectrum $\mathscr{K}(\mathbb{Z}_2, n)$ with zero$^{\text{th}}$ space a $K(\mathbb{Z}_2, n)$ (up to isomorphism in any good stable category) and there is only one non-trivial map from an $(n-1)$-connected spectrum with $\pi_n E = \mathbb{Z}_2$ into $\mathscr{K}(\mathbb{Z}_2, n)$. Thus Top/PL, however it is constructed geometrically, is jut $K(\mathbb{Z}_2, 3)$ as an infinite loop space, and the unique non-trivial map $Top/O \to Top/PL$ is an infinite loop map in precisely one way.

Now define the following spaces (or rather homotopy types, since that is all our data determine). In each case, it is clear that any permissible geometric construction of the space named must yield the specified homotopy type.

(a) PL/O is the fibre of the unique non-trivial map $Top/O \to K(\mathbb{Z}_2, 3)$.
(b) PL is the fibre of the composite $Top \to Top/O \to K(\mathbb{Z}_2, 3)$ and SPL is the fibre of the restriction of this composite to $STop$.
(c) BPL is the fibre of the composite $BTop \to B_1(Top/O) \to K(\mathbb{Z}_2, 3)$ and $BSPL$ is the fibre of the restriction of this composite to $BSTop$. (Here $BTop$ is equivalent to the delooping B_1Top by Proposition VII.3.5.)
(d) F/PL is the fibre of the composite $BPL \to BTop \to BF$.

Clearly the fibre of an infinite loop map is an infinite loop space, and it follows that each space we have constructed has a well-defined infinite loop space structure such that all of the natural maps between these spaces are infinite loop maps. Similar constructions handle PL/G and related maps for other classical groups G.

Coordinate-free spectra

A spectrum T is usually defined to be a sequence of spaces T_i and maps $\sigma_i \colon \Sigma T_i \to T_{i+1}$. Let $\{e_n\}$ be the standard basis for $\mathbb{R}^\infty$ and think of the 1-sphere as the one-point compactification $t\mathbb{R}e_n$ of the subspace $\mathbb{R}e_n$ of $\mathbb{R}^\infty$. Then a change of notation allows us to describe T as a sequence of spaces $T(\mathbb{R}^i)$ and maps $\sigma_i \colon T(\mathbb{R}^i) \wedge t\mathbb{R}e_{i+1} \to T(\mathbb{R}^{i+1})$. Thus the usual notion of spectrum implicitly refers to a fixed chosen basis for $\mathbb{R}^\infty$. Many very real difficulties in the homotopy theory of spectra, in particular the problems associated with the construction of well-behaved smash products, arise from permutations of suspension coordinates. Such permutations can be thought of as resulting from changes of basis for $\mathbb{R}^\infty$, and we shall see in [48] that the coordinate-free definition of spectra to be given here leads to a relatively simple development of the properties of the stable homotopy category.

However, our present concern is with more than just stable homotopy theory. In order to define E_∞ ring spectra, it is essential to work in a category of (omega) spectra which enjoys good properties even before passage to homotopy. The point is that these spectra have very rich internal structure, much of which is lost upon passage to the homotopy category.

The spectra used in the best previous construction of stable homotopy categories are (or are derived from) CW-spectra, namely those spectra T such that T_i is a CW-complex and each σ_i is a cellular inclusion.

Obviously such rigid structures cannot possibly be related to infinite loop spaces before passage to homotopy, and our spectra will be cell-free as well as coordinate-free. Restriction to CW-spectra is in any case undesirable since CW-spectra seldom occur in nature are not closed under such simple and useful constructions as formation of product and loop spectra. In our stable category, desuspension will be given by the loop spectrum functor.

We define coordinate-free prespectra and spectra, show how to pass back and forth between spaces, prespectra, and spectra, and relate coordinate-free to coordinatized spectra in Section II§1. We define the stable homotopy category and discuss ring spectra, connective spectra and localizations and completions of spectra in Section II§2. We shall omit most proofs in these sections; the missing details may be found in [48] and are largely irrelevant to our later work in this book. In Section II§3, we consider cohomology theories and give a rather pedantic analysis of the precise relationships between periodic spaces, periodic spectra, and "periodic connective spectra."

Although exploited in a wholly different way, the idea of using linear isometries to study the stable category is due to Boardman [18]. Puppe [56] independently came to the idea of coordinate-free prespectra.

1. Spaces, prespectra, and spectra

Recall the definition of $\mathscr{I}_*$ from Definition I.1.8, as in Section I§2, fix a continuous sphere-valued functor $t \colon \mathscr{I}_* \to \mathscr{T}$ such that $t\{0\} = S^0$ and $tV \wedge tW = t(V \oplus W)$. In practice, tV will be the one-point compactification of τV for some functor $\tau \colon \mathscr{I} \to \mathscr{I}$, and we shall see in Remark II.1.9 that restriction to the identity functor τ would result in no real loss of generality. Recall that $F(X, Y)$ denotes the space of based maps $X \to Y$ and define the suspension and loop functors based on V to be

$$\Sigma^V X = X \wedge tV \quad \text{and} \quad \Omega^V X = F(tV, X).$$

It will be important to keep in mind the old-fashioned distinction between external and internal direct sums; we write $\oplus$ for the former and $+$ for the latter. We write $V \perp W$ to indicate that two sub inner product spaces of a given inner product space are orthogonal; the notation $+$ will only be used between orthogonal subspaces and will thus carry orthogonality as an implied hypothesis.

Let $\mathscr{I}_*(\mathbb{R}^\infty)$ denote the full subcategory of $\mathscr{I}_*$ the objects of which are finite-dimensional sub inner product spaces of $\mathbb{R}^\infty$ and let $\mathscr{A}\mathscr{T}$ denote the category of based spaces and based homeomorphisms. Let $h\mathscr{a}$ denote the homotopy category associated to a topological category $\mathscr{a}$. Since $\mathscr{I}_n(V, W)$ is homeomorphic to $O(n)$, $(h\mathscr{I}_n)(V, W)$ has precisely two elements if $n > 0$. Remark II.1.10 below indicate a possible simplification of the following definition.

Definition II.1.1. A prespectrum (T, σ) is a function $T \colon \mathscr{I}_*(\mathbb{R}^\infty) \to \mathscr{A}\mathscr{T}$ (on objects and morphisms) which induces a functor $T \colon h\mathscr{I}_*(\mathbb{R}^\infty) \to h\mathscr{T}$ together with based maps $\sigma \colon \Sigma^W TV \to T(V + W)$ for $V \perp W$ which satisfy the following conditions.

(i) Each adjoint map $\widetilde{\sigma} \colon TV \to \Omega^W T(V + W)$ is an inclusion with closed image.

(ii) The following diagrams commute in $\mathscr{T}$, where $V \perp W \perp Z \perp V$:

$$
\begin{array}{ccc}
TV & =\!=\!= & \Sigma^0 TV \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \sigma} \\
TV & =\!=\!= & T(V + \{0\})
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\Sigma^Z \Sigma^W TV & =\!=\!= & \Sigma^{W+Z} TV \\
{\scriptstyle \Sigma^Z \sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\
\Sigma^Z T(V + W) & \xrightarrow{\ \sigma\ } & T(V + W + Z)
\end{array}
$$

(iii) The following diagrams commute in $h\mathscr{T}$, where $f \in \mathscr{I}_*(V, V')$, $g \in \mathscr{I}_*(W, W')$, $V \perp W$ and $V' \perp W'$:

$$
\begin{array}{ccc}
TV \wedge tW \ =\!=\!=\ \Sigma^W TV & \xrightarrow{\ \sigma\ } & T(V + W) \\
{\scriptstyle Tf \wedge tg}\downarrow & & \downarrow{\scriptstyle T(f+g)} \\
TV' \wedge tW' \ =\!=\!=\ \Sigma^{W'} TV' & \xrightarrow{\ \sigma\ } & T(V' + W')
\end{array}
$$

(T, σ) is said to be a spectrum if each $\widetilde{\sigma} \colon TV \to \Omega^W T(V + W)$ is a homeomorphism. A morphism $\theta \colon (T, \sigma) \to (T', \sigma')$ of prespectra consists of maps $\theta \colon TV \to T'V$ such that $\theta \colon T \to T'$ is a natural transformation of functors $h\mathscr{I}_*(\mathbb{R}^\infty) \to h\mathscr{T}$ and such that the following diagrams commute in $\mathscr{T}$ for $V \perp W$:

$$
\begin{array}{ccc}
\Sigma^W TV & \xrightarrow{\ \sigma\ } & T(V + W) \\
{\scriptstyle \Sigma^W \theta}\downarrow & & \downarrow{\scriptstyle \theta} \\
\Sigma^W T'V & \xrightarrow{\ \sigma\ } & T'(V + W)
\end{array}
$$

Let $\mathscr{P}$ denote the category of prespectra and let $\mathscr{S}$ denote its full subcategory of spectra. Let $\nu \colon \mathscr{S} \to \mathscr{P}$ denote the inclusion functor.

The category $\mathscr{S}$ is of primary interest, and $\mathscr{P}$ is to be regarded as a convenient auxiliary category. The pair of terms (prespectrum, spectrum), which was introduced by Kan, is distinctly preferable to the older pair (spectrum, Ω-spectrum) since spectra are the fundamental objects of study and since prespectra naturally give rise to spectra. The use of homeomorphisms, rather than homotopy equivalences, in the definition of spectra is both essential to the theory and convenient in the applications. We have little use for the classical notion of an Ω-spectrum.

Granted the desirability of a coordinate-free theory of spectra, it is clearly sensible to think of the finite-dimensional subspaces of $\mathbb{R}^\infty$ as an indexing set. Thus a prespectrum ought, at least, to consist of spaces TV and maps $\sigma \colon \Sigma^W TV \to T(V + W)$, and it is obviously reasonable to insist that TV be homeomorphic to TW if $\dim V = \dim W$. Our definition merely codifies these specifications in a coherent way. The use of the homotopy category $h\mathscr{I}_*(\mathbb{R}^\infty)$ can be thought of as a systematic device for letting linear isometries keep track of signs, changes of coordinates, and such like complications in the usual theory of spectra. One might be tempted to define prespectra

by requiring T to be a functor $\mathscr{I}_*(\mathbb{R}^\infty) \to \mathscr{HT}$ and σ to be natural, without use of homotopy categories, but such a definition would not allow the construction of spectra from presepctra or of coordinate-free spectra from coordinatized spectra.

We next make precise the categorical interrelationships among $\mathscr{T}$, $\mathscr{P}$, and $\mathscr{S}$ (as was done in [**43**] for ordinary spectra). Observe that we have forgetful, or zero$^{\text{th}}$ space, functors $\mathscr{P} \to \mathscr{T}$ and $\mathscr{S} \to \mathscr{T}$ defined on objects by $(T, \sigma) \to T_0 = T\{0\}$.

Definition II.1.2. Define the suspension prespectrum functor $\Sigma^\infty \colon \mathscr{T} \to \mathscr{P}$ by letting

$$(\Sigma^\infty X)(V) = \Sigma^V X \quad \text{and} \quad (\Sigma^\infty X)(f) = \Sigma^f X \equiv 1 \wedge tf$$

on objects V and morphisms f of $\mathscr{I}_*(\mathbb{R}^\infty)$ and by letting

$$\sigma = 1 \colon \Sigma^W \Sigma^V X \longrightarrow \Sigma^{V+W} X \text{ for } V \perp W.$$

For $\phi \colon X \to X'$, let $\Sigma^\infty \phi = \Sigma^V \phi \colon \Sigma^V X \to \Sigma^V X'$.

Lemma II.1.3. $\Sigma^\infty X$ *is the free prespectrum generated by the space X; that is, for $T \in \mathscr{P}$ and $\phi \colon X \to T_0$, there is a unique map $\widetilde{\phi} \colon \Sigma^\infty X \to T$ of prespectra with zeroth map ϕ.*

Definition II.1.4. Define the associated spectrum functor $\Omega^\infty \colon \mathscr{P} \to \mathscr{S}$ as follows. Let $(T, \sigma) \in \mathscr{P}$. We have identifications

$$(*) \qquad \Omega^{W+Z} T(V + W + Z) = \Omega^W \Omega^Z T(V + W + Z)$$

for $V \perp W \perp Z \perp V$ and we define

$$(\Omega^\infty T)(V) = \varinjlim_{W \perp V} \Omega^W T(V + W)$$

where the limit is taken with respect to the inclusions

$$\Omega^W, \widetilde{\sigma}, \widetilde{\sigma} \colon T(V + W) \longrightarrow \Omega^Z T(V + W + Z).$$

For $V \perp W$, the required homeomorphisms

$$\widetilde{\sigma} \colon (\Omega^\infty T)(V) \longrightarrow \Omega^W (\Omega^\infty T)(V + W)$$

are obtained by passage to limits over Z from the identifications $(*)$. To define $(\Omega^\infty T)(f) \colon (\Omega^\infty T)(V) \to (\Omega^\infty T)(V')$ for $f \colon V \to V'$, choose Z which contains both V and V', let W and W' be the orthogonal complements of V and V' in Z, and observe that there is one and, up to homotopy, only one linear isometry $g \colon W \to W'$ such that $f + g$ is homotopic (through isometries) to the identity of Z. The required homeomorphism $(\Omega^\infty T)(f)$ is obtained by passage to limits over $Z' \perp Z$ from the maps

$$\Omega^{W+Z'} T(Z + Z') \longrightarrow \Omega^{W'+Z'} T(Z + Z')$$

given by composition with $t(g^{-1}+1)$. For a map $\theta \colon T \to T'$ of prespectra, define $\Omega^\infty \theta \colon (\Omega^\infty T)(V) \to (\Omega^\infty T')(V)$ by passage to limits from the maps

$$\Omega^W \theta \colon \Omega^W T(V + W) \longrightarrow \Omega^W T'(V + W).$$

Lemma II.1.5. *The inclusions $TV = \Omega^0 T(V + \{0\}) \to \varinjlim \Omega^W T(V + W)$ define a map $\iota \colon T \to \nu\Omega^\infty T$ of prespectra, and $\Omega^\infty T$ is the free spectrum generated by the prespectrum T; that is, for $E \in \mathscr{S}$ and $\theta \colon T \to \nu E$, there is a unique map $\widetilde{\theta} \colon \Omega^\infty T \to E$ of spectra such that $\widetilde{\theta}\iota = \theta$.*

Definition II.1.6. Define $Q_\infty = \Omega^\infty \Sigma^\infty \colon \mathscr{T} \to \mathscr{S}$ and define $QX = (Q_\infty X)_0$. Observe that QX is then homeomorphic to $\varinjlim \Omega^n \Sigma^n X$. $Q_\infty X$ is the free spectrum generated by X, and we thus have an adjunction

$$\mathscr{T}(X, E_0) \cong \mathscr{S}(Q_\infty X, E), \ X \in \mathscr{T} \text{ and } E \in \mathscr{S}.$$

Finally, we define coordinatized prespectra and spectra and relate them to the coordinate-free variety.

Definition II.1.7. Let $A = \{A_i\}$, $\{0\} = A_0 \subset A_1 \subset \cdots \subset A_i \subset \cdots$, be an increasing sequence of subspaces of $\mathbb{R}^\infty$, with $\mathbb{R}^\infty = \bigcup_{i \geq 0} A_i$. Let B_i denote the orthogonal complement of A_i in A_{i+1}. A prespectrum $\{T_i, \sigma_i\}$ indexed by A is a sequence of based maps T_i and maps $\sigma_i \colon \Sigma^{b_i} T_i = T_i \wedge tB_i \to T_{i+1}$ such that the adjoints $\widetilde{\sigma}_i \colon T_i \to \Omega^{b_i} T_{i+1}$ are inclusions with closed images. $\{T_i, \sigma_i\}$ is a spectrum if each σ_i is a homeomorphism. A map $\{\theta_i\} \colon \{T_i, \sigma_i\} \to \{T_i', \sigma_i'\}$ of prespectra is a sequence of based maps $\theta_i \colon T_i \to T_i'$ such that $\theta_{i+1} \circ \sigma_i = \sigma_i' \circ \Sigma^{b_i} \theta_i$. Let $\mathscr{P}_A$ and $\mathscr{S}_A$ denote the category of prespectra indexed by A and its full subcategory of spectra. Define a forgetful functor $\phi \colon \mathscr{P} \to \mathscr{P}_A$ by letting $\phi(T, \sigma) = \{T_i, \sigma_i\}$, where $T - i = TA_i$ and $\sigma_i = \sigma \colon \Sigma^{b_i} TA_i \to TA_{i+1}$.

With trivial modifications, the previous definitions and lemmas apply to coordinatized prespectra and spectra. The following result would be false on the prespectrum level.

THEOREM II.1.8. *The forgetful functor $\phi \colon \mathscr{S} \to \mathscr{S}_A$ is an equivalence of categories; that is, there is a functor $\psi \colon \mathscr{S}_A \to \mathscr{S}$ and there are natural isomorphisms*

$$\phi\psi \{E_i, \sigma_i\} \cong \{E_i, \sigma_i\} \quad and \quad \psi\phi(E, \sigma) \cong (E, \sigma).$$

Remark II.1.9. Suppose that $tV = \tau V \cup \infty$ for some nontrivial additive functor $\tau \colon \mathscr{I} \to \mathscr{I}$. Then $\tau\mathbb{R}^\infty$ has countably infinite dimension and we may choose an isometric isomorphism $g \colon \tau\mathbb{R}^\infty \to \mathbb{R}^\infty$. Visibly g determines an isomorphism between the category $\mathscr{S}_A$ defined by use of t and the category $\mathscr{S}_{gA}$ defined by use of the one-point compactification of inner product spaces. Therefore, up to equivalence, the category $\mathscr{S}$ is independent of the choice of t (of the specified form).

Remark II.1.10. Theorem II.1.8 implies that explicit use of linear isometries in our definition of a spectrum is quite unnecessary, and the details in Definition II.1.4 show why this is the case. I find the introduction of isometries conceptually helpful, particularly on the prespectrum level (compare Remark IV.1.3), but the reader is free to ignore them throughout.

2. The stable homotopy category

We require small smash products and function spectra, "small" meaning between spaces and spectra rather than between spectra and spectra.

Definition II.2.1. For $X \in \mathscr{T}$ and $T \in \mathscr{P}$, define $T \wedge X \in \mathscr{P}$ by letting

$$(T \wedge X)(V) = TV \wedge X \quad and \quad (T \wedge X)(f) = Tf \wedge 1$$

on objects V and morphisms f of $\mathscr{I}_*(\mathbb{R}^\infty)$ and by letting the structural maps σ between the composites

$$\Sigma^W(TV \wedge X) \cong (\Sigma^W TV) \wedge X \xrightarrow{\sigma \wedge 1} T(V + W) \wedge X.$$

For $E \in \mathscr{S}$, define $E \wedge X \in \mathscr{S}$ by $E \wedge X = \Omega^\infty(\nu E \wedge X)$.

Observe that Ω^∞ and ν can be used similarly to transport to spectra any functor on prespectra which does not preserve spectra. The functors Σ^∞ and Ω^∞ preserve smash products with spaces.

Lemma II.2.2. *For $X \in \mathscr{T}$ an $Y \in \mathscr{T}$, $\Sigma^\infty(Y \wedge X)$ is isomorphic to $(\Sigma^\infty Y) \wedge X$. For $X \in \mathscr{T}$ and $T \in \mathscr{P}$, $\Omega^\infty(\iota \wedge 1) \colon \Omega^\infty(T \wedge X) \to (\Omega^\infty T) \wedge X$ is an isomorphism.*

Definition II.2.3. For $X \in \mathscr{T}$ and $T \in \mathscr{P}$, define $F(X, T) \in \mathscr{P}$ by letting

$$F(X, T)(V) = F(X, TV) \quad and \quad F(X, T)(f) = F(1, Tf)$$

on objects V and morphisms f of $\mathscr{I}_*(\mathbb{R}^\infty)$ and by letting the structural maps σ be the adjoints of the composites

$$F(X, TV) \xrightarrow{F(1, \widetilde{\sigma})} F(X, \Omega^W T(V + W)) \cong \Omega^W F(X, T(V + W)).$$

If T is a spectrum, then so is $F(X, T)$.

Lemma II.2.4. *For $X \in \mathscr{T}$ and $E \in \mathscr{S}$, $\nu F(X, E) = F(X, \nu E)$. For $X \in \mathscr{T}$ and $T \in \mathscr{P}$, $F(X, T)_0 = F(X, T_0)$.*

Lemma II.2.5. *For $X \in \mathscr{T}$, there are natural (adjunction) isomorphisms*

$$\mathscr{P}(T \wedge X, T') \cong \mathscr{P}(T, F(X, T')),\ T, T' \in \mathscr{P}$$

and

$$\mathscr{S}(E \wedge X, E') \cong \mathscr{S}(E, F(X, E')),\ E, E' \in \mathscr{S}.$$

Explicitly, the adjoint $\widetilde{\theta}\colon E \to F(X, E')$ of $\theta\colon E \wedge X \to E'$ has V^{th} map $EV \to F(X, E'V)$ the adjoint in $\mathscr{T}$ of the composite

$$EV \wedge X = (\nu E \wedge X)(V) \xrightarrow{\iota} \nu\Omega^{\infty}(\nu E \wedge X)(V) = (E \wedge X)(V) \xrightarrow{\theta} E'V.$$

Let K^+ denote the union of a space K and a disjoint basepoint.

Definition II.2.6. For Y and Z both in $\mathscr{T}$ or $\mathscr{P}$ or $\mathscr{S}$, define a homotopy $h\colon f_0 \simeq f_1$ between maps $f_i\colon Y \to Z$ to be a map $h\colon Y \wedge I^+ \to Z$ (in the relevant category) such that $h|_Y \wedge \{i\}^+ = f_i$. Note that h could equally well be considered as a map $Y \to F(I^+, Z)$. Let $\pi(Y, Z)$ denote the set of homotopy classes of maps $Y \to Z$.

The basic machinery of elementary homotopy theory, such as the dual Barratt–Puppe sequence sand dual Milnor $\varprojlim^1$ exact sequences, applies equally well in $\mathscr{T}$ and in $\mathscr{S}$, [**48**, I]. Lemmas II.2.2 and II.2.4 imply that Σ^{∞}, Ω^{∞}, Q_{∞}, ν and the zeroth space functors $\mathscr{P} \to \mathscr{T}$ and $\mathscr{S} \to \mathscr{T}$ are all homotopy preserving. Clearly Σ^{∞}, Ω^{∞}, and Q_{∞} are still free functors after passage to homotopy categories. In particular, we have a natural isomorphism

$$\pi(X, E_0) \cong \pi(Q_{\infty}X, E),\ X \in \mathscr{T} \text{ and } E \in \mathscr{S}.$$

Definition II.2.7. For $E \in \mathscr{S}$, define $\Sigma E = E \wedge S^1$ and $\Omega E = F(S^1, E)$. Abbreviate $S = Q_{\infty}S^0$ and define the homotopy groups of E by

$$\pi_r E = \pi(\Sigma^r S, E) \text{ and } \pi_{-r} E = \pi(\Omega^r S, E) \text{ for } r \geq 0.$$

Since $\Sigma^r S \cong Q_{\infty}S^r$, by Lemma II.2.2, $\pi_r E = \pi_r E_0$ for $r \geq 0$. A map $\theta\colon E \to E'$ in $\mathscr{S}$ is said to be a weak equivalence if $\pi_r\theta$ is an isomorphism for all integers r.

The adjunction between Σ and Ω gives natural maps

$$\eta\colon E \longrightarrow \Omega\Sigma E \quad \text{and} \quad \varepsilon\colon \Sigma\Omega E \longrightarrow E.$$

The following result is a version of the Puppe desuspension theorem [**55**].

THEOREM II.2.8. *For all spectra E, η and ε are weak equivalences.*

There is a category $H\mathscr{S}$ and a functor $L\colon h\mathscr{S} \to H\mathscr{S}$ such that L is the identity function on objects, L carries weak equivalences to isomorphisms, and L is universal with respect to the latter property [**48**, II and XI]. We call $H\mathscr{S}$ the stable homotopy category. Its morphisms are composites of morphisms in $h\mathscr{S}$ and formal inverses of weak equivalences in $h\mathscr{S}$. In $H\mathscr{S}$, the functors Σ and Ω become inverse equivalences of categories, and we therefore write $\Omega^r = \Sigma^{-r}$ for all integers r. It will be shown in [**48**] that $H\mathscr{S}$ has all the good properties one could hope for and, despite its wholly different definition is in fact equivalent to the stable categories of Boardman [**18**] and Adams [**7**].

Let $H\mathscr{T}$ denote the category obtained from $h\mathscr{T}$ by formally inverting the weak equivalences and let $[Y, Z]$ denote the set of morphisms in $H\mathscr{T}$ or $H\mathscr{S}$ between spaces or spectra Y and Z. Again, we have

$$[X, E_0] \cong [Q_{\infty}X, E],\ X \in \mathscr{T} \text{ and } E \in \mathscr{S}.$$

Q_{∞} should be regarded as the stabilization functor from spaces to spectra. Let $\mathscr{V}$ denote the category of spaces in $\mathscr{T}$ of the (based) homotopy type of CW-complexes. For $X \in \mathscr{V}$ and $Y \in \mathscr{T}$,

$[X, Y] = \pi(X, Y)$; the categories $h\mathscr{V}$ and $H\mathscr{T}$ are equivalent [**48**, III]. Analogous statements are valid for $H\mathscr{S}$ [**48**, XI].

$H\mathscr{S}$ admits a coherently associative, commutative and unital smash product with unit S [**48**, XI]. Define a (commutative) ring spectrum to be a spectrum E together with an associative (and commutative) product $\phi\colon E \wedge E \to E$ with two-sided unit $e\colon S \to E$. The following lemma will play a vital role in our study of Bott periodicity and Brauer lifting in Section VIII§2.

Lemma II.2.9. *The product ϕ of a ring spectrum E induces a map (again denoted ϕ) from $E \wedge E_0$ to E.*

PROOF. For spaces X, $E \wedge X$ is coherently naturally isomorphic to $E \wedge Q_\infty X$ [**48**, XI]; indeed, such a relationship between small and large smash products is a standard property of any good stable category. Via Definition II.1.6, the identity map of E_0 determines a map $\psi\colon Q_\infty E_0 \to E$ of spectra. The required map is the composite

$$E \wedge E_0 \cong E \wedge Q_\infty E_0 \xrightarrow{1 \wedge \psi} E \wedge E \xrightarrow{\phi} E.$$

$\square$

A spectrum E is said to be *n-connected* if $\pi_r E = 0$ for $r \leq n$ and to be *connective* if it is (-1)-connected. Infinite loop space theory is concerned with connective spectra, and we require the following observations.

Lemma II.2.10. *If C and D are $(q-1)$-connected and $\theta\colon D \to E$ is a map in $H\mathscr{S}$ such that $\pi_i \theta$ is an isomorphism for all $i \geq q$, then $\theta_*\colon [C, D] \to [C, E]$ is an isomorphism.*

PROOF. Let F denote the cofibre of θ. Up to sign, cofiberings and fiberings coincide in $H\mathscr{S}$ [**48**, XI], hence $\pi_i F = 0$ for $i \geq q$. By replacing C by a CW-spectrum, applying induction over its skeleta, and using the $\varprojlim^1$ exact sequence, we find easily that $[C, \Omega^i F] = 0$ for $i \geq 0$. The conclusion follows by the Barratt–Puppe sequence. $\square$

Lemma II.2.11. *For a spectrum E, there exists one and, up to equivalence, only one connective spectrum D and map $\theta\colon D \to E$ in $H\mathscr{S}$ such that $\pi_i \theta$ is an isomorphism for $i \geq 0$. If E is a ring spectrum, then D admits a unique structure of ring spectrum such that θ is a map of ring spectra.*

PROOF. While the existence and uniqueness could be proven by stable techniques, we simply note that the map $\widetilde{\omega}\colon \Omega^\infty T E_0 \to E$ in $\mathscr{S}$ constructed in Proposition VII.3.2 has the desired properties required of θ and that, given $\theta\colon D \to E$ and $\theta'\colon D' \to E'$ as specified, the naturality of $\widetilde{\omega}$ yields the following commutative diagram in $H\mathscr{S}$, in which all arrows with targets other than E are isomorphisms:

$$\begin{array}{ccccc}
\Omega^\infty T D_0 & \xrightarrow{\Omega^\infty T \theta_0} & \Omega^\infty T E_0 & \xleftarrow{\Omega^\infty T \theta_0'} & \Omega^\infty T D_0' \\
\widetilde{\omega}\downarrow & & \downarrow\widetilde{\omega} & & \downarrow\widetilde{\omega} \\
D & \xrightarrow{\theta} & E & \xleftarrow{\theta'} & D'
\end{array}$$

Since $D \wedge D$ is connective, the assertion about ring structures follows directly from the previous lemma. $\square$

Note that our proof not only gives an associated connective spectrum functor on $H\mathscr{S}$, it already gives such a functor on $\mathscr{S}$.

For a (commutative) ring R, the Eilenberg–MacLane spectrum $HR = \mathscr{K}(R, 0)$ is a (commutative) ring spectrum.

Lemma II.2.12. *If E is a connective ring spectrum, then the unique map $d\colon E \to H\pi_0 E$ in $H\mathscr{S}$ which realizes the identity map of $\pi_0 E$ is a map of ring spectra.*

PROOF. $\operatorname{Hom} \pi_0 E, \pi_0 E = H^0(E; \pi_0 E) = [E, H\pi_0 E]$ (see Definition II.3.1), hence d is well-defined. d is a ring map because $\operatorname{Hom}(\pi_0 E \otimes \pi_0 E, \pi_0 E) = [E \wedge E, H\pi_0 E]$. Note that d can be explicitly constructed by application of the functor $\Omega^\infty T$ to the discretization $E_0 \to \pi_0 E_0 = \pi_0 E$. $\square$

Localizations and completions will often be needed in our work. Let T be a set of primes. Recall that an Abelian group A is said to be T-local if it is a module over the localization $\mathbb{Z}_T$ of $\mathbb{Z}$ at T and to be T-complete if $\operatorname{Hom}(\mathbb{Z}[T^{-1}]/\mathbb{Z}, A) = 0$ and the natural (connecting) homomorphism $A \to \operatorname{Ext}(\mathbb{Z}[T^{-1}]/\mathbb{Z}, A)$ is an isomorphism (where $\mathbb{Z}[T^{-1}]$ is the localization of $\mathbb{Z}$ away from T). A (connected) simple space Y is said to be T-local or T-complete if each $\pi_i Y$ is T-local or T-complete. A localization $\lambda \colon X \to X_T$ or completion $\gamma \colon X \to \widehat{X}_T$ of a simple space X at T is a map into a T-local or T-complete space such that

$$\lambda^* \colon [X_T, Y] \longrightarrow [X, Y] \quad \text{and} \quad \gamma^* \colon [\widehat{X}_T, Y] \longrightarrow [X, Y]$$

is an isomorphism for all T-local or T-complete spaces Y or, equivalently, such that (with $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$)

$$\lambda_* \colon H_*(X; \mathbb{Z}_T) \longrightarrow H_*(X_T, \mathbb{Z}_T) \quad \text{or each} \quad \gamma_* \colon H_*(X; \mathbb{Z}_p) \longrightarrow H_*(\widehat{X}_T; \mathbb{Z}_p),\ p \in T$$

is an isomorphism. λ and γ exist (and are unique), and $\widehat{X}_T$ is equivalent to the completion of X_T at T and to $\prod_{p \in T} \widehat{X}_p$, where X_p and $\widehat{X}_p$ denote the localization and completion of X at p. Localizations and completions commute with products, fibrations, and loops and localizations but not completions commute with wedges, cofibrations, suspensions, and smash products. However, the completion at T of $\gamma \wedge \gamma \colon X \wedge X' \to \widehat{X}_T \wedge \widehat{X}'_T$ is an equivalence.

The completions just described are those due to Bousfield and Kan [**23**]; the completions of Sullivan [**73**] are not adequate for our applications in Chapter VIII. A new topological treatment of localizations and completions will be given in [**48**], and it will be seen there that completions are not appreciably more difficult to analyze than localizations. Incidentally, the category $H\mathcal{T}$ (and not its equivalent $h\mathcal{V}$) is the appropriate one in which to study localizations and completions since Postnikov towers with infinitely many non-zero homotopy groups never have the homotopy type of CW-complexes.

We also need localizations and completions of connective spectra. This subject is nowhere treated in the literature, a situation that will be rectified in [**48**]. In $H\mathcal{S}$, the summary above applies verbatim with spaces replaced by connective spectra, the only twist being that the commutation of completions with products, fibrations, and loops implies their commutation with finite wedges, cofibrations, and suspensions, but still not with smash products. Again, the completions at T of $\gamma \wedge \gamma \colon E \wedge E' \to \widehat{E}_T \wedge \widehat{E}'_T$ and, for a simple space X, of $\gamma \wedge \gamma \colon E \wedge X \to \widehat{E}_T \wedge \widehat{X}_T$ are isomorphisms in $H\mathcal{S}$. In particular, the completion at T of a ring spectrum E is a ring spectrum with unit $S \xrightarrow{e} E \xrightarrow{\gamma} \widehat{E}_T$ and product

$$\widehat{E}_T \wedge \widehat{E}_T \xrightarrow{\gamma} (\widehat{\widehat{E}_T \wedge \widehat{E}_T})_T \xrightarrow{(\widehat{\gamma \wedge \gamma})^{-1}} (\widehat{E \wedge E})_T \xrightarrow{\widehat{\phi}} \widehat{E}_T.$$

The zero$^{\text{th}}$ space functor commutes with completions in the sense that the zero$^{\text{th}}$ space of $\widehat{E}_T$ is equivalent to the product of the completion at T of the component of the basepoint of E_0 and the discrete group $\operatorname{Ext}(\mathbb{Z}[T^{-1}]/\mathbb{Z}, \pi_0 E_0)$. When $\pi_0 E_0 = \mathbb{Z}$, the latter group is the T-adic integers $\widehat{\mathbb{Z}}_T = \prod_{p \in T} \widehat{\mathbb{Z}}_{(p)}$.

Finally, we shall need the following pair of results from [**48**]. Taken together, they assert that, under minimal technical hypotheses, a map between T-local infinite loop spaces which completes to an infinite loop map at each prime $p \in T$ is itself an infinite loop map.

THEOREM II.2.13. *Let D and E be 0-connected spectra such that $\pi_* D$ and $\pi_* E$ are of finite type over $\mathbb{Z}_T$ and let $f \colon D_0 \to E_0$ be a map (in $H\mathcal{T}$) whose localization at p is the zeroth map*

of a map $\phi_p\colon D_p \to E_p$ (in $H\mathscr{S}$) for each prime $p \in T$. Assume either that D_0 and E_0 have no T-torsion and f is an H-map or that $D_0 \simeq \mathrm{Tel}(D_0^q)_T$ where each D_0^q is a finite CW-complex and $\varprojlim^1(\Sigma D_0^q, E_0) = 0$. Then f is the zeroth map of a map $\phi\colon D \to E$. Moreover, if $D \simeq \mathrm{Tel}\, D_T^q$ where each D^q is a finite CW-spectrum and $\varprojlim^1(\Sigma D^q, E) = 0$, then there is exactly one such map ϕ which localizes to the given map ϕ_p at each prime $p \in T$.

THEOREM II.2.14. *Let D and E be 0-connected spectra such that $\pi_* D$ and $\pi_* E$ are of finite type over $\mathbb{Z}_{(p)}$ and let $f\colon D_0 \to E_0$ be a map (in $H\mathscr{T}$) whose rationalization is an H-map and whose completion at P is the zeroth map of a map $\widehat{\phi}_p\colon \widehat{D}_p \to \widehat{E}_p$ (in $H\mathscr{S}$). Assume either that D_0 and E_0 have no p-torsion and f is an H-map or that $D_0 \simeq \mathrm{Tel}(D_0^q)_p$ where each D_0^q is a finite CW-complex and $\varprojlim^q(\Sigma D_0^q, E_0) = 0$. Then f is the zeroth map of a map $\phi\colon D \to E$. Moreover, if $D \simeq \mathrm{Tel}\, D_p^q$ where D^q is a finite CW-spectrum and $\varprojlim^q(\Sigma D^q, E) = 0$, then there is exactly one such map ϕ which completes at p to the given map $\widehat{\phi}_p$.*

One pleasant feature of these results is the complete irrelevance of $\varprojlim^1$ terms associated to the spaces D_n and E_n for $n > 0$. As will be discussed in [**48**], results of Anderson [**9, 11**] show that the stable $\varprojlim^1$ terms vanish in the cases relevant to this book.

3. Cohomology; periodic spaces and spectra

Definition II.3.1. For spectra Y and E, define the E-cohomology of Y by
$$E^n Y = [\Omega^n Y, E] = [Y, \Sigma^n E].$$
For a space X, define $E^X = E^n Q_\infty X$; this is what is usually called the reduced E-cohomology of X. In terms of $H\mathscr{T}$, $E^n X$ can be described by
$$E^X = [X, EV] \text{ if } \dim tV = n \geq 0 \quad \text{and} \quad E^{-n} X = [X, \Omega^n E_0] \text{ if } n \geq 0.$$

This description of $E^n X$ will be essential to our treatment of orientation theory in the next chapter. A complete analysis of homology and cohomology theories within the framework given by $H\mathscr{S}$ will be presented in [**48**]. Suffice it to say that all of the familiar machinery is available.

We shall shortly need the following result, which is proven in [**48**]. Observe that, with the standard coordinatization $Y_i = Y\mathbb{R}^i$ and $tV = V \cup \infty$, the restriction of $\theta \in [Y, E]$ to maps $\theta_i \in [Y_i, E_i]$ specifies a homomorphism $E_0 Y \to \varprojlim E^i Y_i$.

Proposition II.3.2. *For all spectra Y and E, the map $E^0 Y \to \varprojlim E^i Y_i$ is an epimorphism with kernel isomorphic to $\varprojlim^1 E^{i-1} Y_i$.*

This result is closely related to Whitehead's analysis of cohomology theories on spaces in terms of less stringent notions of spectra and their maps than we have been using. Define (coordinatized) weak prespectra by deleting the inclusion condition on the $\widetilde{\sigma}_i$ in Definition II.1.7, define weak Ω-prespectra by requiring the $\widetilde{\sigma}_i$ to be weak equivalences, and define weak maps of weak prespectra by requiring only that $\theta_{i+1} \circ \sigma_i \simeq \sigma_i' \circ \Sigma \theta_i$ (and retain the term map for the case when equality holds). Weak Ω-prespectra and weak maps determine (additive) cohomology theories on spaces and morphisms thereof. Two weak maps determine the same morphism if $\theta_i = \theta_i'$ in $H\mathscr{T}$; we then say that θ and θ' are weakly homotopic. Similarly, we say that two maps $\theta, \theta' \in [Y, E]$ are weakly homotopic if $\theta V \simeq \theta' V$ in $H\mathscr{T}$ for each indexing space V. In view of Theorem II.1.8, we see that θ and θ' are weakly homotopic if and only if $\theta - \theta'$ is in the kernel $\varprojlim^1 E^{i-1} Y_i$ of the epimorphism $E^0 Y \to \varprojlim E^i Y_i$.

Let $WH\mathscr{S}$ denote the category of spectra and weak homotopy classes of maps in $H\mathscr{S}$.

Inductive mapping cylinder arguments [**43**, Theorem 4] allow one to replace a weak prespectrum T by a weakly equivalent (coordinatized) prespectrum, functorially up to weak homotopy. One can then use the functor Ω^∞ of Definition II.1.4 (on the coordinatized level) and the functor ψ of

Theorem II.1.8 to obtain an actual spectrum in $\mathscr{S}$. Alternatively, one can use Q_∞ and a direct telescope construction on the spectrum level to pass from weak prespectra to spectra [**48**, XII]. Either way, one obtains the following result. For a spectrum E, we may of course regard the underlying coordinatized prespectrum $\nu\phi E$ as a weak Ω-prespectrum.

THEOREM II.3.3. *There is a functor L from the category of weak prespectra and weak homotopy classes of maps to the category $WH\mathscr{S}$ and there is a map $\kappa\colon T \to \nu\phi LT$ of weak prespectra, natural up to weak homotopy, which is a weak equivalence if T is a weak Ω-prespectrum. Further, for $E \in \mathscr{S}$, there is a natural weak equivalence $\rho\colon L\nu\phi E \to E$ of spectra such that the following composite is the identity map:*

$$\nu\phi E \xrightarrow{\ \kappa\ } \nu\phi L\nu\phi E \xrightarrow{\ \nu\phi\rho\ } \nu\phi E.$$

Finally, there is a natural weak equivalence $\omega\colon \Omega LT \to L\Omega T$ such that the following diagrams commute:

$$
\begin{array}{ccc}
\Omega T & \xrightarrow{\ \Omega\kappa\ } & \Omega\nu\phi LT \\
{\scriptstyle\kappa}\downarrow & & \| \\
\nu\phi L\Omega T & \xleftarrow{\ \nu\phi\omega\ } & \nu\phi\Omega LT
\end{array}
\qquad and \qquad
\begin{array}{ccc}
L\nu\phi\Omega E & == & L\Omega\nu\phi E \\
{\scriptstyle\rho}\downarrow & & \uparrow{\scriptstyle\omega} \\
\Omega E & \xleftarrow{\ \Omega\rho\ } & \Omega L\nu\phi E
\end{array}
$$

Together with standard representability arguments, this result implies that a cohomology theory on spaces extends uniquely to a cohomology theory on spectra and that a morphism of cohomology theories on spaces extends to a morphism of cohomology theories on spectra but not, however, uniquely since there are non-trivial maps in $H\mathscr{S}$ which are weakly null homotopic and thus induce the trivial morphism on cohomology theories on spaces. Formally, $WH\mathscr{S}$ is equivalent to the category of cohomology theories on spaces and $H\mathscr{S} \to WH\mathscr{S}$ corresponds to the forgetful functor from cohomology theories on spectra to cohomology theories on spaces.

We also require the analogous result for products. Recall the notion of a pairing $(T', T'') \to T$ of weak prespectra from Definition IX.2.5. (Our signs differ from Whitehead's [**80**] since we write suspension coordinates on the right.) It is easy to see that a map $\theta\colon E' \wedge E'' \to E$ of spectra determines a pairing $\pi(\theta)\colon (\nu\phi E', \nu\phi E'') \to \nu\phi E$ of weak prespectra (compare Remark IV.1.3 below). Via either of the two lines of proof of the previous theorem, one can verify the following addendum [**48**, XII].

Proposition II.3.4. *A pairing $f\colon (T', T'') \to T$ of weak prespectra determines a map $\phi(f)\colon LT' \wedge LT'' \to LT$ of spectra, unique up to weak homotopy, such that $\kappa \circ f$ and $\pi(\phi(f)) \circ (\kappa, \kappa)$ are weakly homotopic pairings $(T', T'') \to LT$. If $\theta\colon E' \wedge E'' \to E$ is a map of spectra, then $\rho \circ (\phi(\pi(\theta)))$ is weakly homotopic to $\theta \circ (\rho \wedge \rho)$.*

The notion of pairing gives rise to a notion of weak ring prespectrum, and this notion is adequate for the study of products in cohomology theories on spaces [**80**]. Define a weak ring spectrum in $H\mathscr{S}$ by only requiring the associativity and unit laws to hold up to weak homotopy. The proposition and theorem imply that a weak ring prespectrum T determines a weak ring spectrum LT.

The distinction between maps in $H\mathscr{S}$ and morphisms of cohomology theories on spaces and the concomitant distinction between weak ring spectra and ring spectra are folklore. The E_∞ ring spectra to be introduced in Chapter IV are always honest ring spectra, and we shall construct E_∞ ring spectra from E_∞ ring spaces in Chapter VII. Thus, where it applies, our work will circumvent any need for analysis of $\varprojlim^1$ terms. For the periodic K-theories, the relevant $\varprojlim^1$ terms vanish because $KU^{-1}(BG) = 0$ and $KO^{-1}(BG)$ is a finite dimensional vector space over $\mathbb{Z}_2$ for any compact Lie group G [**14**]. For the connective K-theories, the relevant $\varprojlim^1$ terms vanish by results of Anderson [**11**]. We shall keep track of these distinctions in this section but, because of the arguments just given, shall generally ignore them in the rest of the book.

We now turn to the study of periodic spaces and spectra, and we fix an even positive integer d throughout the discussion. As a harmless simplification, we assume henceforward that the zero$^{\text{th}}$ spaces of all spectra lie in the category $\mathscr{V}$ of spaces of the based homotopy type of CW complexes.

Definition II.3.5. For $\mathscr{C} = h\mathscr{V}$, or $\mathscr{C} = WH\mathscr{S}$, define $\Pi\mathscr{C}$, the category of periodic objects in $\mathscr{C}$, to be the category of pairs (X, χ) where $X \in \mathscr{C}$ and $\chi\colon X \to \Omega^d X$ is an isomorphism in $\mathscr{C}$. The morphisms $\zeta\colon (X, \chi) \to (X', \chi')$ are the maps $\zeta\colon X \to X'$ such that $\Omega^d\zeta \circ \chi = \chi' \circ \zeta$.

Proposition II.3.6. *The zero$^{\text{th}}$ space functor from periodic spectra to periodic spaces is an equivalence of categories.*

PROOF. We shall work with coordinatized prespectra (as in Definition II.1.7) taken with each B_i of dimension d and with tV the one-point compactification of V, so that σ_i maps $\Sigma^d T_i$ to T_{i+1} for all $i \geq 0$. Let $(X, \chi) \in \Pi h\mathscr{V}$. Let $X_i = X$, let $\chi_i = \chi\colon X_i \to \Omega^d X_i$, and let $\alpha_i\colon \Sigma^d X_i \to X_{i+1}$ be the adjoint of χ. Then $\{X_i, \alpha_i\}$ is a weak prespectrum and $\{\chi_i\}\colon \{X_i\} \to \Omega^d \{X_i\}$ is a weak map of weak prespectra (because, since d is even, the interchange of coordinates self homeomorphism of $\Omega^d \Omega^d X$ is homotopic to the identity). Define $KX = L\{X_i\} \in \mathscr{S}$ and define $\overline{\chi}\colon KX \to \Omega^d KX$ to be the composite of $L\{\chi_i\}$ and the natural isomorphism $\omega^{-1}\colon L\Omega^d \{X_i\} \to \Omega^d L\{X_i\}$ of Theorem II.3.3. With the evident maps, we thus obtain a functor $K\colon \Pi h\mathscr{V} \to \Pi WH\mathscr{S}$. By the naturality of κ and the first diagram of Theorem II.3.3 the zero$^{\text{th}}$ map of $\kappa\colon \{X_i\} \to \nu\phi L\{X_i\}$ specifies an equivalence of periodic spaces $(X, \chi) \to (KX, \overline{\chi})$. Conversely, given $(E, \xi) \in \Pi WH\mathscr{S}$, write $(X, \chi) = (E_0, \xi_0)$, define $\gamma_0 = 1\colon E_0 \to X$ and define $\gamma_i\colon E_i = E\mathbb{R}^{d_i} \to X$ inductively as the composite

$$E_i \xrightarrow{\xi_i} \Omega^d E_i \xrightarrow{\widetilde{\sigma}_{i-1}^{-1}} E_{i-1} \xrightarrow{\gamma_{i-1}} X.$$

Then $\{\gamma_i\}\colon \nu\phi E \to \{X_i\}$ is a weak map and $\Omega^d\{\gamma_i\} \circ \{\xi_i\} = \{\chi_i\} \circ \{\gamma_i\}$ as weak maps. Define $\gamma\colon E \to KX$ be the composite of $\rho^{-1}\colon E \to L\nu\phi E$ and $L\{\gamma_i\}$. By the naturality of ρ and the second diagram of Theorem II.3.3, $\gamma\colon (E, \xi) \to (KX, \overline{\chi})$ is a weak equivalence of periodic spectra (and of course, as we have used several times, weak equivalences are isomorphisms in $WH\mathscr{S}$). $\square$

We are really interested not in periodic spectra but in "periodic connective spectra," and we write $\mathscr{S}_c$ for the category of connective spectra.

Definition II.3.7. Define $\Pi WH\mathscr{S}_c$, the category of periodic connective spectra, to be the category of pairs (D, δ), where D is a connective spectrum and $\delta\colon D \to \Omega^d D$ is a map in $WH\mathscr{S}$ such that $\delta_0\colon D_0 \to \Omega^d D_0$ is an equivalence of spaces. The morphisms $\zeta\colon (D, \delta) \to (D', \delta')$ are the maps $\zeta\colon D \to D'$ such that $\Omega^d\zeta \circ \delta = \delta' \circ \zeta$ in $WH\mathscr{S}$.

Proposition II.3.8. *The associated connective spectrum functor from periodic spectra to periodic connective spectra is an equivalence of categories.*

PROOF. Given a periodic spectrum (E, ξ), let $\theta\colon D \to E$ be its associated connective spectrum of Lemma II.2.11 and note that Lemma II.2.10 gives a unique map $\delta\colon D \to \Omega^d D$ such that $\Omega^d\theta \circ \delta = \xi \circ \theta$ (since these results for $H\mathscr{S}$ clearly remain valid for $WH\mathscr{S}$). Clearly $\theta_0\colon (D_0, \delta_0) \to (E_0, \xi_0)$ is an equivalence of periodic spaces. Given a periodic connective spectrum (D, δ), the last part of the previous proof applies verbatim to yield a map $\gamma\colon D \to KD_0$ such that $\Omega^d\gamma \circ \delta = \overline{\delta}_0 \circ \gamma$ in $WH\mathscr{S}$. By inspection of the zero$^{\text{th}}$ space level, $\pi_i\gamma$ is an isomorphism for $i \geq 0$. Therefore γ induces a natural isomorphism between the identity functor of $\Pi WH\mathscr{S}_c$ and the composite

$$\Pi WH\mathscr{S}_c \longrightarrow \Pi h\mathscr{V} \xrightarrow{K} \Pi WH\mathscr{S} \longrightarrow \Pi WH\mathscr{S}_c.$$

The conclusion follows formally from Proposition II.3.6. $\square$

THe following consequence has already been used in Chapter I and will be used more deeply in Chapter VIII.

Corollary II.3.9. *Let (D, δ) and (D', δ') be periodic connective spectra and let $\lambda \colon (D_0, \delta_0) \to (D'_0, \delta'_0)$ be a map of periodic spaces. Then there is a unique map $\Lambda \colon (D, \delta) \to (D', \delta')$ of periodic connective spectra with zeroth map λ. If λ is an equivalence, then Λ is a weak equivalence of spectra.*

PROOF. Λ is given by Lemma II.2.10 as the unique map (up to weak homotopy) such that the following diagram commutes in $WH\mathscr{S}$:

$$
\begin{array}{ccc}
D & \xrightarrow{\ \Lambda\ } & D' \\
\gamma \downarrow & & \downarrow \gamma \\
KD_0 & \xrightarrow{\ K\lambda\ } & KD'_0
\end{array}
$$

$\square$

Note that, when λ is an equivalence, only one of δ_0 and δ'_0 need be assumed to be an equivalence. The corollary characterizes the periodic connective spectrum associated to a periodic space. We also need a multiplicative elaboration applicable to periodic ring spectra.

Definition II.3.10. A ring space is a space X together with a basepoint 0 and unit point 1, products $\oplus$ and $\otimes$, and an additive inverse map such that the ring axioms hold up to homotopy and 0 is a strict zero for $\otimes$ (so that $\otimes$ factors through $X \wedge X$); X is said to be commutative if $\otimes$ is homotopy commutative. By [**26**, I.4.6], X is equivalent to $X_0 \times \pi_0 X$ as an H-space under $\oplus$, where X_0 denotes the 0-component. Define $\Pi Rh\mathscr{V}$, the category of periodic ring spaces, to be the subcategory of $\Pi h\mathscr{V}$ whose objects are pairs (X, χ) such that X is a commutative ring space and $\chi \colon X \to \Omega^d X$ is adjoint to the composite

$$
X \wedge S^d \xrightarrow{1 \wedge b} X \wedge X \xrightarrow{\otimes} X
$$

for some $[b] \in \pi_d X = \pi_d(X_0, 0)$; the morphisms $\zeta \colon (X, \chi) \to (X', \chi')$ are the maps $\zeta \colon X \to X'$ of ring spaces such that $\zeta_*[b] = [b']$. Note that X is automatically an H-map with respect to $\oplus$ and is determined by its restriction $X_0 \to \Omega_0^d X_0$ to basepoint components and by $\pi_0 \chi \colon \pi_0 X \to \pi_0 \Omega^d X = \pi_d X$; since $(\pi_0 \chi)[a] = [a][b]$ for $[a] \in \pi_0 X$, $\pi_d X$ must be the free $\pi_0 X$-module generated by $[b]$.

Definition II.3.11. For $\mathscr{C} = WH\mathscr{S}$ or $\mathscr{C} = WH\mathscr{S}_c$, define $\Pi R\mathscr{C}$, the category of periodic ring objects in $\mathscr{C}$, to be the subcategory of $\Pi\mathscr{C}$ whose objects are the pairs (E, ξ) such that E is a weak commutative ring spectrum and $\xi \colon E \to \Omega^d E$ is adjoint to the composite

$$
E \wedge S^d \xrightarrow{1 \wedge b} E \wedge E_0 \xrightarrow{\phi} E
$$

for some $[b] \in \pi_d E_0$, where ϕ is as constructed in Lemma II.2.9; the morphisms $\zeta \colon (E, \xi) \to (E', \xi')$ are the maps $\zeta \colon E \to E'$ of weak ring spectra such that $\zeta_*[b] = [b']$. By Lemmas II.2.4 and II.2.5, the zeroth space functor $\mathscr{C} \to h\mathscr{V}$ induces a functor $\Pi R\mathscr{C} \to \Pi Rh\mathscr{V}$.

We have the following complement to Proposition II.3.6.

Proposition II.3.12. *The zeroth space functor from periodic ring spectra to periodic ring spaces is an equivalence of categories.*

PROOF. Given $(X, \chi) \in \Pi Rh\mathscr{V}$, the following diagram is homotopy commutative (where b determines χ and τ denotes the transposition):

$$
\begin{array}{ccccccc}
X \wedge S^d \wedge X & \xrightarrow{1 \wedge b \wedge 1} & X \wedge X \wedge X & \xrightarrow{\otimes \wedge 1} & X \wedge X & & \\
1 \wedge \tau \downarrow & & 1 \wedge \tau \downarrow & \searrow^{1 \wedge \otimes} & & \searrow^{\otimes} & \\
X \wedge X \wedge S^d & \xrightarrow{1 \wedge 1 \wedge b} & X \wedge X \wedge X & \xrightarrow{1 \wedge \otimes} & X \wedge X & \xrightarrow{\otimes} & X \\
\| & & & \searrow^{\otimes \wedge 1} & & \nearrow^{\otimes} & \\
X \wedge X \wedge S^d & \xrightarrow{\otimes \wedge 1} & X \wedge S^d & \xrightarrow{1 \wedge b} & X \wedge X & &
\end{array}
$$

Therefore $\otimes\colon X_i \wedge X_j = X \wedge X \to X = X_{i+j}$ defines a pairing $(\{X_i\}, \{X_i\}) \to \{X_i\}$ and thus determines a weak ring spectrum structure on KX by Proposition II.3.4. That result also implies that $\kappa_0\colon X \to K_0 X$ is a map of ring spaces. Conversely, given $(E, \xi) \in \Pi RWH\mathscr{S}$, it is straightforward to verify that the map $\{\gamma_i\}\colon \nu\phi E \to \{X_i\}$, $X_i = E_0$, in the proof of Proposition II.3.6 carries the pairing induced by the product on E to that just constructed. Therefore $\gamma\colon E \to KE_0$ is a map of weak ring spectra. $\square$

The analogous complement to Proposition II.3.8 is a direct consequence of Lemma II.2.11 (which clearly remains valid for weak ring spectra) and application of the last part of the argument just given to $(D, \delta) \in \Pi RWH\mathscr{S}_c$.

Proposition II.3.13. *The associated connective spectrum functor from periodic ring spectra to periodic connective ring spectra is an equivalence of categories*

Corollary II.3.14. *Let (D, δ) and (D', δ') be periodic connective ring spectra and let $\lambda\colon (D_0, \delta_0) \to (D'_0, \delta'_0)$ be a map of periodic ring spaces. Then there is a unique map $\Lambda\colon (D, \delta) \to (D', \delta')$ of periodic connective ring spectra with zeroth map λ.*

PROOF. The maps $K\lambda$ and γ in the proof of Corollary II.3.9 are maps of weak ring spectra, and it follows from Lemma II.2.10 that Λ is so as well. $\square$

Remark II.3.15. Let the periodic connective ring spectrum (D, δ) be determined by $[b] \in \pi_d D_0$. Suppose that $\psi\colon D_0 \to D_0$ is a map of ring spaces such that $\psi_*[b] = n[b]$, where n is a unit of the ring $\pi_0 D_0$. Then the adjoint δ'_0 of

$$D_0 \wedge S^d \xrightarrow{1 \wedge \left(\frac{1}{n}\right)b} D_0 \wedge D_0 \xrightarrow{\otimes} D_0$$

is an equivalence. Since $\psi_*\left(\frac{1}{n}[b]\right) = [b]$, the corollary yields a map $\Psi\colon (D, \delta') \to (D, \delta)$ of periodic connective ring spectra with zeroth map ψ.

<center>CHAPTER III</center>

<center># Orientation theory</center>

The notion of orientability with respect to an extraordinary cohomology theory is central to bundle theory. We shall here use the coordinate-free spectra of Chapter II to relate orientation theory to the geometric classifying spaces of Chapter I. We shall think of a monoid-valued $\mathscr{I}_*$-functor G which maps to F as specifying a theory of sphere bundles (orthogonal, topological, etc.). In Section III§1, we shall use the general theory of fibrations developed in [47] to make rigorous a folklore treatment of orientation theory for GV-bundles oriented with respect to a commutative ring spectrum E. In Section III§2, we shall use the two-sided bar construction discussed in Section I§2 to give a precise geometric description of a classifying space $B(GV; E)$ for E-oriented GV-bundles. That it does indeed classify will be deduced from the much more general classification theorems for fibrations and bundles with additional structure established in [47], and several other consequences of the general theorems there will also be discussed.

1. Elementary orientation theory

It is folklore that the theory of Thom complexes and orientations where is particularly smoothly if one starts with spherical fibrations $\xi\colon D \to X$ given a given cross-section $\sigma\colon X \to D$ such that σ is a cofibration. One then defines the Thom complex $T\xi$ to be the quotient space D/X. For an n-plane bundle, the idea is to think of the n-sphere bundle obtained by applying one-point compactification to each fibre. The Thom complex obtained in this way will usually agree with that obtained by one-point compactification of the total space and will always agree with that obtained by taking the quotient of the unit disc bundle by its boundary $(n-1)$-sphere bundle.

Clearly, if the homotopy type of $T\xi$ is to be an invariant of the fibre homotopy equivalence class of ξ, then the latter notion must be defined in terms of section preserving fibrewise maps (and homotopies). In turn, if homotopic maps $X' \to X$ are to induce fibre homotopy equivalent fibrations from ξ, ten the covering homotopy property must also be formulated in terms of section preserving fibrewise maps. It is then not immediately obvious how much of the standard theory of fibrations goes through; for example, the usual procedure for replacing a spherical quasifibration by a spherical fibration clearly fails.

in [47, §1–3], the basic theory of fibrations is redeveloped with fibres and maps of fibres constrained to lie in any preassigned category of spaces $\mathscr{F}$. Let V be a finite dimensional real inner product space and take $\mathscr{F}$ to be the subcategory of $\mathscr{T}$ which consists of the spaces of the (based) homotopy type of tV and their (based) homotopy equivalences. The basepoints of fibres are required to define cross-sections which are fibrewise cofibrations (see [47, 5.2]). This condition both allows our proposed construction of Thom complexes and is necessary to circumvent the problem with quasifibrations mentioned above [47, §5]. We shall call spherical fibrations of the sort just specified "FV-bundles".

Now assume given a grouplike monoid-valued $\mathscr{I}_*$-functor G together with a morphism $G \to F$ of monoid-valued $\mathscr{I}_*$-functors, as in Example I.2.5. Define a GV-bundle to be an FV-bundle together with a reduction of its structural monoid to GV. The precise meaning of a "reduction" in this generality is specified in [47, 10.4], and the cited definition show that GV-bundles are naturally equivalent to Steenrod fibre bundles with group GV and fibre tV when G is group-valued and GV acts effectively on tV.

<center>29</center>

In this context, fibrewise joins are replaced by fibrewise smash products. Explicitly, if ξ and ψ are GV and GW bundles over X and Y with total spaces $D\xi$ and $D\psi$, define $\xi \wedge \psi$ to be the $G(V \oplus W)$-bundle over $X \times Y$ with total space $D(\xi \wedge \psi) = (D\xi \times D\psi)/(\approx)$, where the equivalence identifies the wedge $(\sigma x, \psi^{-1}y) \vee (\xi^{-1}x, \sigma y)$ to the point $(\sigma x, \sigma y)$ for each $(x, y) \in X \times Y$. The projection $\xi \wedge \psi$ is induced from $\xi \times \psi$ and the section is induced from $\sigma \times \sigma$. $\xi \wedge \psi$ is in fact a $G(V \oplus W)$-bundle because it is an $F(V \oplus W)$-bundle by [**47**, 5.6] and because it inherits a reduction from those of ξ and ψ by [**47**, 5.6 and 10.4] and use of the Whitney sums given in Definition I.2.2 on the bar constructions which appear in [**47**, 10.4]. We have an evident homeomorphism

$$T\xi \wedge T\psi \xrightarrow{\cong} T(\xi \wedge \psi).$$

When $X = Y$, define the Whitney sum $\xi \oplus \psi$ to be the $G(V \oplus W)$-bundle over X induced from $\xi \wedge \psi$ by the diagonal map $\Delta \colon X \to X \times X$. We then have a homeomorphism

$$T(\xi \oplus \psi) \xrightarrow{\cong} T\xi^*(\psi)/T\psi,$$

where $\xi^*(\psi)$ is the GW-bundle over $D\xi$ induced from ψ by $\xi \colon D\xi \to X$; of course, ψ is the GW-bundle over X induced from $\xi^*(\psi)$ by $\sigma \colon X \to D\xi$ (since $\xi\sigma = 1$), and the GW-bundle map over σ induces the inclusion used to define the quotient on the right.

Let $E \in \mathcal{S}$ be a commutative ring spectrum and recall the definition of E^*X from Definition II.3.1.

Definition III.1.1. A GV-bundle ξ is E-orientable if there exists a class $\mu \in E^nT\xi$, $n = \dim tV$, such that μ restricts to a generator of the free π_*E-module $E^*T\chi$ for each fibre χ of ξ (where fibres are thought of as GV-bundles over points of the base space).

Remark III.1.2. Let $\theta \colon D \to E$ be a map of commutative ring spectra. Clearly ξ is E-orientable if it is D-orientable. Conversely, if D is connective and $\pi_i\theta$ is an isomorphism for $i \geq 0$, then ξ is D-orientable if it is E-orientable (because $\theta_* \colon D^nT\xi \to E^nT\xi$ is an isomorphism since $T\xi$ is $(n-1)$-connected). By Lemmas II.2.11 and II.2.12, it follows that orientation theory depends only on connective spectra and that a bundle ξ is $H\pi_0E$-orientable if it is E-orientable.

Henceforward, write R for $\pi_0E = \pi_0E_0$. Recall that $(HR)^*(X)$ is the ordinary reduced cohomology $\widetilde{H}^*(X, R)$. By an R-orientation (or orientation if $R = \mathbb{Z}$) of a GV-bundle ξ, we understand a class $\mu \in \widetilde{H}^n(T\xi; R)$ such that μ restricts to a generator of the free R-module $\widetilde{H}^n(T\chi; R)$ for each fibre χ; the pair (ξ, μ) is then said to be an R-oriented GV-bundle. Since we can identify $\widetilde{H}^n(T\chi; \pi_*E)$ with $E^*T\chi$, μ restricts to a definite fundamental class $E^nT\chi$ for each fibre χ.

Definition III.1.3. An E-orientation of an R-oriented GV-bundle ξ is a class $\mu \in E^nT\xi$, $n = \dim tV$, such that μ restricts to the fundamental class of $E^nT\chi$ for each fibre χ; the pair (ξ, μ) is then said to be an E-oriented GV-bundle.

Thus E-orientations are required to be consistent with preassigned R-orientations. The following proof of the Thom isomorphism theorem should help motivate this precise definition. Let X^+ denote the union of X and a disjoint basepoint.

THEOREM III.1.4. *Let (ξ, μ) be an E-oriented GV-bundle over a finite dimensional CW-complex X. Then the cup product with μ defines an isomorphism $E^*X^+ \to E^*T\xi$. Therefore $E^*T\xi$ is the free E^*X^+-module generated by μ.*

PROOF. The cup product is determined by the reduced diagonal $T\xi \to X^+ \wedge T\xi$ (which is induced via $\xi \wedge 1$ from the ordinary diagonal $D\xi^+ \to D\xi^+ \wedge D\xi^+$ of the total space). Now $\smile \mu$ induces a morphism of Atiyah–Hirzebruch spectral sequences which, on the E_2-level, is the isomorphism

$$\smile \mu \colon \widetilde{H}^*(X^+; \pi_*E) \longrightarrow \widetilde{H}^*(T\xi; \pi_*E)$$

determined by the preassigned R-orientation of ξ.   □

Of course, the finite-dimensionality of X serves only to ensure convergence of the spectral sequences.

The following remarks summarize other basic facts about orientations; the proofs are immediate from the definitions, the previous theorem, and the facts about Thom complexes recorded above.

Remark III.1.5. Let X and Y be (finite-dimensional) CW-complex.

(i) The trivial GV-bundle $\varepsilon = \varepsilon V \colon X \times tV \to X$ satisfies $T\varepsilon = X^+ \wedge tV$. The image under suspension of $1 \in E^0 X^+$ is an E-orientation of ε; it is called the canonical orientation and is denoted μ_0.

(ii) If (ψ, ν) is an E-oriented GV-bundle over Y and $f \colon X \to Y$ is a map, then $(Tf)^*(\nu)$ is an E-orientation of $f^*(\psi)$, where $Tf \colon Tf^*\psi \to T\psi$ is the induced map of Thom complexes. If, further, f is a cofibration, then the cup product with ν induces an isomorphism

$$E^*(Y/X) \xrightarrow{\cong} E^*(T\psi/T(\psi|_X))$$

(by the long exact cohomology sequences and the five lemma).

(iii) If (ξ, μ) and (ψ, ν) are E-oriented GV and GW-bundles over X and Y, then $(\xi \wedge \psi, \mu \wedge \nu)$ is an E-oriented $G(V \oplus W)$-bundle over $X \times Y$, where $\mu \wedge \nu$ is the image of $\mu \otimes \nu$ under the external product [**48**, XII]

$$E^*T\xi \otimes E^*T\psi \longrightarrow E^*(T\xi \wedge T\psi) = E^*T(\xi \wedge \psi).$$

When $X = Y$, $\mu \oplus \nu$ denotes the induced E-orientation $(T\Delta)^*(\mu \wedge \nu)$ of $\xi \oplus \psi$.

(iv) If (ψ, ν) and $(\xi \oplus \psi, \omega)$ are E-oriented W and $G(V \oplus W)$-bundles over X, where ξ is a GV-bundle over X, then the image μ of 1 under the composite isomorphism

$$E^* X^+ \xrightarrow{\smile \omega} E^*T(\xi \oplus \psi) = E^*(T\xi^*(\psi)/T\psi) \xrightarrow{(\smile \nu)^{-1}} E^*(D\xi/X) = E^*T\xi$$

is the unique E-orientation of ξ such that $\mu \oplus \nu = \omega$.

(v) If ξ and μ are stably equivalent GV and GW-bundles over X, so that $\xi \oplus \varepsilon(W \oplus Z)$ is equivalent to $\psi \oplus \varepsilon(V \oplus Z)$ for some Z, then ξ is E-orientable if and only if ψ is E-orientable.

2. Classification of E-oriented GV-bundles

We retain the notations of the previous section and assume that all spaces in sight are in the category srW of spaces of the homotopy type of CW-complexes. By [**46**, A.6], $B(Y, G, X)$ is in $\mathscr{W}$ if Y, G, and X are in $\mathscr{W}$.

Let SGV denote the component of the identity element of GV.

Let FR denote the group of units of the ring $R = \pi_0 E_0$ and let $FE \subset E_0$ denote the union of the corresponding components. Define $d \colon FE \to \pi_0 FE = FR$ to be the discretization map. let $SFE \subset FE$ denote the component corresponding to the identity element of R. When E is the sphere spectrum $Q_\infty S^0$, the space $E_0 = QS^0$ coincides with $\widetilde{F}$; in particular, $FE = F$ and $SFE = SF$. In the general case, we may take the unit $e \colon Q_\infty S^0 \to E$ to be an honest map in $\mathscr{S}$ rather than just a map in $H\mathscr{S}$ (by Definition II.2.7), and we also write e for the composite $GV \to FV \subset F \xrightarrow{e} FE$, $V \subset \mathbb{R}^\infty$.

By Definition II.1.1 of a spectrum, we have a homeomorphism $\widetilde{\sigma} \colon E_0 \to F(tV, EV)$ for each finite-dimensional sub inner product space V of $\mathbb{R}^\infty$. We restrict attention to such V, and we identify FE with a subspace of $F(tV, EV)$ via $\widetilde{\sigma}$. We are given a morphism of monoids $GV \to FV \subset F(tV, tV)$, and the composition of maps defines a right action of GV on FE and of SGV on SFE. Define

$$B(GV; E) = B(FE, GV, *) \quad \text{and} \quad B(SGV; E) = B(SFE, SGV, *).$$

We then have the following commutative "orientation diagram", in which the maps i are the evident inclusions and $B(GV;R)$ is defined to be $B(FR, GV, *)$:

$$
\begin{array}{ccccccc}
SGV & \xrightarrow{\ e\ } & SFE & \xrightarrow{\ \tau\ } & B(SGV; E) & \xrightarrow{\ q\ } & BSGV \\
{\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow \\
GV & \xrightarrow{\ e\ } & FE & \xrightarrow{\ \tau\ } & B(GV; E) & \xrightarrow{\ q\ } & BGV \\
\| & & {\scriptstyle d}\downarrow & & {\scriptstyle Bd}\downarrow & & \| \\
GV & \xrightarrow{\ de\ } & FR & \xrightarrow{\ \tau\ } & B(GV; R) & \xrightarrow{\ q\ } & BGV
\end{array}
$$

The rows are quasifibration sequences by [**47**, 7.9], and the maps q can be thought of as universal determinant bundles.

We shall interpret geometrically the functors an natural transformations represented on the homotopy category $h\mathscr{W}$ by the spaces and maps of the orientation diagram by quoting appropriate results of [**47**]. Recall that brackets denoted unbased homotopy classes in that paper but denote based homotopy classes here.

We note first that, by [**47**, 9.8], $[X^+, BFV]$ is naturally isomorphic to the set of equivalence classes of FV-bundles over X. Next, by [**47**, 11.1 and 10.4], $[X^+, B(GV \backslash FV, FV, *)]$ is naturally isomorphic to the set of equivalence classes of FV-bundles over X with a reduction of the structural monoid to GV. Here, by [**47**, 8.9], $B(GV \backslash FV, FV, *)$ is homotopy equivalent to BGV and the maps

$$q\colon B(GV \backslash FV, FV, *) \longrightarrow BFV \quad \text{and} \quad Bj\colon BGV \longrightarrow BFV$$

can be used interchangeably, $j\colon G \to F$. Of course, if G is group-valued and GV acts effectively on tV, then BGV also classifies Steenrod fibre bundles over X with group GV and fibre tV [**47**, 9.10]; here the map Bj induces the transformation obtained by sending a fibre bundle to its underlying spherical fibration, whereas q induces the transformation from GV-bundles to FV-bundles obtained by forgetting the reduction of the structural monoid. Given $i\colon H \to G$, BHV can be regarded as classifying either FV-bundles or GV-bundles with a reduction of their structural monoid to HV.

THEOREM III.2.1. *For $X \in \mathscr{W}$, $[X^+, B(GV; E)]$ is naturally isomorphic to the set of equivalence classes of E-oriented GV-bundles over X under the relation of orientation preserving GV-bundle equivalence.*

PROOF. First, let $G = F$. An orientation μ of an FV-bundle $\xi\colon D \to X$ can be described as a homotopy class of maps $D \to EV$ such that, for any map $\psi\colon tV \to D$ which is a based homotopy equivalence into some fibre, the composite $\mu\psi\colon tV \to EV$ lies in FE. Here the basepoints of fibres determine the cross-section of ξ, and μ factors through $T\xi$ because $\mu\chi$ is a based map for each inclusion $\chi\colon \xi^{-1}(x) \subset D$. The condition $\mu\chi \in FE$ also ensures that the restrictions of $\mu \in E^nT\xi$ to $E^nT\chi$ is a generator of $E^*T\chi$ over π_*E. In the language of [**47**, 10.1, 10.2 and 10.6], μ is an FE-structure defined with respect to the admissible pair (FE, EV). Therefore the result for $G = F$ is a special case of [**47**, 11.1]. For general G, an orientation of a GV-bundle depends only on the underlying FV-bundle (and not on the reduction), hence the result for F implies the result for G by [**47**, 11.3]. Alternatively, when G is group-valued and GV acts effectively on tV, we could appeal to the bundle-theoretic result [**47**, 11.4] rather than to the quoted fibration-theoretic results.   □

The proofs of [**47**, 11.1 and 11.4] give explicit universal E-oriented GV-bundles (π, θ) with base $B(GV; E)$ and with π classified by $q\colon B(GV; E) \to BGV$ [**47**, 11.2]. Therefore q induces he obvious forgetful transformation from E-oriented GV-bundles to GV-bundles.

If FE happens to admit a structure of topological monoid such that $e\colon GV \to FE$ is a map of monoids, so that $B(GV; E) = FE/GV$ is homotopy equivalent of the fibre of $Be\colon BGV \to BFE$, then $\alpha^*(Be) \in [X^+, BFE]$ is the only obstruction to the E-orientability of the GV-bundle classified by $\alpha\colon X \to BGV$. As far as I know, the only examples are $E = HR$ and $E = S$ (both discussed

below). Nevertheless, a similar obstruction will be constructed much more generally in the next chapter.

Example III.2.2. Let $E = HR$. It is not hard to construct a model for HR such that $e\colon GV \to FHR$ is a morphism of monoids. Rather than give details, we note that $Bd\colon B(GV; HR) \to B(GV; R)$ is a homotopy equivalence, since $d\colon FHR \to FR$ is, and we can thus use the middle and bottom rows of the orientation diagram interchangeably. Clearly $de\colon GV \to FR$ is a morphism of monoids, $B(GV; R)$ is equivalent to the fibre of $B(de)$, and $B(de)$ factors through $Bd\colon BGV \to B\pi_0GV$. $BSGV$ is contained in $B(GV; R)$, and we have the following commutative diagram:

$$
\begin{array}{ccccc}
BSGV & \xrightarrow{\ i\ } & BGV & \xrightarrow{\ Bd\ } & K(\pi_0GV, 1) \\
\downarrow & & \| & & \downarrow \\
B(GV; R) & \xrightarrow{\ q\ } & BGV & \xrightarrow{\ B(de)\ } & K(FR, 1)
\end{array}
$$

The GV-bundle classified by $\alpha\colon X \to BGV$ is R-orientable if and only if $\alpha^*B(de) = 0$, and this holds if $\pi_0GV = \{1\}$ or if $\pi_0GV = \mathbb{Z}_2$ and either char $R = 2$ or the first Stiefel–Whitney class $w_1 = \alpha^*Bd \in H^1(X; \mathbb{Z}_2)$ is zero. By the diagram, $BSGV \to B(GV; R)$ is a homotopy equivalence if $\pi_0GV = FR$.

Let (π_0, θ_0) be the (integrally) oriented GV-bundle classified by the inclusion of $BSGV$ in $B(GV; \mathbb{Z})$. Via $\alpha \to \alpha^*(\pi_0, \theta_0)$, $[X^+, BSGV]$ is naturally isomorphic to the set of equivalence classes of SGV-bundles, by which we understand GV-bundles with a canonical orientation. In other words, Theorem III.2.1 allows us to choose compatible preferred orientations simultaneously on all GV-bundles with a reduction of their structural monoid to SGV.

Definition III.2.3. An E-oriented SGV-bundle is an E-oriented GV-bundle and and SGV-bundle such that the preassigned R-orientation is that induced from the canonical orientation.

The map $Bd\colon B(GV; E) \to B(GV; R)$ induces the evident forgetful transformation from E-oriented to R-oriented GV-bundles and is an isomorphism on components. Our original definition of an E-orientation takes cognizance of the fact that the set of equivalence classes of E-oriented GV-bundles over X is the union of the inverse images under $(Bd)_*$ of the elements of the set of R-oriented GV-bundles over X. The image of $B(SGV; E)$ under $Bd \circ i$ is precisely $BSGV \subset B(GV; R)$. This implies the following corollary.

Corollary III.2.4. For $X \in \mathscr{W}$, $[X^+, B(SGV; E)]$ is naturally isomorphic to the set of equivalence classes of E-oriented SGV-bundles over X.

We complete the analysis of the upper two rows of the orientation diagram in the following remarks (compare [**47**, 11.3]).

Remark III.2.5. (i) $[X^+, FE]$ is isomorphic to the set of E-orientations of the trivial GV-bundle εV over X. Indeed, given $\alpha\colon X \to FE$, its adjoint $X \times tV \to EV$ gives the corresponding E-orientation μ_α. α has image in SFE if and only if μ_α restricts to the canonical fundamental class of each fibre of εV.
(ii) The maps τ of the orientation diagram induce the transformations which send an E-orientation μ of εV to the equivalence class of the pair $(\varepsilon V, \mu)$
(iii) $[X^+, GV]$ is isomorphic to the set of homotopy classes of GV-bundle equivalences $\varepsilon V \to \varepsilon V$. Indeed, given $\alpha\colon X \to GV$, its adjoint $X \times tV \to tV$ gives the second coordinate of the corresponding GV-bundle map ν_α. α has image in SGV if and only if $T\nu_\alpha$ preserves the canonical orientation of εV.
(iv) The maps e of the orientation diagram induce the transformations which send a GV-bundle map ν to the E-orientation $(T\nu)^*(\mu_0)$, where μ_0 is the canonical E-orientation of εV.

There is an analogy between orientations and trivializations that plays an important role in the applications. Suppose given morphisms $H \to G \to F$ of monoid-valued $\mathscr{I}_*$-functors, where H is group-valued. By [**47**, 10.3 and 11.1] (and, for cases such as $H = Spin$, where HV fails to act effectively on tV, [**47**, 10.4 and 11.3]), $[X^+, GV/HV]$ is naturally isomorphic to the set of equivalence classes of GV-trivialized HV-bundles over X.

Remark III.2.6. $Be\colon GV/HV = B(GV, HV, *) \to B(FE, HV, *) = B(HV; E)$ induces the transformation which sends (ξ, ζ) to $(\xi, (T\zeta)^*(\mu_0))$, where μ_0 is the canonical E-orientation of the trivial GV-bundle. The following diagram commutes:

$$
\begin{array}{ccccccc}
HV & \longrightarrow & GV & \xrightarrow{\ \tau\ } & GV/HV & \xrightarrow{\ q\ } & BHV \\
\| & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle Be} & & \| \\
HV & \xrightarrow{\ e\ } & FE & \xrightarrow{\ \tau\ } & B(HV; E) & \xrightarrow{\ q\ } & BHV
\end{array}
$$

In the upper row, τ induces the transformation which sends $\nu\colon \varepsilon V \to \varepsilon V$ to $(\varepsilon V, \nu)$ and q induces the evident forgetful transformation.

Example III.2.7. Let $E = S$, so that $FE = F$ and $B(GV; E) = F/GV$. Since $Be\colon BGV \to BF$ induces the transformation which sends a GV-bundle ξ to its stable fibre homotopy equivalence class, this class is the obstruction to the S-orientability of ξ. This fact can also be seen directly since, if $V = \mathbb{R}^m$ and $tV = S^m$, an S-orientation $\mu\colon T\xi \to QS^m$ has adjoint a map $D\xi \times S^n \to S^{n+m}$ for n suitably large (if the base of ξ is compact), and the latter map is the second coordinate of a stable trivialization of ξ.

Finally, we relate Theorem III.2.1 to fibrewise smash products and discussi ts naturality in E.

Remark III.2.8. Let V and W be orthogonal finite dimensional sub inner product spaces of $\mathbb{R}^\infty$. It is an easy consequence of the definition of the smash product on $H\mathscr{S}$ given in [**48**] that the product on E determines maps $\phi\colon FE \times FE \to FE$ and $\phi\colon EV \wedge EW \to E(V + W)$ (depending on an appropriate linear isometry $\mathbb{R}^\infty \oplus \mathbb{R}^\infty \to \mathbb{R}^\infty$) such that the following diagram is commutative:

$$
\begin{array}{ccccc}
FE \times FE & \xrightarrow{\widetilde{\sigma} \times \widetilde{\sigma}} & F(tV, EV) \times F(tW, EW) & \xrightarrow{\ \wedge\ } & F(tV \wedge tW, EV \wedge EW) \\
\downarrow{\scriptstyle \phi} & & & & \downarrow{\scriptstyle F(1,\phi)} \\
FE & & \xrightarrow{\hspace{5cm}\widetilde{\sigma}\hspace{5cm}} & & F(t(V + W), E(V + W))
\end{array}
$$

(at least if the product on E is given by an honest map in $\mathscr{S}$; possible required formal inverses of weak equivalences would mildly complicate the argument to follow). If $\{V_i\}$ and $\{W_i\}$ are expanding sequences such that $V_i \perp W_i$ and $\mathbb{R}^\infty = (\bigcup_i V_i) + (\bigcup_i W_i)$, then there is a sequence of maps $\phi_i\colon EV_i \wedge EW_i \to E(V_i + W_i)$ so compatible with $\phi\colon FE \times FE \to FE$. The diagram and the definitions of ω on F and of the right actions γ by the GV on FE imply that the following diagram also commutes:

$$
\begin{array}{ccccc}
FE \times GV \times FE \times GW & \xrightarrow{1 \times t \times 1} & FE \times FE \times GV \times GW & \xrightarrow{\phi \times \omega} & FE \times G(V + W) \\
\downarrow{\scriptstyle \gamma \times \gamma} & & & & \downarrow{\scriptstyle \gamma} \\
FE \times FE & & \xrightarrow{\hspace{4cm}\phi\hspace{4cm}} & & FE
\end{array}
$$

Therefore, since the bar construction commutes with products, ϕ and ω induce

$$
B(\omega; \phi)\colon B(GV; E) \times B(GW; E) \longrightarrow B(G(V + W); E).
$$

Let (ξ, μ) and (ψ, ν) be E-oriented GV and GW-bundles over X and Y classified by $\overline{\alpha}$ and $\overline{\beta}$. Then $(\xi \wedge \psi, \mu \wedge \nu)$ is classified by $B(\omega; \phi)(\overline{\alpha} \times \overline{\beta})$. The proof is based on the observation that [**47**, 5.6, 7.4 and 11.1] imply that $B(\omega; \phi)$ can be covered by a map of E-oriented $G(V + W)$-bundles from the

fibrewise smash product of the universal bundles over $B(GV; E)$ and $B(GW; E)$ to the universal bundle over $B(G(V + W); E)$.

Remark III.2.9. Let $\zeta \colon E \to E'$ be a map of commutative ring spectra. By the definition of a map of spectra, Definition II.1.1, and of the actions γ, the following diagrams are commutative:

$$
\begin{array}{ccc}
FE & \xrightarrow{\widetilde{\sigma}} & F(tV, EV) \\
\zeta \downarrow & & \downarrow F(1,\zeta) \\
FE' & \xrightarrow{\widetilde{\sigma}} & F(tV, E'V)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
FE \times GV & \xrightarrow{\zeta \times 1} & FE' \times GV \\
\gamma \downarrow & & \downarrow \gamma \\
FE & \xrightarrow{\zeta} & FE'
\end{array}
$$

Therefore ζ induces a map $B\zeta \colon B(GV; E) \to B(GV; E')$, and the following diagram is obviously commutative:

$$
\begin{array}{ccccc}
FE & \xrightarrow{\tau} & B(GV; E) & \xrightarrow{q} & BGV \\
\zeta \downarrow & & \downarrow B\zeta & & \| \\
FE' & \xrightarrow{\tau} & B(GV; E') & \xrightarrow{q} & BGV
\end{array}
$$

The construction of the universal E and E' oriented GV bundles (π, θ) and (π', θ') in [**47**, 11.1] shows that the map of Thom complexes $T\zeta \colon T\pi \to T\pi'$ induced by $B\zeta$ is such that the diagram

$$
\begin{array}{ccc}
T\pi & \xrightarrow{T\zeta} & T\pi' \\
\theta \downarrow & & \downarrow \theta' \\
EV & \xrightarrow{\zeta} & E'V
\end{array}
$$

is homotopy commutative. We conclude that $B\zeta$ induces the transformation which sends an E-oriented GV-bundle (ξ, μ) to the E'-oriented GV-bundle $(\xi, \zeta\mu)$, where we have also written ζ for the cohomology operation it determines.

<center>CHAPTER IV</center>

E_∞ **ring spectra**[*]

In the previous chapter, the basics of orientation theory were developed for a cohomology theory represented by a commutative ring spectrum E. In order to analyze the obstruction to orientability, and for many other purposes, it is desirable to have a more structured notion of a ring spectrum.

To see what is wanted, consider the unit space $FE \subset E_0$ and the component SFE of the identity element of $\pi_0 E$. The product $\phi\colon E \wedge E \to E$ and unit $e\colon S \to E$ determine H-space structures on FE and SFE. Recall that, when $E = S$, $FE = F$ and $SFE = SF$. When $E = kO$, Adams pointed out in [**5**, §7] that the group of kO-oriented spherical fibrations over X ought to play a key role in the analysis of $J(X)$ and that the obstruction to kO-orientability ought to be directly related to the d-invariant. Now, the d-invariant can be thought of as induced from the H-map $e\colon SF \to BO_\otimes = SFkO$, and Sullivan pointed out in [**72**, §6] that if e were to admit a delooping $Be\colon SFE \to BBO_\otimes$, then the fibre of Be ought to be equivalent to $B(SF;kO)$ and Be therefore ought to be the universal obstruction to kO-orientability.

Thus one wants at least sufficient structure on E to ensure that FE admits a classifying space (or delooping). One's first thought is to insist that FE admits a structure of topological monoid. One cannot require ϕ to be associative and unital, without passage to homotopy, since the smash product of spectra is itself only associative and unital after passage to homotopy. However, one can ignore the smash product, revert to Whitehead's notion of a ring spetrum defined in terms of a pairing of spectra [**80**], and assume that the given pairing is strictly associative and unital. This is perfectly satisfactory in theory, and has been used by Patterson and Stong in an investigation of the orientability of bundles [**52**], but is at present of little use in practice since the only known strictly associative ring spectra are S (under the composition pairing) and the Eilenberg–Mac Lane spectra HR, where R is a ring.

We shall define the notion of an E_∞ ring spectrum in Section IV§1. When E is such a spectrum, FE and SFE will not only have deloopings, they will actually be infinite loop spaces. Paradoxically, although the implications of an E_∞ ring structure are thus much stronger than the implications of strict associativity, it is not hard to construct E_∞ ring spectra. Indeed, among other examples, we shall see in Section IV§2 that the various Thom spectra MG, including such exotic ones as $MTop$ and MF, are E_∞ ring spectra as they occur in nature. In Chapter VII, the first author will give a machine for the construction of E_∞ ring spectra from spaces with appropriate internal structure. This machine will construct E_∞ ring spectra which represent the various connective K-theories (geometric and algebraic) and the ordinary cohomology theories.

In Section IV§3, we study E-orientation theory when E is an E_∞ ring spectrum. Here the obstruction to E-orientability takes on a conceptual form: it is a degree one "Stiefel–Whitney" class in the cohomology theory determined by the infinite loop space FE. We shall give a number of examples to show how this obstruction can be calculated in favorable cases. Moreover, the classifying space $B(G;E)$ for E-oriented stable G-bundles ($G = O$, U, F, Top, etc.) is itself an infinite loop space, and the Thom spectrum $M(G;E)$ associated to the universal E-oriented GV-bundles, $V \subset \mathbb{R}^\infty$, is again an E_∞ ring spectrum. Thus we have new cohomology theories of E-oriented G-bundles and corresponding new cobordism theories. Our work raises many unanswered questions. Can these new theories be calculated in interesting cases? What is the relationship

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[*](by J.P. May, F. Quinn, and N. Ray)

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between the cohomology theory determined by an E_∞ ring spectrum E and that determined by FE? What are the implications for $\pi_* E$ of the existence of an E_∞ ring structure on E? All that we show here is that the representing spectra of most of the interesting cohomology theories have an enormously richer internal structure than has yet been studied and that this structure gives rise to a collection of potentially powerful new theories.

1. E_∞ ring prespectra and spectra

One way to prove that a space is an infinite loop space is to display an action of an E_∞ operad on it Section VI§1. We think of spectra as already carrying additive structures, and we shall superimpose multiplicative structures by means of actions by operads. Since spectra, as defined in Definition II.1.1 are indexed on the finite-dimensional sub inner product spaces of $\mathbb{R}^\infty$, it is natural to give the linear isometries operad $\mathscr{L}$ of Definition I.1.2 a privileged role in our theory. For the examples in this chapter, it would suffice to use only $\mathscr{L}$, but it is essential for later chapters to allow more general operads. Thus we assume given an E_∞ operad $\mathscr{G}$ and a morphism of operads $\mathscr{G} \to \mathscr{L}$. By abuse, we shall think of elements of $\mathscr{G}(j)$ as linear isometries via the given map to $\mathscr{L}(j)$.

Actions by operads refer to chosen basepoints. We think of actions by $\mathscr{G}$ on spaces as multiplicative, and the relevant basepoint is denoted by 1. We do not want to impose additive structures on spaces, but we do want to impose zeroes. Thus let $\mathscr{T}_e$ denote the category of spaces X together with cofibrations $e\colon S^0 \to X$ where $S^0 = \{0, 1\}$. By a $\mathscr{G}$-space with zero, or $\mathscr{G}_0$-space, we understand a $\mathscr{G}$-space (X, ξ) such that $X \in \mathscr{T}_e$ and $\xi_j(g, x_1, \ldots, x_j) = 0$ if any $x_i = 0$; in other words, $\xi_j\colon \mathscr{G}(j) \times X^j \to X$ is required to factor through equivariant half-smash $\mathscr{G}(j) \times_{\Sigma_j} X \wedge \cdots \wedge X/\mathscr{G}(j) \times_{\Sigma_j} *$ defined with respect to the basepoint 0. Let $\mathscr{G}[\mathscr{T}_e]$ denote the category of $\mathscr{G}$-spaces with zero.

The spaces TV of a prespectrum have given basepoints, which we denote by 0; all wedges and smash products used below are to be taken with respect to these basepoints. By a unit for $T \in \mathscr{P}$ or for $E \in \mathscr{S}$ we understand a map $e\colon \Sigma^\infty S^0 \to T$ in $\mathscr{P}$ or $e\colon Q_\infty S^0 \to E$ in $\mathscr{S}$ such that the resulting (and, by Lemma II.1.3 and Definition II.1.7, determining) map $S^0 \to T_0$ or $S^0 \to E_0$ is a cofibration. Let $\mathscr{P}_e$ and $\mathscr{S}_e$ denote the categories of prespectra with units and spectra with units (and morphisms which preserve units). The constructions and results of Section II§1 extend immediately to the categories $\mathscr{T}_e$, $\mathscr{P}_e$, and $\mathscr{S}_e$.

The reader is advised to review the definitions of operads (Definition VI.1.2), of actions by operads (Definition VI.1.3), of the linear isometries operad (Definition I.1.2), and of prespectra and spectra (Definition II.1.1) before proceeding to the following definition.

Definition IV.1.1. A $\mathscr{G}$-prespectrum (T, σ, ξ) is a unital prespectrum (T, σ) together with maps

$$\xi_j(g)\colon TV_1 \wedge \cdots \wedge TV_j \longrightarrow Tg(V_1 \oplus \cdots \oplus V_j)$$

for $j \geq 0$, $g \in \mathscr{G}(j)$, and $V_i \in \mathscr{I}_*(\mathbb{R}^\infty)$, where $\xi_0(*)$ is to be interpreted as the inclusion $e\colon S^0 \to T_0$, such that the following conditions are satisfied.

(a) If $g \in \mathscr{G}(k)$, $h_r \in \mathscr{G}(j_r)$ for $1 \leq r \leq k$, and $j = j_1 + \cdots + j_k$, then the following diagram is commutative.

$$
\begin{array}{ccc}
TV_1 \wedge \cdots \wedge TV_j & \xrightarrow{\xi_j(\gamma(g; h_1, \ldots, h_k))} & T\gamma(g; h_1, \ldots, h_k)(V_1 \oplus \cdots \oplus V_j) \\
{\scriptstyle \xi_{j_1}(h_1) \cdots \xi_{j_k}(h_k)} \Big\downarrow & & \Big\| \\
TW_1 \wedge \cdots \wedge TW_k & \xrightarrow{\qquad \xi_k(g) \qquad} & Tg(W_1 \oplus \cdots \oplus W_k)
\end{array}
$$

where $W_r = h_r(V_{j_1 + \cdots + j_{r-1} + 1} \oplus \cdots \oplus V_{j_1 + \cdots + j_r})$ (or $\{0\}$ if $j_r = 0$).

(b) $\xi_1(1)\colon TV \to TV$ is the identity map.

(c) If $g \in \mathscr{G}(j)$ and $\tau \in \Sigma_j$, then the following diagram is commutative:

$$
\begin{array}{ccc}
TV_1 \wedge \cdots \wedge TV_j & \xrightarrow{\xi_j(g\tau)} & Tg\tau(V_1 \oplus \cdots \oplus V_j) \\
\downarrow{\tau} & & \| \\
TV_{\tau^{-1}(1)} \wedge \cdots \wedge TV_{\tau^{-1}(j)} & \xrightarrow{\xi_j(g)} & Tg(V_{\tau^{-1}(1)} \oplus \cdots \oplus V_{\tau^{-1}(j)})
\end{array}
$$

(d) For fixed V_i and W, ξ_j is continuous in g as g ranges through the subspace of $\mathscr{G}(j)$ which consists of those elements such that $g(V_1 \oplus \cdots \oplus V_j) = W$.

(e) If $g \in \mathscr{G}(j)$ and $V_i \perp W_i$, then the following diagram is commutative:

$$
\begin{array}{ccc}
TV_1 \wedge tW_1 \wedge \cdots \wedge TV_j \wedge tW_j & \xrightarrow{\sigma \wedge \cdots \wedge \sigma} & T(V_1 + W_1) \wedge \cdots \wedge T(V_j + W_j) \\
\| & & \downarrow{\xi_j(g)} \\
TV_1 \wedge \cdots \wedge TV_j \wedge t(W_1 \oplus \cdots \oplus W_j) & & Tg((V_1 + W_1) \oplus \cdots \oplus (V_j + W_j)) \\
\downarrow{\xi_j(g) \wedge tg} & & \| \\
Tg(V_1 \oplus \cdots \oplus V_j) \wedge tg(W_1 \oplus \cdots \oplus W_j) & \xrightarrow{\sigma} & T(g(V_1 \oplus \cdots \oplus V_j) + g(W_1 \oplus \cdots \oplus W_j))
\end{array}
$$

(where t is a sphere-valued functor on $\mathscr{I}_*$ as in Section II§1).

(f) If $g \in \mathscr{G}(1)$, then $\xi_1(g) \colon TV \to TgV$ is a homeomorphism in the homotopy class $T(g|_V)$, and every morphism $f \in \mathscr{I}_*(\mathbb{R}^\infty)$ is obtainable by restriction from some $\gamma \in \mathscr{G}(1)$. ((f) could be deleted; see Remark II.1.10.)

(T, σ, ξ) is a $\mathscr{G}$-spectrum if (T, σ) is a spectrum. A morphism $\psi \colon (T, \sigma, \xi) \to (T', \sigma', \xi')$ of $\mathscr{G}$-prespectra is a morphism $\psi \colon (T, \sigma) \to (T', \sigma')$ of unital prespectra such that the following diagrams are commutative:

$$
\begin{array}{ccc}
TV_1 \wedge \cdots \wedge TV_j & \xrightarrow{\xi_j(g)} & Tg(V_1 \oplus \cdots \oplus V_j) \\
{\psi \wedge \cdots \wedge \psi}\downarrow & & \downarrow{\psi} \\
T'V_1 \wedge \cdots \wedge T'V_j & \xrightarrow{\xi'_j(g)} & T'g(V_1 \oplus \cdots \oplus V_j)
\end{array}
$$

Let $\mathscr{G}[\mathscr{P}_e]$ denote the category of $\mathscr{G}$-prespectra and let $\mathscr{G}[\mathscr{S}_e]$ denote its full subcategory of $\mathscr{G}$-spectra. Let $\nu \colon \mathscr{G}[\mathscr{S}_e] \to \mathscr{G}[\mathscr{P}_e]$ denote the inclusion functor.

Definition IV.1.2. An E_∞ ring prespectrum (or spectrum) is a $\mathscr{G}$-prespectrum (or $\mathscr{G}$-spectrum) over any E_∞ operad $\mathscr{G}$ with a given morphism of operads $\mathscr{G} \to \mathscr{L}$.

We have not defined and do not need any notion of a morphism between E_∞ ring spectra over different operads.

Think of a prespectrum (T, σ) as determining an underlying space, the wedge over all $V \in \mathscr{I}_*(\mathbb{R}^\infty)$ of spaces TV. Then conditions (a), (b), and (c) are precisely the algebraic identities required for the ξ_j to give this space a structure of $\mathscr{G}_0$-space. Condition (d) describes how to weave in the topology of $\mathscr{G}$, but we should add that we only know how to make effective use of the topology when V_i and W are all $\{0\}$. The last two conditions relate the ξ_j to the internal structure of (T, σ). In practice, (f) is used to define the maps $T(g|_V)$, and the force of the definition lies in condition (e).

In [**48**, XI], a smash product functor $\wedge_g \colon \mathscr{S} \times \mathscr{S} \to \mathscr{S}$ is defined for each element $g \in \mathscr{L}(2)$; all such functors become equivalent in the stable homotopy category $H\mathscr{S}$. Our definition ensures that, for each $\mathscr{G}$-spectrum E and each $g \in \mathscr{G}(2)$, there is a well-defined map $E \wedge_g E \to E$ in $\mathscr{S}$ which gives E a structure of commutative ring spectrum in $H\mathscr{S}$.

Although irrelevant to our theory, a comparison with Whitehead's notion of a ring spectrum may be illuminating.

Remark IV.1.3. Let (T, σ, ξ) be a $\mathscr{G}$-prespectrum and let $g \in \mathscr{G}(2)$. Let $\mathbb{R}^\infty$ and $\mathbb{R}^\infty \oplus \mathbb{R}^\infty$ have orthonormal bases $\{e_i\}$ and $\{e'_i, e''_i\}$. Assume that tV is the one-point compactification of V and let $T_i = T\mathbb{R}^i$ and $\sigma_i = \sigma : \Sigma T_i \wedge t\mathbb{R}e_{i+1} \to T_{i+1}$. Consider the following diagram for any $p \geq 0$ and $q \geq 0$, where $d : \mathbb{R}ge'_{p+1} \to \mathbb{R}ge'_{q+1}$ is the obvious linear isometry and $f : g(\mathbb{R}^p \oplus \mathbb{R}^{q+1}) \to \mathbb{R}^{p+q+1}$ is any linear isometry:

The left rectangle commutes trivially, the two trapezoids commute by (e) in the definition of a $\mathscr{G}$-prespectrum, and the two left triangles commute while the remaining triangle homotopy commute by the very definition of a prespectrum. This looks just like Whitehead's diagram (see Definition IX.2.5), except that we haven't mentioned signs. The point is that, to get a pairing in Whitehead's sense, we must use fixed chose isometries $f_{pq} : g(\mathbb{R}^p \oplus \mathbb{R}^q) \to \mathbb{R}^{p+q}$. If $f = f_{p,q+1}$ in our diagram, then $f_{p+1,q}$ may lie in the opposite component from $f_{p,q+1}(1 \oplus d)$ in the space of linear isometries $g(\mathbb{R}^{p+1} \oplus \mathbb{R}^q) \to \mathbb{R}^{p+q+1}$. Of course, our theory requires no such choices, and the linear isometries in the definition of a prespectrum efficiently keep track of all such changes of coordinates.

The interest lies in E_∞ ring spectra, but it is E_∞ spaces and ring prespectra which occur in nature. We next show that the relationships between categories $\mathscr{T}_e$, $\mathscr{P}_e$, and $\mathscr{S}_e$ derived in Section II§1 restrict to give similar relationships between the categories $\mathscr{G}[\mathscr{T}_e]$, $\mathscr{G}[\mathscr{P}_e]$, and $\mathscr{G}[\mathscr{S}_e]$. Clearly we have forgetful functors $\mathscr{G}[\mathscr{P}_e] \to \mathscr{G}[\mathscr{T}_e]$ and $\mathscr{G}[\mathscr{S}_e] \to \mathscr{G}[\mathscr{T}_e]$ defined on objects by $(T, \sigma, \xi) \to (T_0, \xi|_{T_0})$.

Lemma IV.1.4. *The functor* $\Sigma^\infty : \mathscr{T}_e \to \mathscr{P}_e$ *restricts to a functor* $\Sigma^\infty : \mathscr{G}[\mathscr{T}_e] \to \mathscr{G}[\mathscr{P}_e]$, *and there is a natural isomorphism*

$$\mathscr{G}[\mathscr{T}_e](X, T_0) \cong \mathscr{G}[\mathscr{P}_e](\Sigma^\infty X, T), \ X \in \mathscr{G}[\mathscr{T}_e] \ and \ T \in \mathscr{G}[\mathscr{P}_e].$$

PROOF. For $(X, \xi) \in \mathscr{G}[\mathscr{T}_e]$, $g \in \mathscr{G}(j)$, $x_i \in X$, and $v_i \in tV_i$, define

$$\xi_j(g)(x_1 \wedge v_1 \wedge \cdots \wedge x_j \wedge v_j) = \xi_j(g)(x_1 \wedge \cdots \wedge x_j) \wedge tg(v_1 \wedge \cdots \wedge v_j).$$

Then ξ gives $\Sigma^\infty X$ a structure of $\mathscr{G}$-prespectrum, and the remaining verifications are trivial. (For (f), each $\xi_i(g) : X \to X$ must be a homeomorphism.) $\square$

Lemma IV.1.5. $\Sigma^\infty S^0$ *is a* $\mathscr{G}$-*prespectrum, and* $e : \Sigma^\infty S^0 \to T$ *is a morphism of* $\mathscr{G}$-*prespectra for every* $\mathscr{G}$-*prespectrum* T.

PROOF. With $\xi_j(g) = 1$ on $S^0 \wedge \cdots \wedge S^0 = S^0$ for each $g \in \mathscr{G}(j)$, S^0 is a $\mathscr{G}$-space with zero such that $e : S^0 \to X$ is a morphism of $\mathscr{G}$-spaces with zero for all $X \in \mathscr{G}[\mathscr{T}_e]$. The conclusion follows by taking $X = T_0$ and applying the previous lemma. $\square$

Lemma IV.1.6. *The functor $\Omega^\infty\colon \mathscr{P}_e \to \mathscr{S}_e$ restricts to a functor $\Omega^\infty\colon \mathscr{G}[\mathscr{P}_e] \to \mathscr{G}[\mathscr{S}_e]$, and there is a natural isomorphism*

$$\mathscr{G}[\mathscr{P}_e](T, \nu E) \cong \mathscr{G}[\mathscr{S}_e](\Omega^\infty T, E),\ T \in \mathscr{G}[\mathscr{P}_e]\ and\ E \in \mathscr{G}[\mathscr{S}_e].$$

PROOF. By Definition II.1.4, $(\Omega^\infty T)(V) = \varinjlim \Omega^W T(V + W)$, $W \perp V$. Let $(T, \sigma, \xi) \in \mathscr{G}[\mathscr{P}_e]$, $g \in \mathscr{G}(j)$, and $f_i \in \Omega^{w_i} T(V_i + W_i)$ and define a map $\xi_j(g)(f_1 \wedge \cdots \wedge f_j)$ by commutativity of the diagram

$$
\begin{array}{ccc}
tg(W_1 \oplus \cdots \oplus W_j) & \xrightarrow{\ \xi_j(g)(f_1 \wedge \cdots \wedge f_j)\ } & T(g(V_1 \oplus \cdots \oplus V_j) + g(W_1 \oplus \cdots \oplus W_j)) \\
\ \downarrow{\scriptstyle (tg)^{-1}} & & \| \\
t(W_1 \oplus \cdots \oplus W_j) & & Tg((V_1 + W_1) \oplus \cdots \oplus (V_j + W_j)) \\
\| & & \uparrow{\scriptstyle \xi_j(g)} \\
tW_1 \wedge \cdots \wedge tW_j & \xrightarrow{\ f_1 \wedge \cdots \wedge f_j\ } & T(V_1 + W_1) \wedge \cdots \wedge T(V_j + W_j)
\end{array}
$$

$\xi_j(g)\colon (\Omega^\infty T)(V_1) \wedge \cdots \wedge (\Omega^\infty T)(V_j) \to (\Omega^\infty T)g(V_1 \oplus \cdots \oplus V_j)$ is obtained by passage to limits, and these maps are easily verified to make $\Omega^\infty T$ a $\mathscr{G}$-spectrum. With $W_i = \{0\}$ in our diagram, we see that the map $\iota\colon T \to \nu\Omega^\infty T$ of Lemma II.1.5 is a morphism of $\mathscr{G}$-prespectra, and the rest is clear. $\qquad\square$

The previous two lemmas imply the following result

Lemma IV.1.7. $S = \Omega^\infty \Sigma^\infty S^0$ *is a $\mathscr{G}$-spectrum, and $e\colon S \to E$ is a morphism of $\mathscr{G}$-spectra for every $\mathscr{G}$-spectrum E.*

$Q_\infty = \Omega^\infty \Sigma^\infty$ and Lemmas IV.1.4 and IV.1.6 can be composed.

Lemma IV.1.8. *The functor $Q_\infty\colon \mathscr{T}_e \to \mathscr{S}_e$ restricts to a functor $Q_\infty\colon \mathscr{G}[\mathscr{T}_e] \to \mathscr{G}[\mathscr{S}_e]$ and there is a natural isomorphism*

$$\mathscr{G}[\mathscr{T}_e](X, E_0) \cong \mathscr{G}[\mathscr{S}_e](Q_\infty X, E),\ X \in \mathscr{G}[\mathscr{T}_e]\ and\ E \in \mathscr{G}[\mathscr{S}_e].$$

The following immediate consequence of this lemma will be needed in Chapter VII. Recall that $QX = (Q_\infty X)_0$.

Lemma IV.1.9. *The monad (Q, μ, η) in $\mathscr{T}_e$ restricts to a monad in $\mathscr{G}[\mathscr{T}_e]$; for $E \in \mathscr{G}[\mathscr{S}_e]$, the natural map $QE_0 \to E_0$ gives E_0 a structure of Q-algebra in $\mathscr{G}[\mathscr{T}_e]$.*

As we shall see in Chapter VII, the lemma implies that E_0 is an "E_∞ ring space," which is a space with two E_∞ space structures so interrelated that the underlying H-space structures satisfy the distributivity laws up to all possible higher coherence homotopies. Moreover, we shall see that a connective $\mathscr{G}$-spectrum E can be reconstructed (up to homotopy) from the E_∞ ring spectrum E_0.

Lemma IV.1.8 gives the following class of examples. Recall that any infinite loop space is an E_∞ space (Theorem VII.2.1) and any grouplike E_∞ space is an infinite loop space (Proposition VII.3.2).

Example IV.1.10. For any $\mathscr{G}$-space (X, ξ), without zero, construct a $\mathscr{G}$-space (X^+, ξ) with zero by adjoining a disjoint basepoint 0 to X and extending ξ in the evident way. $Q_\infty X^+$ is then a $\mathscr{G}$-spectrum, and the inclusion of X in QX^+ is a morphism of $\mathscr{G}$-spaces. If $\mathscr{G}'$ is any E_∞ operad and $\mathscr{G} = \mathscr{G}' \times \mathscr{L}$, then any $\mathscr{G}'$-space is a $\mathscr{G}$-space via the projection $\mathscr{G} \to \mathscr{G}'$, while the projection $\mathscr{G} \to \mathscr{L}$ allows $\mathscr{G}$ to be used in the present theory. Therefore $Q_\infty X^+$ is an E_∞ ring spectrum for any E_∞ space X.

2. $\mathscr{I}_*$ prefunctors and Thom spectra

As explained in Section I§1, to construct an $\mathscr{L}$-space it is often simplest to first construct an $\mathscr{I}_*$-functor. Analogously, to construct an $\mathscr{L}$-prespectrum, it is often simplest to first construct an $\mathscr{I}_*$-prefunctor.

Definition IV.2.1. An $\mathscr{I}_*$-prefunctor (T, ω, e) is a continuous functor $T \colon \mathscr{I}_* \to \mathscr{T}$ together with a commutative, associative, and continuous natural transformation $\omega \colon T \times T \to T \circ \oplus$ (of functors $\mathscr{I}_* \times \mathscr{I}_* \to \mathscr{T}$) and a continuous natural transformation $e \colon t \to T$ such that

(a) $\omega \colon TV \times TW \to T(V \oplus W)$ factors through $TV \wedge TW$.

(b) The composite $TV \wedge tW \xrightarrow{1 \wedge e} TV \wedge TW \xrightarrow{\omega} T(V \oplus W)$ has adjoint an inclusion with closed image and coincides with the identity map of TV when $W = \{0\}$.

(c) The diagram $\quad$ $\begin{array}{ccc} t(V \oplus W) & \xrightarrow{\ e\ } & T(V \oplus W) \\ \| & & \uparrow{\omega} \\ tV \wedge tW & \xrightarrow{e \wedge e} & TV \wedge TW \end{array}$ $\quad$ is commutative.

A morphism $\Phi \colon (T, \omega, e) \to (T', \omega', e')$ of $\mathscr{I}_*$-prefunctors is a continuous natural transformation $\Phi \colon T \to T'$ such that $\omega'(\Phi \times \Phi) = \Phi\omega$ and $e' = \Phi e$. The $\mathscr{I}_*$-functor t with $\omega \colon tV \times tW \to t(V \oplus W)$ the projection is also an $\mathscr{I}_*$-prefunctor with e the identity. Condition (c) asserts that $e \colon t \to T$ is a morphism of $\mathscr{I}_*$-prefunctors.

Lemma IV.2.2. An $\mathscr{I}_*$-prefunctor (T, ω, e) naturally determines an $\mathscr{L}$-prespectrum (T, σ, ξ).

PROOF. The continuity of T ensures that its restriction to sub inner product spaces of $\mathbb{R}^\infty$ induces a functor $h\mathscr{I}_*(\mathbb{R}^\infty) \to h\mathscr{T}$. Define $\sigma = \omega(1 \wedge e) \colon TV \wedge tW \to T(V + W)$ for orthogonal pairs (V, W) of subspaces of $\mathbb{R}^\infty$. Then (a)–(c) and the associativity of ω ensures that (T, σ) is a prespectrum and that $e \colon \Sigma^\infty S^0 \to T$ is a morphism of prespectra. For $V_i \subset \mathbb{R}^\infty$ and $g \in \mathscr{L}(j)$, define

$$\xi_j(g) \colon TV_1 \wedge \cdots \wedge TV_j \longrightarrow Tg(V_1 \oplus \cdots \oplus V_j)$$

to be the composite

$$TV_1 \wedge \cdots \wedge TV_j \xrightarrow{\ \omega\ } T(V_1 \oplus \cdots \oplus V_j) \xrightarrow{T(g|_{V_1 \oplus \cdots \oplus V_j})} Tg(V_1 \oplus \cdots \oplus V_j).$$

It is straightforward to verify that (T, σ, ξ) is then an $\mathscr{L}$-prespectrum. $\qquad\square$

As pointed out to us by Becker, Kochman, and Schultz, there is a class of $\mathscr{I}_*$-functors which leads via Lemmas IV.1.6 and IV.2.2 to certain of the E_∞ ring spectra $Q_\infty X^+$ of Example IV.1.10.

Example IV.2.3. Let X be an Abelian topological monoid with product ϕ and unit η. Define an $\mathscr{I}_*$-prefunctor (TX, ω, e) by

$$(TX)(V) = X^+ \wedge tV \quad \text{and} \quad (TX)(f) = 1 \wedge tf \text{ for } f \colon V \to V',$$

with ω and e given by the maps

$$\phi^+ \wedge 1 \colon X^+ \wedge tV \wedge X^+ \wedge tW \cong (X \times X)^+ \wedge tV \wedge tW \longrightarrow X^+ \wedge t(V \oplus W)$$

and

$$\eta^+ \wedge 1 \colon tV \cong \{1\}^+ \wedge tV \longrightarrow X^+ \wedge tV.$$

Note that t is recovered as the special case $T\{1\}$.

The $\mathscr{L}$-spectrum determined by t is $Q_\infty S^0$. The derived $\mathscr{L}$-space structure on the zero$^{\text{th}}$ space QS^0 coincides with the $\mathscr{L}$-space structure derived from $\mathscr{I}_*$-functor $\widetilde{F}$ defined in Example I.2.5. This phenomenon generalizes to arbitrary $\mathscr{I}_*$-prefunctors.

Definition IV.2.4. Let (T, ω, e) be an $\mathscr{I}_*$-prefunctor and write M for the $\mathscr{L}$-spectrum derived by application of the functor Ω^∞ to the associated $\mathscr{L}$-prespectrum T. Define an $\mathscr{I}_*$-functor $\widetilde{F}T$ by

$$\widetilde{F}TV = \Omega^V TV \quad \text{and} \quad \widetilde{F}Tf = \Omega^{f^{-1}} Tf \text{ for } f\colon V \to V',$$

with $\omega\colon \widetilde{F}TV \times \widetilde{F}TW \to \widetilde{F}T(V \oplus W)$ given by the composite

$$\Omega^V TV \times \Omega^W TW \xrightarrow{\wedge} \Omega^{V \oplus W}(TV \wedge TW) \xrightarrow{\Omega^{V \oplus W}\omega} \Omega^{V \oplus W} T(V \oplus W).$$

As explained in Definition I.1.6 and Proposition I.1.9, $\widetilde{F}T$ determines an $\mathscr{L}$-space (also denoted by $\widetilde{F}T$) by passage to limits over $V \subset \mathbb{R}^\infty$. A trivial comparison of definitions shows that $\widetilde{F}T = M_0$ as an $\mathscr{L}$-space. There are evident sub $\mathscr{I}_*$-functors FT and SFT of $\widetilde{F}T$ which give rise to the sub $\mathscr{L}$-spaces FM and SFM of M_0.

We next display the Thom spectra as $\mathscr{I}_*$-prefunctors. Recall the discussion of the two-sided geometric bar construction as an $\mathscr{I}_*$-functor from Definitions I.2.1 and I.2.2.

Construction IV.2.5. Let $G \to \widetilde{F}$ be a morphism of monoid-valued $\mathscr{I}_*$-functors. Then G acts from the left by evaluation on the $\mathscr{I}_*$-functor t and from the left and right on the trivial $\mathscr{I}_*$-functor $*$. Let Y be any $\mathscr{I}_*$-functor on which G acts from the right. The map $p\colon B(YV, GV, tV) \to B(YV, GV, *)$ is a quasi-fibration if G is grouplike and a GV-bundle of G is group-valued. p admits a cross-section σ induced from the morphism of GV-space $* \to tV$ and has fibre $\tau\colon tV \to B(YV, GV, tV)$ over the basepoint of $B(YV, GV, *)$. Moreover, p, σ, and τ are all morphisms of $\mathscr{I}_*$-functors. Define an $\mathscr{I}_*$-prefunctor $(T(G; Y), \omega, e)$ by

$$T(G; Y)(V) = B(YV, GV, tV)/B(YV, GV, *),$$

with ω and e induced from the Whitney sum of $B(Y, G, t)$ and from τ. Write $T(G; Y)$ for the associated $\mathscr{L}$-prespectrum and write $M(G; Y)$ for the derived $\mathscr{L}$-spectrum $\Omega^\infty T(G; Y)$. In an evident sense, T and M are functorial on pairs (G, Y). Abbreviate $T(G; *) = TG$ and $M(G; *) = MG$. TG and MG are called the Thom prespectrum and spectrum of G.

Clearly, Me coincides with Q_*S^0. When G is group-valued we could just as well define MG by use of the associated sphere bundles to the principal bundles

$$G(V \oplus V)/e \times GV \longrightarrow G(V \oplus V)/GV \times GV.$$

However, since these bundles are not universal (because their total spaces are not contractible), it seems preferable even in the classical case to use the bar construction.

When $G = F$, p must be replaced by an appropriate fibration $DFV \to BFV$ in order to obtain a universal FV-bundle (spherical fibration with cross-section). Here we could replace TFV by DFV/BFV; the new TFV would again determine an $\mathscr{I}_*$-prefunctor and the old TFV would be deformation retracts of the new ones via deformations which define morphisms of $\mathscr{I}_*$-prefunctors for each parameter value. A similar remark applies to the general case $T(G; Y)$ when G maps to F.

In view of Example I.2.5, all the usual cobordism theories except for PL theory are thus represented by $\mathscr{L}$-spectra. While it may be possible to handle MPL and $MSPL$ in an ad hoc manner based on the triangulation theorem, as BPL and $BSPL$ were handled in Section I§2, it is certainly preferable to treat these within a general framework of axiomatic bundle theory. Such a treatment will be given by the second author in [**64**].

We note one other important example to which our theory does not yet apply, namely the Brown–Peterson spectrum. The point is that our theory requires a good concrete geometric model, not merely a homotopy theoretical construction, and no such model is presently known for BP.

For general Y in Construction IV.2.5, Lashof's treatment [**36**] of the Pontryagin–Thom construction implies that if G is group-valued and maps to O, then $p_*M(G; Y)$ gives the cobordism groups of G-manifolds with a "Y-structure" on their stable normal bundles. In full generality,

when G maps to F, define a G-normal space to be a normal space in the sense of [**63**, 1.1] with a reduction of the structural monoid of its spherical fibration to G [**47**, 10.4]; then $\pi_* M(G; Y)$ gives the cobordism groups of G-normal spaces with a "Y-structure" on their spherical fibrations. An intrinsic bundle or fibration theoretic interpretation of a Y-structure requires an appropriate classification theorem, and general results of this nature are given in [**47**, §11]. When G maps to G', a G'-structure is a G'-trivialization [**47**, 10.3]. When H maps to G, and $H\backslash G$-structure is a reduction of the structural monoid to H [**47**, 10.4]. When $Y = FM$ is as in Definition IV.2.4 and is regarded as a right $\mathscr{I}_*$-functor over G via composition of maps,

$$FMV \times GV \longrightarrow F(tV, TV) \times F(tV, tV) \longrightarrow F(tV, TV) = FMV,$$

a Y-structure is an M-orientation by Section III§2 and Remark IV.3.5 below.

Note that the map of $\mathscr{L}$-prespectra $\iota\colon TG \to \nu\Omega^\infty TG = \nu MG$ defines explicit MG-orientations $\iota\colon TGV \to MGV$ of the universal GV-bundle (via the equivalence of TFV and DFV/BFV when $G = F$). Thus any GV-bundle admits an MG-orientation. The following lemma reflects the fact that a G-normal space (or G-manifold) with an $H\backslash G$-structure admits an FMH-structure (that is, an MH-orientation).

Lemma IV.2.6. *Let $H \to G \to \widetilde{F}$ be morphisms of monoid-valued $\mathscr{I}_*$-functors. Then there is a morphism $j\colon H\backslash G \to \widetilde{F}MH$ of right $\mathscr{I}_*$-functors over G such that j coincides with the given morphism $G \to \widetilde{F}$ when $H = e$ and the following diagram commutes:*

$$
\begin{array}{ccc}
G & \xrightarrow{\ \tau\ } & H\backslash G \\
{\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle j} \\
\widetilde{F} & \xrightarrow{\ e\ } & \widetilde{F}MH
\end{array}
$$

If G maps to F or SF then $H\backslash G$ maps to FMH or $SFMH$.

PROOF. $H\backslash G = B(*, H, G)$ and we have the commutative diagram

$$
\begin{array}{ccccccc}
B(*, HV, GV) \times tV & \cong & B(*, HV, GV \times tV) & \xrightarrow{\ B\lambda\ } & B(*, HV, tV) & \longrightarrow & THV \\
{\scriptstyle \tau\times 1}\big\uparrow & & & & {\scriptstyle \tau}\big\uparrow & \nearrow_{e} & \\
GV \times tV & & \xrightarrow{\hspace{4.5cm}\lambda\hspace{4.5cm}} & & tV & &
\end{array}
\quad ,
$$

where λ is the evaluation map. $j\colon HV\backslash GV \to F(tV, THV) = \widetilde{F}MH$ is defined to be the adjoint of the top composite. $\qquad\square$

As will be discussed and interpreted geometrically in [**66**], the maps $j\colon H\backslash G \to FMH$ induce the bordism J-homomorphism.

We record a number of natural maps of $\mathscr{I}_*$-prefunctors in the following remarks. The same letters will be used for the derived morphisms of $\mathscr{L}$-prespectra and $\mathscr{L}$-spectra. The cobordism interpretations should be clear from the discussion above.

Remark IV.2.7.   (i) For (G, Y) as in Construction IV.2.5, the morphism of $\mathscr{I}_*$-functors $q\colon B(Y, G, t) \to B(*, G, t)$ induces a morphism of $\mathscr{I}_*$-prefunctors $q\colon T(G; Y) \to TG$.

  (ii) In the notation of the previous proof, the maps $B\lambda$ induce a morphism of $\mathscr{I}_*$-functors $\varepsilon = \varepsilon(B\lambda)\colon B(H\backslash G, G, t) \to B(*, H, t)$ which in turn induces a morphism of $\mathscr{I}_*$-prefunctors $\varepsilon\colon T(G; H\backslash G) \to TH$.

 (iii) If G maps to F and if T is an $\mathscr{I}_*$-prefunctor, then the evaluation maps $F(tV, TV) \times tV \to TV$ induce maps $\varepsilon\colon B(FMV, GV, tV) \to TV$ which in turn induce a morphism $\varepsilon\colon T(G, FM) \to T$ of $\mathscr{I}_*$-prefunctors.

The maps of the previous lemma and remarks give considerable information about the structure of $M(G; MG)$.

Remark IV.2.8. For an $\mathscr{L}$-spectrum M derived from an $\mathscr{I}_*$-prefunctor, write

$$M(G;M) = \Omega^\infty T(G;FM) \quad \text{and} \quad M(SG;M) = \Omega^\infty T(SG;SFM).$$

Let $H \to G \to F$ be morphisms of monoid-valued $\mathscr{I}_*$-functors. The following diagrams commute because they already do so on the level of $\mathscr{I}_*$-prefunctors:

$$
\begin{array}{ccc}
& M(G;H\backslash G) & \\
{}^\varepsilon\swarrow \quad \downarrow{\scriptstyle Mj} \quad \searrow{}^q & & \\
MH \qquad MG & \text{and} & \\
{}_\varepsilon\nwarrow \quad \uparrow \quad \nearrow{}_q & & \\
& M(G;MH) &
\end{array}
$$

When $H = G$, $H\backslash G = B(*,G,G)$ is contractible and the upper maps ε and q are isomorphisms in $H\mathscr{S}$ (because the maps ε and q on the prespectrum level are weak homotopy equivalences for each V). We conclude that, in $H\mathscr{S}$, the lower maps q split off a direct factor MG or MSG (via $Mj \circ \varepsilon^{-1}$) such that the restriction to this factor of the lower map ε is the identity.

3. Orientation theory for E_∞ ring spectra

Let $\mathscr{G}$ be an E_∞ operad with given map to $\mathscr{L}$ and consider $\mathscr{L}$-spaces as $\mathscr{G}$-spaces via this map. We shall write BX for the first de-looping of a grouplike $\mathscr{G}$-space X (Section VII§3 or [46]). This is a harmless abuse of notation since BX is equivalent as an infinite loop space to the usual classifying space of X if X happens to be a topological monoid in the category of $\mathscr{G}$-spaces (by Proposition VII.3.5).

Let E be a $\mathscr{G}$-spectrum. By Lemma IV.1.7, $e\colon S \to E$ is a morphism of $\mathscr{G}$-spectra and thus $e\colon F \to FE$ is a morphism of $\mathscr{G}$-spaces. Let R denote the commutative ring $\pi_0 E$.

Let $j\colon G \to F$ be a morphism of monoid-valued $\mathscr{I}_*$-functor, and let j also denote the derived map of $\mathscr{G}$-spaces (Proposition I.1.9 and Definition I.1.6). Write e for any of the composites $GV \to G \xrightarrow{j} F \xrightarrow{e} FE$, $V \subset \mathbb{R}^\infty$. Consider the orientation diagram constructed in Section III§2. It features the classifying space $B(GV;E) = B(FE,GV,*)$ for E-oriented GV-bundles. Here FE is identified as a subspace of $\Omega^V EV$ via the homeomorphism $\tilde{\sigma}\colon E \to \Omega^V EV$ and is a right GV-space by composition of maps. By the definition of a spectrum, Definition II.1.1, the following diagram is commutative if V and W are orthogonal subspaces of $\mathbb{R}^\infty$:

$$
\begin{array}{ccc}
E_0 & \xrightarrow{\tilde{\sigma}} & \Omega^{V+W}E(V+W) \\
{\scriptstyle\tilde{\sigma}}\downarrow & & \| \wr \\
\Omega^V EV & \xrightarrow{\Omega^V\tilde{\sigma}} & \Omega^V\Omega^W E(V+W)
\end{array}
.
$$

Therefore the identification of FE as a subspace of $\Omega^V EV$ is consistent as V varies, and FE inherits a right action by $G = G\mathbb{R}^\infty$ from the right actions by the GV. Moreover, the action $FE \times G \to FE$ is itself a morphism of $\mathscr{G}$-spaces. Indeed, a comparison of (e) of Definition IV.1.1 with Remark I.1.10 shows that this follows from the cancellation $(xg^{-1})(gy) = xy$ for $g\colon V \to V'$, $x \in \Omega^V EV$, and $y \in \Omega^V tV = FV$. Now recall the discussion of the two-sided geometric bar construction as a $\mathscr{G}$-space from Section I§2 (or [46, §3]). Recall too that an Abelian monoid is a $\mathscr{G}$-space for any $\mathscr{G}$ and tat the discretization $d\colon X \to \pi_0 X$ of a $\mathscr{G}$-space is always a map of $\mathscr{G}$-spaces [45, §3]. By passage to limits from the orientation diagrams for $V \subset \mathbb{R}^\infty$, we conclude the following result.

THEOREM IV.3.1. *All spaces are grouplike $\mathcal{G}$-spaces and all maps are $\mathcal{G}$-maps in the stable orientation diagram*

$$
\begin{array}{ccccccc}
SG & \xrightarrow{e} & SFE & \xrightarrow{\tau} & B(SG;E) & \xrightarrow{q} & BSG \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
G & \xrightarrow{e} & FE & \xrightarrow{\tau} & B(G;E) & \xrightarrow{q} & BG \\
\| & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle Bd} & & \| \\
G & \xrightarrow{de} & FR & \xrightarrow{\tau} & B(G;R) & \xrightarrow{q} & BG
\end{array}
$$

This diagram is therefore equivalent to the diagram obtained by application of the zeroth space functor to a diagram of connective spectra in which the rows are fiberings (that is, are equivalent in the stable category to fibration sequences [**48**, XI]). On the level of spaces, the stable E-orientation sequence now extends infinitely in both directions:

$$\cdots \longrightarrow \Omega B(G;E) \longrightarrow G \xrightarrow{e} FE \xrightarrow{\tau} B(G;E) \xrightarrow{q} BG \xrightarrow{Be} BFE \longrightarrow BB(G;E) \longrightarrow \cdots$$

Given $H \to G$, we also have the infinite sequence

$$\cdots \longrightarrow \Omega(G/H) \longrightarrow H \longrightarrow G \xrightarrow{\tau} G/H \xrightarrow{q} BH \longrightarrow BG \longrightarrow B(G/H) \longrightarrow \cdots$$

and a map of this sequence into the E-orientation sequence of H (because the maps of Remark III.2.6 are $\mathcal{G}$-maps when $V = \mathbb{R}^\infty$).

Since $B(G;E)$ is equivalent to the fibre of $Be\colon BG \to BFE$, an easy diagram chase shows that $B(GV;E)$ is equivalent to the fibre of the following composite, which we again denote by Be:

$$BGV \longrightarrow BG \xrightarrow{Bj} BF \xrightarrow{Be} BFE \ .$$

Definition IV.3.2. Let ξ be a GV-bundle classified by $\alpha\colon X \to BGV$. Define $w(\xi;E)$ to be the element $\alpha^*(Be)$ of the group $[X^+, BFE]$. $w(\xi;E)$ is called the E-theory Stiefel–Whitney class of ξ and is the obstruction to its E-orientability. If FE also denotes the (reduced) cohomology theory represented by the spectrum determined by FE, then $w(\xi;E)$ can be regarded as an element of $FE^1(X^+)$.

Previously, the obstruction to E-orientability was studied by analysis of the Atiyah–Hirzebruch spectral sequence. Larry Taylor [**74**] has given a number of results so obtainable, and we are indebted to him for several very helpful conversations. When E is an E_∞ ring spectrum, these results are immediate consequences of the definition and ordinary obstruction theory. To see this, assume that X is a finite dimensional CW-complex. If $w(\xi;E)\colon X \to BFE$ is null homotopic on the $(n-1)$-skeleton X^{n-1}, then we have an obstruction set

$$w_n(\xi;E) \subset H^n(X; \pi_n BFE) = H^n(X; \pi_{n-1} FE)$$

to the existence of a null homotopy of $w(\xi;E)$ on X^n. Clearly $w_1(\xi;E)$ is the usual Stiefel–Whitney obstruction to the R-orientability of ξ of Example III.2.2. For $n > 0$,

$$\pi_n FE = \pi_N SFE = \pi_n(E_0, 1) \cong \pi_n(E_0, 0) = \pi_n E,$$

where the isomorphism is given by translation from the 1-component to the 0-component. Let ψ be a GW-bundle over Y classified by $\beta\colon Y \to BGW$. Then $\xi \wedge \psi$ is classified by the composite of $\alpha \times \beta$ and $\omega\colon BGV \times BGW \to BG(V \oplus W)$. Take V and W to be orthogonal subspaces of $\mathbb{R}^\infty$. Then the diagram

$$
\begin{array}{ccc}
BGV \times BGW & \xrightarrow{\omega} & BG(V+W) \\
& & \downarrow \\
BG \times BG & \xrightarrow{\phi} & BG
\end{array}
$$

is homotopy commutative, where ϕ is the product given by the $\mathcal{G}$-space structure. Since $Be\colon BG \to BFE$ is an H-map, we conclude that $w_n(\xi \wedge \psi; E)$ is defined and contains $w_n(\xi; E) + w_n(\psi; E)$ if $w_n(\xi; E)$ and $w_n(\psi; E)$ are defined. Clearly w_n is natural in X, in the sense that if $f\colon X' \to X$ is a map and if $w_n(\xi; E)$ is defined, then $w_n(f^*\xi; E)$ is defined and contains $f^*w_n(\xi; E)$. Similarly, if $\theta\colon E \to E'$ is a morphism of E_∞ ring spectra, then $w_n(\xi; E')$ contains $\theta_* w_n(\xi; E)$. Of course, the E-orientability of ξ implies its E'-orientability under the much weaker assumption that θ is a morphism of ring spectra in $H\mathcal{S}$.

Since $\pi_* BF$ is finite in each degree and X is finite dimensional, $a(Bj \circ \alpha) = 0$ for some positive integer a. Therefore $aw_n(\xi; E) \subset w_n(a\xi; E) = 0$ if $w_n(\xi; E)$ is defined. Thus, if ξ is R-orientable and if $H^n(X; \pi_{n-1}E)$ is torsion free for $n > 1$, then ξ is E-orientable. For example, if $E = MU$ or if E is a ring spectrum into which MU maps (such as MO, MSO, KU, etc.) and if $H^{2n+1}(X; \mathbb{Z})$ is torsion free for $n \geq 1$, then ξ is E-orientable if it is R-orientable.

When X is finite, $w(\xi; E)$ will be null homotopic if and only if its localizations at all primes, or at and away from a set T of primes, are null homotopic. Let E_T and $E[T^{-1}]$ denote the localization of E at and away from T. Suppose that $G = O$. Then 4ξ admits a symplectic structure, hence is MSp-orientable. If $w_1\xi = 0$, it follows that ξ is $MSp[1/2]$-orientable and thus that ξ is MSp-orientable if and only if it is $(MSp)_2$-orientable. Since the same statement holds for ring spectra into which MSp maps and these include most of the interesting Thom spectra MG and KO and KU, E-orientability of vector bundles is generally only a problem at the prime 2.

By the definition of $w(\xi; E)$ and the third author's result [65] that $(ej)_*\colon \pi_n O \to \pi_n MSp$ is zero for $n \geq 2$, any vector bundle over S^n, $n \geq 3$, is E-orientable for any ring spectrum E into which MSp maps.

Clearly applications like this can be multiplied ad infinitum, and our context gives a conceptually satisfactory and computationally efficient framework for the analysis of E-orientability.

Returning to the stable orientation diagram, we note that if $E = M$ happens to be derived from an $\mathscr{I}_*$-prefunctor, then, by Section I§2 and use of the $\mathscr{I}_*$-functor FM of Definition IV.2.4, this diagram is derived by passage to $\mathscr{L}$-spaces from a commutative diagram of $\mathscr{I}_*$-functors. Analogously, although we cannot construct an $\mathscr{I}_*$-prefunctor like $T(G; FM)$ for a general $\mathcal{G}$-spectrum E, we can construct a Thom prespectrum $T(G; E)$ by direct appeal to Definition IV.1.1.

Construction IV.3.3. Define a $\mathcal{G}$-spectrum $T(G; E)$ as follows.

$$T(G; E)(V) = B(FE, GV, tV)/B(FE, GV, *)$$

and, for $f\colon V \to V'$, $T(G; E)(f)$ is induced from $B(1, Gf, tf)$. $e\colon tV \to T(G; E)(V)$ is induced from $\tau\colon tV \to B(FE, GV, tV)$. For V orthogonal to W, $\sigma\colon T(G; E)(V) \wedge tW \to T(G; E)(V + W)$ is induced from the composite

$$B(FE, GV, tV) \times tW \cong B(FE, GV, tV \times tW) \xrightarrow{B(1, i, \omega)} B(FE, G(V + W), t(V + W)),$$

where $i\colon GV \to G(V + W)$ is the natural inclusion. With these maps, $T(G; E)$ is a unital prespectrum. For $g \in \mathcal{G}(j)$, the maps

$$\xi_j(g)\colon T(G; E)(V_1) \wedge \cdots \wedge T(G; E)(V_j) \longrightarrow T(G; E)(g(v_1 \oplus \cdots \oplus V_j))$$

are induced from the composites

$$B(FE, GV_1, tV_1) \times \cdots \times B(FE, GV_j, tV_j)$$

$$\|\rlap{\!2}$$

$$B((FE)^j, GV_1 \times \cdots \times GV_j, tV_1 \times \cdots \times tV_j)$$

$$\downarrow{\scriptstyle B(\xi_j(g), Gg\circ\omega, tg\circ\omega)}$$

$$B(FE, Gg(V_1 \oplus \cdots \oplus V_j), tg(V_1 \oplus \cdots \oplus V_j)).$$

The verification that these maps are well-defined and give $T(G;E)$ a structure of $\mathcal{G}$-prespectrum is tedious, but quite straightforward. Write $T(SG;E)$ for the $\mathcal{G}$-prespectrum defined similarly but with G and FE replaced by SG and SFE. Write $M(G;E)$ and $M(SG;E)$ for the $\mathcal{G}$-spectra derived by application of the functor Ω^∞.

By [36], [63], and Section III§2, $\pi_*M(G;E)$ gives the cobordism groups of (normally) E-oriented G-manifolds when G maps to O and of E-oriented G-normal spaces in general. In the following remarks we record certain morphisms of $\mathcal{G}$-spectra, show the consistency of our two definitions of $M(G;E)$ when E is derived from an $\mathcal{I}_*$-prefunctor, and discuss the structure of $M(G;E)$ when MG maps to E.

Remark IV.3.4.   (i) The maps $q\colon B(FE,GV,tV) \to B(*,GV,tV)$ induce a morphism $q\colon T(G;E) \to TG$ of $\mathcal{G}$-prespectra and thus a morphism $q\colon M(G;E) \to MG$ of $\mathcal{G}$-spectra.

  (ii) If H maps to G, the maps $Be\colon B(GV,HV,tV) \to B(FE,HV,tV)$ induce a morphism $Te\colon T(H;G) \to T(H;E)$ of $\mathcal{G}$-prespectra and thus a morphism $Me\colon M(H;G) \to M(H;E)$ of $\mathcal{G}$-spectra.

 (iii) The evaluation maps $FE \times tV \xrightarrow{\widetilde{\sigma} \times 1} F(tV,EV) \times tV \to EV$ induce maps $\varepsilon\colon B(FE,GV,tV) \to EV$ which in turn induce a morphism $\varepsilon\colon T(G;E) \to \nu E$ of $\mathcal{G}$-prespectra and thus, by Lemma IV.1.6, a morphism $\varepsilon\colon M(G;E) \to E$ of $\mathcal{G}$-spectra.

Remark IV.3.5. Let $E = M$ be derived from an $\mathcal{I}_*$-prefunctor T. Then $M_0 = \varinjlim \Omega^V TV$ and the natural maps $\Omega^V TV \to M_0$ induce maps $B(FMV,GV,tV) \to B(FM,GV,tV)$ which in turn induce a morphism of $\mathcal{G}$-prespectra from $T(G;FM)$ of Construction IV.2.5 and Lemma IV.2.2 to $T(G;M)$ of Construction IV.3.3. In view of the limits used in the definition of Ω^∞, the induced map of $\mathcal{G}$-spectra is and identification. The maps ε and q on $M(G;M)$ given in Remark IV.2.7 and the previous remarks coincide.

Remark IV.3.6. Let $\gamma\colon MG \to E$ be a morphism of $\mathcal{G}$-spectra. Then the following diagram is commutative:

$$
\begin{array}{ccc}
MG & \xleftarrow{\ \varepsilon\ } & M(G;MG) \\
\big\downarrow{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle M\gamma} \qquad\searrow^{q} \\
& & \qquad\qquad MG \\
E & \xleftarrow{\ \varepsilon\ } & M(G;E) \qquad\nearrow_{q}
\end{array}
$$

We conclude from Remark IV.2.8 that, in $H\mathcal{S}$, $q\colon M(G;E) \to MG$ splits off a direct factor MG (via $M\gamma \circ Mj \circ \varepsilon^{-1}$) such that $\varepsilon\colon M(G;E) \to E$ restricts on this factor to the given map γ. We have an analogous result with G replaced by SG.

On kO-oriented bundle theories

One purpose of this chapter is to lay the foundation for an analysis of Adams' study of the groups $J(X)$ and Sullivan's study of topological bundle theory from the point of view of infinite loop space theory. For this purpose, it is essential to understand which portions of their work depend on the geometric (and representation theory) and which portions follow by purely formal manipulations on the classifying space level. It turns out that substantial parts of their results can be obtained by elementary chases of a pair of large diagrams focusing on the classifying space $B(SF; kO)$ for kO-oriented spherical fibrations and on $BTop$. The functors and natural transformations represented on finite-dimensional CW-complexes by the spaces and maps in these diagrams are easily described, and it is simple to interpret the information obtained on the classifying space level in bundle-theoretic terms.

For the construction and analysis of both diagrams, we shall take the following data as given. (More precise formulations of the data will be given later.)

(1) The Adams operations ψ^r and their values on $KO(S^n)$ [**1**].
(2) The validity of the Adams conjecture [**2, 17, 57, 73**].
(3) The splitting of BO when localized at an odd prime [**6, 53**].

For the first diagram, we shall also take as given

(4) The Atiyah–Bott–Shapiro kO-orientation of $Spin$ bundles and the values of the derived cannibalistic classes ρ^r on $KSpin[1/r](S^n)$ [**13, 3**].

Note that these results do not depend on Adams' last two $J(X)$ papers [**4, 5**]. For the second diagram, we shall also take as given

(5) The Sullivan $kO[1/2]$-orientation of $STop$ bundles, the fact that the induced map $F/Top[1/2] \to BO_\otimes[1/2]$ is an equivalence, and the values of the derived cannibalistic classes θ^r on $KO[1/2, 1/r](S^n)$ [**71, 72**]

After stabilizing the classification theory for oriented bundles and fibrations developed in [**47**], we explain what we mean by an orientation of a stable bundle theory with respect to a cohomology theory in Section V§1. We construct certain general diagrams which relate oriented bundle theories to cohomology operations and to larger bundle theories in Section V§2.

We construct our main diagram in Section V§3. By chasing its localizations, we derived splittings at east prime p of various spaces in the diagram, such as $B(SF; kO)$, $SF/Spin$, and SF, in Section V§4. These splittings, and chases, imply many of Adams' calculations in [**5**]. The splittings of SF and F/O were noted by Sullivan (unpublished), but the recognition of the role played by $B(SF; kO)$ and, following from this, the recognition that the analysis at the prime 2 is formally identical to that at the odd primes appear to be new.

In Section V§5, we prove a version of the main theorem of [**4**] and so recalculate the groups $J(X)$. We also introduce bundle theoretic analogs δ and ε of the d and e invariants studied by Adams in [**5**]. δ gives the obstruction to kO-orientability of stable spherical fibrations (and its form depends on application of Section IV§3 to the E_∞ ring spectrum kO). ε is defined on the group $Q(SF; kO)(X)$ of kO-orientable stable spherical fibrations over X and takes values in a certain group $JSpin_\otimes(X)$. Its restriction to $JSpin(X)$ is an isomorphism. Therefore $JSpin(X)$ is a direct summand of $Q(SF; kO)(X)$; the complementary summand is the group of j-oriented stable

spherical fibrations for a certain spectrum j. This analysis should be regarded as a generalization of that carried out by Adams [**5**] for the case when X is an i-sphere with $i > 2$.

In Section V§6, we construct our second main diagram. It looks just like the first one, and its analysis is exactly the same; only the interpretation changes. Chases of its localizations give Sullivan's splittings of $BTop$ and Top/O at odd primes. These splittings, and chases, imply the odd primary part of Brumfiel's calculations [**15**]. Away from the prime 2, $Q(SF; kO)(X)$ is isomorphic to $JTop(X)$, the δ invariant becomes the obstruction to the existence of a Top-structure on a stable spherical fibration, and the ε-invariant yields Sullivan's analysis of $JTop(X)$.

Very little of this theory depends on the use of infinite loop spaces. However, the machinery developed in this book shows that all spaces in sight are infinite loop spaces. This extra structure is essential to the applications. Characteristic classes for spherical fibrations, kO-oriented spherical fibrations, and topological bundles can only be described, at present, in terms of homology operations, and these operations are invariants of the infinite loop space structure. Thus it is important to know which are the splittings described here are only homotopical and which are as of infinite loop spaces. The problem, then, is to determine which of the maps displayed in our diagrams are infinite loop maps and which parts of the diagrams commute on the infinite loop space level. In Section V§7, by combining results of this book with recent results of Adams and Priddy [**8**] and still more recent results of Madsen, Snaith, and Tornehave [**42**] and Ligaard [**38**], we shall nearly complete the infinite loop analysis of our diagrams.

1. E-orientations of stable bundle theories

For the reader's convenience, we quickly summarize those notations from Chapter III to be used in this chapter. We then establish notations for stable bundle theories and explain what we mean by an orientation of such a theory with respect to a commutative ring spectrum E.

Let $j\colon G \to F$ be a morphism of grouplike monoid-valued $\mathscr{I}_*$-functors (Definitions I.1.8 and I.2.1 and Example I.2.5) and let V and W be orthogonal finite-dimensional sub inner product spaces of $\mathbb{R}^\infty$. BGV and $B(GV; E)$ classify GV-bundles and E-oriented GV-bundles over CW-complexes, and $\omega\colon BGV \times BGW \to BG(V + W)$ and $B(\omega, \phi)\colon B(GV; E) \times B(GW; E) \to B(G(V + W); E)$ induce the (external) fibrewise smash product and (internal) Whitney sum (Section III§1, Theorem III.2.1, and Remark III.2.8). There is an explicit quasi-fibration sequence

$$GV \xrightarrow{\ e\ } FE \xrightarrow{\ \tau\ } B(GV; E) \xrightarrow{\ q\ } BGV \ ,$$

the bundle-theoretic interpretation of which is given in Remark III.2.5. If $i\colon H \to G$ is a morphism of grouplike monoid-valued $\mathscr{I}_*$-functors, there is another explicit quasi-fibration sequence

$$GV \xrightarrow{\ \tau\ } GV/HV \xrightarrow{\ q\ } BHV \xrightarrow{\ Bi\ } BGV$$

and a map $Be\colon GV/HV \to B(HV; E)$, interpreted in Remark III.2.6, such that $qBe = q$. The maps τ and q of the two quasi-fibration sequences above are defined in the same way in terms of the bar construction and have analogous interpretations in terms of transformations to and from bundles with additional structure, hence the duplicative notation. If $\zeta\colon E \to E'$ is a map of ring spectra, there is a map $B\zeta\colon B(GV; E) \to B(GV; E')$, interpreted in Remark III.2.9, such that $qB\zeta = q$. We write SG instead of G when all bundles are given with a canonical integral orientation, and then all E-orientations are required to be consistent with the preassigned integral orientation.

Write G, G/H, BG, and $B(G; E)$ for the spaces obtained by passage to limits over $V \subset \mathbb{R}^\infty$. The first three are infinite loop spaces (by Chapter I) and the last is at least a grouplike H-space (by Remark III.2.8) and is an infinite loop space if E is an E_∞ ring spectrum (by Theorem IV.3.1).

Henceforward, restrict attention to connected finite-dimensional CW-complexes X as base spaces of GV-bundles. GV and GV' bundles ξ and ξ' over X are said to be stably equivalent if $\xi \oplus \varepsilon W$ is equivalent to $\xi' \oplus \omega W'$ for some $W \perp V$ and $W' \perp V'$ such that $V + W = V' + W'$.

Write $\{\xi\}$ for the stable equivalence class of ξ and call $\{\xi\}$ a stable G-bundle. Let $\widetilde{K}G(X)$ denote the set of stable G-bundles over X. Then $\widetilde{K}G(X)$ is classified by BG, and the image of $[X^+, BGV]$ in $[X^+, BG]$ depends only on the dimension of V. The product on BG induces the external and internal operations $\wedge$ and $\oplus$ on stable G-bundles described in terms of the fibrewise smash product and Whitney sum by $\{\xi\} \wedge \{\psi\} = \{\xi \wedge \psi\}$ and $\{\xi\} \oplus \{\psi\} \{\xi \oplus \psi\}$, where ξ and ψ are representative GV and GW bundles with $V \perp W$.

Let $KG(X)$ denote the Grothendieck group constructed from the $G(n)$-bundles for $n \geq 0$ (or, with the additional relations $\varepsilon V = \varepsilon V'$ when $\dim V = \dim V'$, from the GV-bundles for $V \subset \mathbb{R}^\infty$). We have identifications

$$\widetilde{K}G(X) = [X^+, BG] = [X^+, BG \times \{0\}] \subset [X^+, BG \times \mathbb{Z}] = KG(X).$$

Let $JG(X)$ denote the image of $\widetilde{K}G(X)$ in $\widetilde{K}F(X)$ under neglect of G-reduction (or, equivalently if G is group-valued, passage to fibre homotopy equivalence). Thus

$$JG(X) = (Bj)_*[X^+, BG] \subset [X^+, BF], \quad Bj \colon BG \to BF.$$

Adams writes $\widetilde{J}(X)$ for $JO(X)$ and $J(X)$ for $JO(X) \times \mathbb{Z} \subset KF(X)$; his notation is more logical, but less convenient since $JO(X) \times \mathbb{Z}$ has no geometric ring structure and is therefore uninteresting. Of course, $J = j_* \colon \pi_* SO \to \pi_* SF = \pi_*^S$ is the classical J-homomorphism, where π_*^S denotes the stable stems (i.e., stable homotopy groups of spheres).

E-oriented GV and GV'-bundles (ξ, μ) and (ξ', μ') over X are said to be stably equivalent if $(\xi \oplus \varepsilon W, \mu \oplus \mu_0 W)$ is equivalent to $(\xi' \oplus \varepsilon W', \mu' \oplus \mu_0 W')$ for some $W \perp V$ and $W' \perp V'$ such that $V + W = V' + W'$. Write $\{\xi, \mu\}$ for the stable equivalence class of (ξ, μ) and call $\{\xi, \mu\}$ an E-oriented stable G-bundle. Let $\widetilde{K}(G; E)(X)$ denote the set of E-oriented stable G-bundles over X. $\widetilde{K}(G; E)(X)$ is classified by $B(G; E)$, the image of $[X^+, B(GV; E)]$ in $[X^+, B(G; E)]$ depends only on the dimension of V, and the product on $B(G; E)$ induces the external and internal operations $\wedge$ and $\oplus$ on E-oriented stable G-bundles given by the fibrewise smash product and Whitney sum.

A Grothendieck group $K(G; E)(X)$ can be defined, but is uninteresting. Let $Q(G; E)(X)$ denote the image of $\widetilde{K}(G; E)(X)$ in $\widetilde{K}G(X)$ under neglect of orientation. Thus

$$Q(G; E)(X) = q_*[X^+, B(G; E)] \subset [X^+, BG], \quad q \colon B(G; E) \longrightarrow BG.$$

Definition V.1.1. An E-orientation of G is an H-map $g \colon BG \to B(G; E)$ such that qg is homotopic to the identity map.

Given g, its composite with $Bj \colon B(G; E) \to B(F; E)$ will again be denote by g and the following diagram will be homotopy commutative:

$$
\begin{array}{ccc}
BG & \xrightarrow{\ \ Bj\ \ } & BF \\
& g \searrow \quad \nearrow q & \\
& B(F; E) &
\end{array}
$$

In particular, $JG(X) \subset Q(F; E)(X)$ if G admits an E-orientation.

The proof that $B(GV; E)$ classifies E-oriented GV-bundles gives an explicit universal E-oriented GV-bundle (π, θ); in an evident sense, the (π, θ) are compatible as V varies. Let $\{\xi\} \in \widetilde{K}GX$ be classified by $\alpha \colon X \to BG$. If $g\alpha$ factors (up to homotopy) through $B(GV; E)$, as necessarily holds for some V by the finite dimensionality of X, then the element $\{(g\alpha)^*(\pi), (g\alpha)^*(\theta)\}$ of $\widetilde{K}(G; E)(X)$ is independent of the choice of V and projects to $\{\xi\}$ in $\widetilde{K}G(X)$. If we write $\mu(g)$ for all orientations so determined by g, then the requirement that g be an H-map ensures the validity of the product formula

$$\{\xi, \mu(g)\} \wedge \{\psi, \mu(g)\} \equiv \{\xi \wedge \psi, \mu(g) \wedge \mu(g)\} = \{\xi \wedge \psi, \mu(g)\}$$

in $\widetilde{K}(G; E)(X \times Y)$ for all $\{\xi\} \in \widetilde{K}G(X)$ and $\{\psi\} \in \widetilde{K}G(Y)$.

We are particularly interested in the fine structure preserved by infinite loop maps, by which we understand maps in $H\mathcal{T}$ which are equivalent to the zero$^{\text{th}}$ maps of morphisms in $H\mathcal{S}$ (see Section II§2). The reader who does not share our interest may skip to the next section. The discussion there will proceed on two levels, one based on E-orientations and the other based on the following notion.

Definition V.1.2. Let E be an E_∞ ring spectrum. An E-orientation $g\colon BG \to B(G;E)$ is said to be perfect if g is an infinite loop map and if $qg = 1$ as infinite loop maps.

As will be discussed in Section V§7, there now exist homotopical proofs in several important cases of the somewhat weaker assertion that the composite of g and the natural map $B(G;E) \to B(F;E)$ is an infinite loop map whose composite with $q\colon B(F;E) \to BF$ is equal as an infinite loop map to $Bj\colon BG \to BF$. An easy Barratt–Puppe sequence argument in $\mathcal{S}$ (justified by [**48**, I]) shows that G admits a perfect E-orientation if and only if $e\colon G \to FE$ is the trivial infinite loop map.

When an E-orientation g of G is the limit over $V \subset \mathbb{R}^\infty$ of maps $gV\colon BGV \to B(GV;E)$, the gV induce maps $TGV \to T(GV;E) \xrightarrow{\varepsilon} EV$. In practice, these maps define a morphism of prespectra $TG \to \nu E$ and therefore induce a morphism $\gamma\colon MG \to E$ of spectra. On the coordinatized level, appropriate compatibility of the $g\mathbb{R}^n$, which destabilizes the statement that g is an H-map, will ensure that γ is a morphism of ring spectra in Whitehead's sense and therefore, as explained in and after Proposition II.3.4, that γ is a morphism of (weak) ring spectra in $H\mathcal{S}$. The verification that γ is actually a morphism of E_∞ ring spectra lies very much deeper and has not yet been carried out in the interesting cases.

Conversely, let $\gamma\colon MG \to E$ be a morphism of ring spectra and consider the following homotopy commutative diagram:

$$
\begin{array}{ccccc}
B(GV\backslash GV, GV, *) & \xrightarrow{\ Bj\ } & B(FMGV, GV, *) & \xrightarrow{\ B\gamma\ } & B(FE, GV, *) \\
\Big\downarrow{\scriptstyle\varepsilon} & & {\scriptstyle gV} & & \Big\downarrow{\scriptstyle q} \\
BGV & = & = & = & BGV
\end{array}
$$

Here $j\colon GV\backslash GV \to FMGV$ is defined in Lemma IV.2.6. ε is a homotopy equivalence (if $GV \in \mathscr{W}$), and we define $gV = B\gamma \circ Bj \circ \varepsilon^{-1}$ for some chosen homotopy inverse ε^{-1}. Comparison of Lemma IV.2.6 with the proof of [**47**, 11.1] shows that gV classifies the E-oriented GV-bundle (π, μ), where $\pi\colon DGV \to BGV$ is the universal GV-bundle and μ is its E-orientation induced via γ from the MG-orientation $\iota\colon TGV \to MGV$. The gV define an E-orientation g of G by passage to limits over V, and Remark IV.3.6 (with $\mathscr{G}$-structure ignored) shows how to recover γ from the diagram of Thom spectra over the displayed diagram of classifying spaces. By Lemma IV.2.6, $e\colon G \to FE$ factors through $G\backslash G$. Since $G\backslash G$ is contractible through maps of $\mathscr{L}$-spaces, by [**45**, 9.9 and 12.2], e is the trivial infinite loop map and g is a perfect E-orientation of G when γ is a morphism of $\mathscr{G}$-spectra.

2. Cannibalistic classes and the comparison diagram

We construct some key commutative diagrams and record a few (presumably well-known) technical lemmas here.

We write ϕ for the product on all (multiplicative) H-spaces in sight, and all H-spaces in sight will have a homotopy inverse map χ. Of course, ϕ and χ induce addition and the additive inverse on homotopy groups. Since the stable category is additive, ϕ and χ on infinite loop spaces are infinite loop maps. For an H-space Y and two maps $f, g\colon X \to Y$, define $f/g = \phi(f \times \chi g)\Delta$. (We shall write $f - g$ instead of f/g when we choose to think of ϕ as additive.) f/g is an H-map if f and g are H-maps and Y is homotopy commutative (as will always be the case). If X and Y

are infinite loop spaces and f and g are infinite loop maps, then f/g is an infinite loop map; in particular $1/1 = \phi(1 \times \chi)\Delta$ is the trivial infinite loop map.

We agree to abbreviate weak homotopy equivalence to equivalence. By an equivalence of infinite loop spaces, we understand the zero$^{\text{th}}$ map of a weak homotopy equivalence of spectra. The following result is an immediate consequence of the quasifibration sequence

$$ G \xrightarrow{e} FE \xrightarrow{\tau} B(G; E) \xrightarrow{q} BG \ . $$

Lemma V.2.1. *Let g be an E-orientation of G. Then the composite*

$$ FE \times BG \xrightarrow{\tau \times g} B(G; E) \times B(G; E) \xrightarrow{\phi} B(G; E) $$

is an equivalence of H-spaces. If g is perfect, then $\phi(\tau \times g)$ is an equivalence of infinite loop spaces.

Our main concern in this section is with the comparison of different E-orientations of the same underlying stable G-bundle. Thus let $\{\xi, \mu\}$ and $\{\xi, \nu\}$ be E-oriented stable G-bundles over X classified by $\overline{\alpha}$ and $\overline{\beta}$. Since $q\overline{\alpha} \simeq q\overline{\beta} \colon X \to BG$ and since $q \colon B(G; E) \to BG$ commutes up to homotopy with ϕ and χ,

$$ q(\overline{\alpha}/\overline{\beta}) = q\varphi(\overline{\alpha} \times \chi\overline{\beta})\Delta \simeq \phi(q\overline{\alpha} \times \chi q\overline{\beta})\Delta \simeq \phi(1 \times \chi)\Delta q\overline{\alpha} \simeq *. $$

Since $\tau \colon FE \to B(G; E)$ is canonically equivalent to the fibre of q, this null homotopy determines a map $\delta \colon X \to FE$ such that $\tau\delta \simeq \overline{\alpha}/\overline{\beta}$. Clearly $\phi(\tau\delta \times \overline{\beta})\Delta \simeq \overline{\alpha}$. Since $\tau\delta$ classifies $\{\varepsilon_0, \delta\}$, where the unit $\delta \in E^0(X^+)$ is regarded as an orientation of the trivial $G(0)$-bundle $\varepsilon_0 \colon X \times S^0 \to X$ (with Thom complex X^+), and since, by the explicit definition of $\delta \oplus \nu$ in [Remark III.1.5](#),

$$ \{\varepsilon_0, \delta\} \oplus \{\xi, \nu\} = \{\varepsilon_0 \oplus \xi, \delta \oplus \nu\} = \{\xi, \delta \smile \nu\} \,, $$

we conclude that $\{\xi, \mu\} = \{\xi, \delta \smile \nu\}$.

In our application of this difference construction, we shall be given a classifying space Y for some class of bundles with additional structure and we shall be given maps $a, b \colon Y \to B(G; E)$ such that $qa \simeq qb$. The classifying maps $\overline{\alpha}$ and $\overline{\beta}$ above will be $a\gamma$ and $b\gamma$ for a classifying map $\gamma \colon X \to Y$ and δ will be $d\gamma$ for a map $d \colon Y \to FE$ such that $\tau d \simeq a/b$. Note that the null homotopy $q(a/b) \simeq *$, hence also d and the homotopy $\tau d \simeq a/b$, are explicitly and canonically determined by the homotopies

$$ qa \simeq qb, \quad q\phi \simeq \phi(q \times q), \quad q\chi \simeq \chi q, \quad \text{and} \quad \phi(1 \times \chi)\Delta \simeq *. $$

If a and b are H-maps, then so is d. If E is an E_∞ ring spectrum, a and b are infinite loop maps, and $qa = qb$ as infinite loop maps, then d is an infinite loop map and $\tau d = a/b$ as infinite loop maps.

The theory of cannibalistic classes fits nicely into this framework. Let $\psi \colon E \to E$ be a map of ring spectra. Then $qB\psi = q$, $B\psi \colon B(G; E) \to B(G; E)$, and there results a canonical H-map $c(\psi) \colon B(G; E) \to FE$ such that $\tau c(\psi) \simeq B\psi/1$; if $B\psi$ is an infinite loop map, then so is $c(\psi)$. We call $c(\psi)$ the universal cannibalistic class determined by ψ. If $\{\xi, \mu\}$ is an E-oriented stable G-bundle over X classified by $\overline{\alpha}$, write

$$ c(\psi) = c(\psi)\{\xi, \mu\} = \overline{\alpha}^* c(\psi) \in E^0(X^+). $$

Define ψ on $\widetilde{K}(G; E)(X)$ by $\psi\{\xi, \mu\} = \{\xi, \psi\mu\}$ and note that $\psi\{\xi, \mu\}$ is classified by $B\psi \circ \overline{\alpha}$. The discussion above shows that

$$ \psi\{\xi, \mu\} = \{\xi, c(\psi) \smile \mu\} \in \widetilde{K}(G; E)(X). $$

Of course, given an E-orientation g of G, we can define cannibalistic classes $\rho = \rho\{\xi\}$ for stable G-bundles $\{\xi\}$ by $\rho\{\xi\} = c(\psi)\{\xi, \mu(g)\}$. The fact that these classes are represented by the composite $c(\psi)g$ on the universal level will play an essential role in our theory. We note next that $\psi/1$ also factors through $c(\psi)$.

Proposition V.2.2. *The following diagram is homotopy commutative:*

$$
\begin{array}{ccc}
FE & \xrightarrow{\ \tau\ } & B(G;E) \\
{\scriptstyle \psi/1}\downarrow & {\scriptstyle c(\psi)}\nearrow & \downarrow{\scriptstyle B\psi/1} \\
FE & \xrightarrow{\ \tau\ } & B(G;E)
\end{array}
$$

If E is an E_∞ ring spectrum and ψ on FE and $B\psi$ on $B(G;E)$ are infinite loop maps, then the diagram determines a corresponding diagram on the level of spectra.

PROOF. $B\psi/1 \simeq \tau \circ c(\psi)$ by the definition of $c(\psi)$, and $(B\psi/1)\tau \simeq \tau(\psi/1)$ since $B\psi \circ \tau = \tau \circ \psi$. By construction, $c(\psi)$ is natural in G. When $G = e$ is trivial, $\tau = 1$ and $\psi = B\psi$ on $FE = B(e;E)$, hence $\psi/1 \simeq c(\psi)$. Now $c(\psi) \circ \tau = \psi/1$ follows by an obvious diagram chase. The last statement holds by the general observations above. □

The main interest often lies not in an E-oriented bundle theory but in its relationship to a larger bundle theory. Thus assume given morphisms $H \xrightarrow{i} G \xrightarrow{j} F$ of monoid-valued $\mathscr{I}_*$-functors and assume given an E-orientation g of H. We also write g for its composite with $Bi\colon B(H;E) \to B(G;E)$. There are now two natural E-orientations of G-trivialized stable H-bundles in sight, namely those given by the maps

$$
G/H \xrightarrow{q} BH \xrightarrow{g} B(H;E) \quad \text{and} \quad G/H \xrightarrow{Be} B(H;E).
$$

Their quotient H-map gq/Be factors as τf, where $f\colon G/H \to FE$ is an H-map by virtue of Lemma V.2.1 and is an infinite loop map if g is perfect. With these notations, we have the following result.

Proposition V.2.3. *The first tree squares of the following "comparison diagram" are homotopy commutative:*

$$
\begin{array}{ccccccccc}
G & \xrightarrow{\ \tau\ } & G/H & \xrightarrow{\ q\ } & BG & \xrightarrow{\ Bi\ } & BG & \xrightarrow{\ B\tau\ } & B(G/H) \longrightarrow \cdots \\
{\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \| & & \downarrow{\scriptstyle Bf} \\
G & \xrightarrow{\ e\ } & FE & \xrightarrow{\ \tau\ } & B(G;E) & \xrightarrow{\ q\ } & BG & \xrightarrow{\ Be\ } & BFE \longrightarrow \cdots
\end{array}
$$

If g is perfect, the diagram extends infinitely to the right and determines a corresponding diagram on the level of spectra.

PROOF. By Remark III.2.6 and the fact that $q\tau \simeq *$, we have

$$
\tau f\tau \simeq (\phi gq \times \chi Be)\Delta\tau \simeq \chi(Be)\tau \simeq \chi\tau e \simeq \tau\chi e \simeq \tau e\chi,
$$

where the first map τ takes FE to $B(H;E)$ and is thus the injection of a factor by Lemma V.2.1. Therefore $f\tau \simeq e\chi$. We have the commutative diagram

$$
\begin{array}{ccc}
G/H & \xrightarrow{\quad Be \quad} & B(H;E) \\
{\scriptstyle Bi}\downarrow & {\scriptstyle \tau}\nearrow & \downarrow{\scriptstyle Bi} \\
 & FE & \\
 & {\scriptstyle \tau}\searrow & \\
G/G & \xrightarrow{\quad Be \quad} & B(G;E)
\end{array}
$$

in which G/G is contractible through infinite loop maps. Thus $BiBe$ is the trivial infinite loop map and (since Bi commutes with ϕ and χ) we have

$$
\tau f = (Bi)\tau f = (Bi)\phi(gq \times \chi Be)\Delta \simeq (Bi)gq = gq.
$$

The third square homotopy commutes by the definition of an orientation. □

The reader familiar with Barratt–Puppe sequences will wonder why the sign given by χ appears. If one writes down explicit equivalences of the two rows with honest fibration sequences, starting from BG and working left, one produces two homotopy equivalences $G \to \Omega BG$. These turn out to differ by χ. Of course, given g, Barratt–Puppe sequence arguments (e.g. [**48**, I§2]) produces a map f, not uniquely determined, such that the two left squares homotopy commute. Conversely, given Bf such that the right square homotopy commutes, t here exists g such that the rest of the diagram homotopy commutes. These statements remain valid with H and G replaced by HV and GV, V finite dimensional, in which case the explicit construction of f fails for lack of an H-space structure on $B(GV; E)$.

We shall need some observations concerning localizations of the comparison diagram at a set of primes T. We restrict to the integrally oriented case in order to deal with connected spaces.

The localization E_T is again a commutative ring spectrum, and $SF(E_T) \simeq (SFE)_T$. I do not know if localizations of E_∞ ring spectra are E_∞ ring spectra, but any infinite loop space information derived from the E_∞ structure is preserved under localizations. We write λ generically for localization at T.

Lemma V.2.4. *For any G, the following composite is a localization at T:*

$$B(SG; E) \xrightarrow{B\lambda} B(SG; E_T) \xrightarrow{\lambda} B(SG; E_T)_T.$$

When $G = F$, $BSF \simeq (BSF)_T \times BSF[T^{-1}]$ and the map

$$(\lambda, q) \colon B(SF; E_T) \longrightarrow B(SF; E)_T \times BSF[T^{-1}]$$

is an equivalence (of infinite loop spaces if E is an E_∞ ring spectrum).

PROOF. In view of the following homotopy commutative diagram, this is immediate from the fact that localization preserves fibrations of connected H-spaces:

$$
\begin{array}{ccccccccc}
SG & \xrightarrow{e} & SFE & \xrightarrow{\tau} & B(SG; E) & \xrightarrow{q} & BSG & \xrightarrow{Be} & BSFE \\
\| & & \downarrow{\lambda} & & \downarrow{B\lambda} & & \| & & \downarrow{\lambda} \\
SG & \xrightarrow{e} & SFE_T & \xrightarrow{\tau} & B(SG; E_T) & \xrightarrow{q} & BSG & \xrightarrow{Be} & BSFE_T \\
\downarrow{\lambda} & & \| & & \downarrow{\lambda} & & \downarrow{\lambda} & & \| \\
SG_T & \xrightarrow{e_T} & SFE_T & \xrightarrow{\tau_T} & B(SG; E_T)_T & \xrightarrow{q_T} & BSG_T & \xrightarrow{(Be)_T} & BSFE_T
\end{array}
$$

$\square$

When $G = F$ in the comparison diagram (in which case we rename $H = G$), f can sometimes be intrinsically characterized in terms of g.

Lemma V.2.5. *Let g be an E_T-orientation of G and assume that the following two conditions hold.*

   (i) $H_*(SF/SG)$, $H_*(SFE)$, and $\pi_*(SFE)$ have no T-torsion.
   (ii) $H_*(SF/SG)$ and $H_*(SFE; \mathbb{Q})$ are of finite type (over $\mathbb{Z}$ and $\mathbb{Q}$, respectively).

Then $f \colon SF/SG \to SFE_T$ is the unique H-map such that the second square of the comparison diagram homotopy commutes.

PROOF. Given another such H-map f', f/f' factors through SF and therefore induces the zero map on homotopy. Thus $f_* = f'_*$ on homotopy. As pointed to me by Frank Adams, the following pair of lemmas complete the proof. $\square$

Lemma V.2.6. *Let X and Y be connected homotopy associative H-spaces, with $H_*(X; \mathbb{Q})$ of finite type. If two H-maps $X \to Y$ induce the same homomorphism $\pi_*X \otimes \mathbb{Q} \to \pi_*Y \otimes \mathbb{Q}$, then they induce the same homomorphism $H_*(X; \mathbb{Q}) \to H_*(Y; \mathbb{Q})$.*

PROOF. By Milnor and Moore [**50**, Appendix], the Hurewicz homomorphism $h\colon \pi_*X \to H_*X$ induces a monomorphism upon tensoring with $\mathbb{Q}$, and the image of this monomorphism generates $H_*(X;\mathbb{Q})$ as an algebra. $\qquad\square$

Lemma V.2.7. *Let X be a connected CW-complex and let Y be a connected homotopy associative T-local H-space. Assume that the following two conditions hold.*

    (i) H_*X, H_*Y, and π_*Y have no T-torsion.
    (ii) H_*X and $H_*(Y;\mathbb{Q})$ are of finite type (over $\mathbb{Z}$ and $\mathbb{Q}$, respectively).

Then two maps $f, f'\colon X \to Y$ are homotopic if they induce the same homomorphism $H_*(X;\mathbb{Q}) \to H_*(Y;\mathbb{Q})$.

PROOF. The hypotheses imply that f and f' induce the same homomorphism on integral homology. Suppose that f and f' are homotopic on the $(n-1)$-skeleton of X. If $k \in H^n(X;\pi_nY)$ is the obstruction to the extension of the restriction to the $(n-2)$-skeleton of a given homotopy, then

$$h\langle k, x\rangle = f_*(x) - f'_*(x) = 0 \text{ for } x \in H_*X,$$

where $\langle,\rangle\colon H^n(X;\pi_nY) \otimes H_nX \to \pi_nY$ is the Kronecker product (the first equality holding by explicit chain level calculation from the definitions). Since $h\colon \pi_*Y \to H_*Y$ is a monomorphism, $\langle k, x\rangle = 0$. Since $\mathrm{Ext}(H_{n-1}X, \pi_nY) = 0$ (because H_*X is of finite type, H_*X and π_*Y have no T-torsion, and π_*Y is a $\mathbb{Z}_T$-module), $k = 0$ by the universal coefficient theorem. (See Lemma VIII.1.1 for a simpler proof when $Y = \Omega Z$.) $\qquad\square$

The following analog of Lemma V.2.7 was also pointed out to me by Frank Adams.

Lemma V.2.8. *Let X and Y be spaces of the homotopy type of BSO_T, where T is any set of primes. Then two maps $f, f'\colon X \to Y$ are homotopic if they induce the same homomorphism $H_*(X;\mathbb{Q}) \to H_*(Y;\mathbb{Q})$.*

PROOF. Let $A = \varinjlim BT(n)$, where $T(n)$ is a maximal torus in $SO(n)$, and let $i\colon A \to BSO$ be the evident inclusion. Consider the diagram

$$
\begin{array}{ccc}
H^{**}(BSO;\mathbb{Q}) & \xrightarrow{\;i^{**}\;} & H^{**}(A;\mathbb{Q}) \\
{\scriptstyle\mathrm{ch}}\big\uparrow & & {\scriptstyle\mathrm{ch}}\big\uparrow \\
\widetilde{KO}(BSO) & \xrightarrow{\;c\;} \widetilde{KU}(BSO) & \xrightarrow{\;i^*\;} \widetilde{KU}(A)
\end{array}
$$

Clearly i^{**}, and ch on $\widetilde{KU}(A)$, are monomorphisms. Thus ch on $\widetilde{KU}(BSO)$ is a monomorphism. By Anderson [**9**, p. 38] or [**14**], c is also a monomorphism (in fact an isomorphism). These statements remain true after localization of BSO and the representing spaces BO and BU at T. It follows that f and f' induce the same map of localized real K-theory and therefore, since $BO \simeq BSO \times BO(1)$, that $f^* = (f')^*\colon [Y, Y] \to [X, Y]$. $\qquad\square$

In Chapter VIII, we shall use the fact that this result remains valid, by the same proof, for completions at T. The following observation explains why rational information determines the behavior with respect to self-maps of the 2-torsion in π_*BSO_T.

Lemma V.2.9. *Let $f\colon BSO_T \to BSO_T$ be a map, where $2 \in T$. Let $a_j \in \mathbb{Z}_T$ be such that $f_*(x) = a_j x$ for all $x \in \pi_{4j}BSO_T \cong \mathbb{Z}_T$.*

    (i) If $0 \neq y \in \pi_2 BSO_T$, then $f_*(y) = a_1 y$.
    (ii) If $0 \neq y \in \pi_{8j+k}BSO_T$, $j \geq 1$ and $k = 1$ or $k = 2$, then $f_*(y) = a_{2j}y$.

In both statements, the coefficients are understood to be reduced mod 2.

PROOF. For (i), let $p_n \in H_n(BSO; \mathbb{Z}_2)$ be the unique non-zero primitive element and recall that
$$\mathrm{Sq}^1_* p_3 = p_2, \quad \mathrm{Sq}^2_* p_5 = p_3, \quad \text{and} \quad \mathrm{Sq}^1_* p_5 = p_4.$$
Clearly $f_*(y) = 0$ if and only if $f_*(p_2) = 0$, and the displayed equations show that $f_*(p_2) = 0$ if and only if $f_*(p_4) = 0$. By an obvious argument with the cover $BSpin_T \to BSpin_T$ of α, $f_*(p_4) = 0$ and only if $a_1 \equiv 0 \bmod 2$. For (ii), simply recall that if $x \colon S^{8j} \to BSO_T$ generates $\pi_{8j} BSO_T$, then $y = x \circ \eta$ or $y = x \circ \eta^2$, where $\eta^k \colon S^{8j+k} \to S^{8j}$ is the non-trivial map. $\qquad\square$

Finally, the following result, which I learned from Anderson and Snaith, implies that Lemma V.2.8 remains true when $X \simeq BSpin_T$ and $Y \simeq BSO_T$.

Lemma V.2.10. *The natural map $\pi \colon BSpin \to BSO$ induces isomorphisms on real and complex K-theory. Therefore $\pi^* \colon [BSO, BSO] \to [BSpin, BSO]$ is an isomorphism.*

PROOF. There are no phantom maps here (since $KU^{-1}(BG) = 0$ and $KO^{-1}(BG)$ is finite for a compact Lie group G), and it suffices to consider the inverse systems of complete (real and complex) representation rings of $Spin(n)$ and $SO(n)$. We may as well consider only odd n, where
$$RO(SO(2m+1)) = R(SO(2m+1)) = P\{\lambda_1, \ldots, \lambda_m\}$$
maps surjectively to $R(SO(2m-1))$ and injectively to
$$RO(Spin(2m+1)) \subset R(Spin(2m+1)) = P\{\lambda_1, \ldots, \lambda_m, \Delta_m\}.$$
Since Δ_m restricts to $2\Delta_{m-1}$ in $R(Spin(2m-1))$ by a check of characters, the exceptional spinor representations make no contribution to the inverse limit. $\qquad\square$

3. The kO-orientation of $Spin$ and the J-theory diagram

The main examples involve K-theory. I do not know if there exist E_∞ ring spectra which represent KO^* and KU^*, but explicit E_∞ ring spectra kO and kU which represent the associated connective theories are constructed in Chapter VII. Write $BU_\otimes = SFkU$ and $BO_\otimes = SFkO$; informally, these infinite loop spaces are the 1-components of $BU \times \mathbb{Z}$ and $BO \times \mathbb{Z}$.

Lemma V.3.1. *For $G = O$ and $G = U$, $BG_\otimes$ is equivalent as an infinite loop space to $BG(1) \times BSG_\otimes$.*

PROOF. $BO(1) = K(\mathbb{Z}_2, 1)$ and $BU(1) = K(\mathbb{Z}, 2)$ admit unique infinite loop space structures, the natural map $BG_\otimes \to BG(1)$ obtained by killing π_1 or π_2 (i.e., w_1 or c_1) is automatically an infinite loop map, and we define $BSG_\otimes$ and $\pi \colon BSG_\otimes \to BG_\otimes$, as an infinite loop space and map, to be its fibre. By Remark VI.4.5, the natural inclusion $\eta \colon BG(1) \to BG_\otimes$ (which classifies the canonical line bundle) is an infinite loop map. The composite
$$BG(1) \times BSG_\otimes \xrightarrow{\eta \times \pi} BG_\otimes \times BG_\otimes \xrightarrow{\phi} BG_\otimes$$
is the desired equivalence. $\qquad\square$

Atiyah, Bott, and Shapiro [13] have constructed a kO-orientation of $Spin$ and a kU-orientation of $Spin^c$. Thus, if $Spin$ maps to G, we have well-defined H-maps
$$\gamma \colon BSpin \longrightarrow B(G; kO) \quad \text{and} \quad f \colon G/Spin \longrightarrow BO_\otimes.$$
The fibre of the quasi-fibration $B(SO, Spin, *) \to B(SO, SO, *)$ is precisely $B\mathbb{Z}_2 = BO(1)$, and this gives an explicit equivalence $i \colon BO(1) \to SO/Spin$. Similarly, if $Spin^c$ maps to G, we have H-maps
$$g \colon BSpin^c \longrightarrow B(G; kO) \quad \text{and} \quad f \colon G/Spin^c \longrightarrow BO_\otimes,$$
and we have an explicit equivalence $i \colon BU(1) \to SO/Spin^c$.

Proposition V.3.2. *The following composites are equivalences:*

$$BSO_\otimes \times BSpin \xrightarrow{\tau \times g} B(SO;kO) \times B(SO;kO) \xrightarrow{\phi} B(SO;kO)$$

and

$$BSU_\otimes \times BSpin^c \xrightarrow{\tau \times g} B(SO;kU) \times B(SO;kU) \xrightarrow{\phi} B(SO;kU).$$

PROOF. The result is proven by easy chases of the relevant comparison diagram of Proposition V.2.3. In the real case, the salient facts are that $BSpin$ is equivalent to the fibre of $w_2 \colon BSO \to K(\mathbb{Z}_2, 2)$ and that $e_* \colon \pi_1 SO \to \pi_1 BO_\otimes$ is an isomorphism (because the obstruction $w_2(\xi; kO)$ of Section IV§3 can be non-zero or by direct calculation on mod 2 homology). In the complex case, the salient facts are that $BSpin^c$ is equivalent to the fibre of $w_3 \colon BSO \to K(\mathbb{Z}, 3)$ or (equivalently by a comparison of fibrations) to the fibre of $w_2 \otimes 1 + 1 \otimes \iota \colon BSO \times K(\mathbb{Z}, 2) K(\mathbb{Z}_2, 2)$ and that $\pi_2 BSpin^c = \mathbb{Z}$. $\square$

Corollary V.3.3. *The composite $fi \colon BO(1) \to SO/Spin \to BO_\otimes$ is homotopic to the natural inclusion $\eta \colon BO(1) \to BO_\otimes$.*

PROOF. Consider the following diagram, where $\zeta \colon BO(1) \to SO$ is any map which is non-trivial on π_1:

$\tau\zeta \simeq i$ and $e\chi\zeta \simeq \eta$ because both composites induce isomorphisms on π_1 and because the component of e in $BSO_\otimes$ is null homotopic by the splitting of $B(SO;kO)$. $f\tau \simeq e\chi$ by the comparison diagram. $\square$

The argument fails in the (less interesting) complex case, and I have not verified whether or not $fi \simeq \eta$ in that case.

We turn to the study of kO-oriented spherical fibrations. The rest of the section will be concerned with the construction and analysis of the "J-theory diagram," which is obtained by superimposing diagrams involving cannibalistic classes and the Adams conjecture on an elaboration of the comparison diagram for

$$g \colon BSpin \longrightarrow B(SF;kO) \quad \text{and} \quad f \colon SF/Spin \longrightarrow BO_\otimes.$$

Of course, this map f restricts on $SO/Spin$ to that just discussed, and we have the following observation.

Lemma V.3.4. *The natural map $SF/Spin \to F/O$ and the composite of f and $w_1 \colon BO_\otimes \to BO(1)$ are the components of an equivalence $SF/Spin \to F/O \times BO(1)$ of infinite loop spaces.*

Define $O_\otimes = \Omega BO_\otimes$ and $(SF;kO) = \Omega B(SF;kO)$. By abuse, write G and ΩBG interchangeably when $G = Spin$, SF, etc. For a map $\theta \colon X \to Y$, write π and ι generically for the projection $F\theta \to X$ from the (homotopy theoretic) fibre and for the inclusion $\Omega Y \to F\theta$. Define $(SF;kO)/Spin$ to be the fibre of f (which is equivalent to the fibre of g). This space classifies stable $Spin$-bundles with trivializations as kO-oriented stable spherical fibrations. Just as if $\Omega g \colon Spin \to (SF;kO)$ were derived from a morphism of monoid-valued $\mathscr{I}_*$-functors, write

$$q \colon (SF;kO)/Spin \longrightarrow BSpin \quad \text{and} \quad \tau \colon (SF;kO) \longrightarrow (SF;kO)/Spin$$

for $q\pi$ and for a map τ (obtained by Barratt–Puppe sequence arguments) such that $\pi\tau \simeq \tau\Omega q$ and $\tau\Omega\tau \simeq \iota$ in the diagram on the following page. With these notations, the solid arrow potion of this diagram exists and is homotopy commutative by Proposition V.2.3.

At the right of the diagram $BSpin_\otimes$ is defined (as an infinite loop space) to be the fibre of $w_2\colon BSO_\otimes \to K(\mathbb{Z}_2, 2)$.

We claim that, with dotted arrows inserted, this *J*-theory diagram exists and is homotopy commutative when $r \geq 2$ and all spaces in sight are localized away from r. To see this, first recall the following calculations of Adams [**1**, 5.1 and 5.2]. (See also Lemma V.2.9.)

THEOREM V.3.5. *ψ^r is a natural ring homomorphism on $KO(X)$ such that $\psi^r\xi = \xi^r$ on line bundles ξ. Let $x \in \widetilde{K}O(S^i) = \pi_i BO$, $i > 0$. If $i = 4j$, $\psi^r x = r^{2j}x$; if $i \equiv 1$ or $2 \bmod 8$ and r is odd, $\psi^r x = x$.*

If follows by Remark II.3.15 that ψ^r determines a morphism of ring spectra $kO[1/r] \to kO[1/r]$ and thus that $c(\psi^r)\colon B(SF; kO) \to BO_\otimes$ is defined away from r. Since $e_*\colon \pi_i SF \to \pi_i BO_\otimes$ is an isomorphism for $i = 1$ (by Corollary V.3.3) and for $i = 2$ (by translation from $Q_0 S^0 \to BO$, where the π_2's are generated by the smash and tensor product squares of the generators of the π_1's), $B(SF; kO)$ is 2-connected. Therefore $c(\psi^r)$ lifts uniquely to a map, still denoted $c(\psi^r)$, into $BSpin_\otimes$. BC^r is defined to be the fibre of this map, and C^r is defined to be ΩBC^r. For r even, $BO \simeq BSO \simeq BSpin$ localized away from r. For r odd, Theorem V.3.5 implies that $\psi^r/1\colon BO_\otimes \to BO_\otimes$ is trivial on $BO(1)$ (because the square of a line bundle is trivial) and annihilates $\pi_2 BSO_\otimes$. The splitting $SO \simeq Spin \times \mathbb{R}P^\infty$ determined by the fibration $Spin \to SO \to SO/Spin$ and any map $\zeta\colon \mathbb{R}P^\infty \to SO$ which is non-trivial on π_1 shows that $\theta\colon SO/Spin \to BSpin$ is null-homotopic. Therefore $\psi^r - 1\colon BO \to BO$ lifts uniquely to a map $\psi^r - 1\colon BO \to BSpin$. Similarly, $\psi^r/1\colon BO_\otimes \to BO_\otimes$ lifts uniquely to a map $\psi^r/1\colon BO_\otimes \to BSpin_\otimes$, and $\psi^r/1 \simeq c(\psi^r)\tau$ by Proposition V.2.2.

Define $\rho^r = c(\psi^r)g$; thus ρ^r is the Adams–Bott cannibalistic class. Recall the following calculations of Adams [**3**, p. 166]. (See also Lemma V.2.9.)

THEOREM V.3.6. *Let $x \in \pi_i BSpin[1/r]$, $i > 0$. If $i = 4j$, $\rho^r x = 1 + \frac{1}{2}(r^{2j} - 1)\alpha_{2j}x$; if $i \equiv 1$ or $2 \bmod 8$, $\rho^r x = 1$ for $r \equiv \pm 1 \bmod 8$ and $\rho^r x = 1 + x$ for $r \equiv \pm 3 \bmod 8$.*

Here $x \to 1 + x$ denotes the translation isomorphism from $\pi_* BSpin$ to $\pi_* BSpin_\otimes$ (and similarly for BO and BSO below).

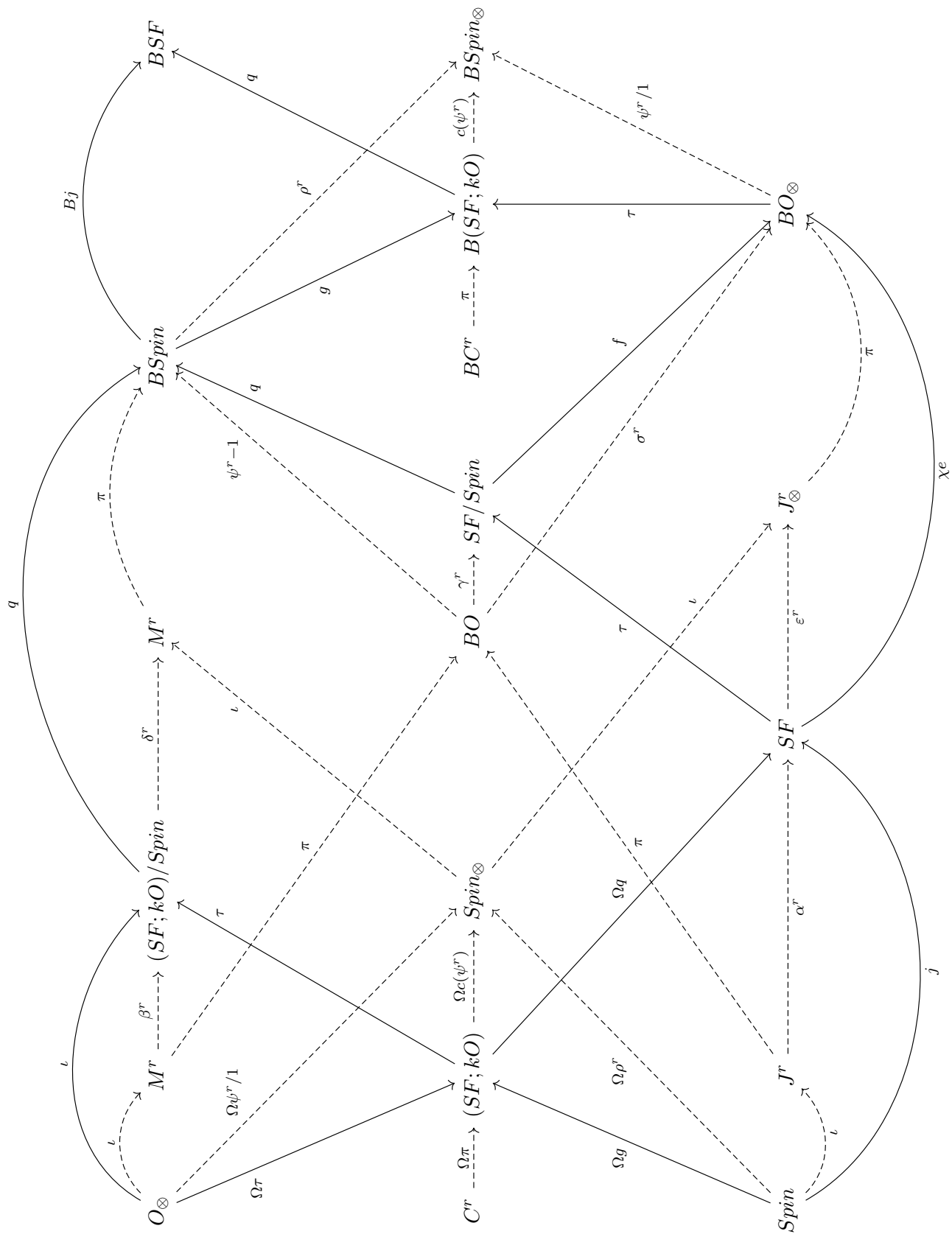The numbers $\alpha_{2j} = (-1)^{j+1} B_j/2j \in \mathbb{Z}[1/r]$ are analyzed in [**3**, §2].

The composite $BO \xrightarrow{\psi^r - 1} BSpin \xrightarrow{Bj} BSF$ is null homotopic away from r, by Quillen [**58**], Sullivan [**73**], or Becker and Gottlieb [**17**], since this statement is just a reformulation of the Adams conjecture. Therefore there exists $\gamma^r\colon BO \to SF/Spin$ such that $q\gamma^r \simeq \psi^r - 1$. γ^r is not uniquely determined. In particular, we can and do insist that its restriction to the translate of $BO(1)$ be the trivial map when $r \equiv \pm 1 \bmod 8$ and the non-trivial map to $SO/Spin \subset SF/Spin$ when $r \equiv \pm 3 \bmod 8$.

Define $\sigma^r = f\gamma^r\colon BO \to BO_\otimes$. By Remark V.3.7 below, the fibres of σ^r and of $\rho^r\colon BSpin \to BSpin_\otimes$ are equivalent; by abuse, we denote both by M^r. Define J^r and $J^r_\otimes$ to be the fibres of $\psi^r - 1\colon BO \to BSpin$ and of $\psi^r/1\colon BO_\otimes \to BSpin_\otimes$. Standard Barratt–Puppe sequence arguments then give maps α^r, β^r, δ^r, and ε^r such that $(1, \alpha^r, \gamma^r)$, $(1, \beta^r, \gamma^r)$, $(\Omega c(\psi^r), \delta^r, 1)$ and $(\Omega c(\psi^r), \varepsilon^r, 1)$ are maps of fibrations. This completes the construction of the diagram.

Remark V.3.7. Localize all spaces in sight at any set of primes T and consider the following diagram, where θ is an H-map and ξ is any map:

$$\begin{array}{ccc} BSO & \xrightarrow{\psi^r - 1} & BSO \\ \xi\downarrow & & \downarrow\theta \\ BSO_\otimes & \xrightarrow{\psi^r/1} & BSO_\otimes \end{array}$$

BSF

$BSpin_\otimes$

$BSpin$

$B(SF;kO)$

$BO_\otimes$

$BC^r \xdashrightarrow{\pi} B(SF;kO) \xdashrightarrow{c(\psi^r)} BSpin_\otimes$

$SF/Spin$

$BO \xdashrightarrow{\gamma^r} SF/Spin$

$J^r_\otimes$

M^r

$(SF;kO)/Spin$

$M^r \xdashrightarrow{\beta^r} (SF;kO)/Spin \xdashrightarrow{\delta^r} M^r$

SF

$Spin_\otimes$

$C^r \xdashrightarrow{\Omega\pi} (SF;kO) \xdashrightarrow{\Omega c(\psi^r)} Spin_\otimes$

J^r

$O^\otimes$

$Spin$

Arrow labels: q, Bj, ρ^r, $\psi^r/1$, τ, g, f, σ^r, π, ψ^r-1, ι, χ^e, ε^r, Ωq, $\Omega\rho^r$, Ωg, $\Omega\psi^r/1$, α^r, j, $\Omega\tau$

If $\xi = \theta$,the diagram is homotopy commutative by Lemma V.2.8 (or, if $2 \notin T$, Lemma V.2.7) since, regardless of what θ does, $\theta(\psi^r - 1)$ and $(\psi^r/1)\theta$ induce the same map on rational homology by Lemma V.2.6 and Theorem V.3.5. In particular, away from s, $\rho^s(\psi^r - 1) \simeq (\psi^r/1)\rho^s$. Conversely, if the diagram is homotopy commutative, then $\xi_* = \theta_*$ on rational homology (by the known behavior of $\psi^r - 1$ and $\psi^r/1$ on rational homology) and thus $\xi \simeq \theta$ by the cited lemmas. In particular, away from 2 and r, the maps σ^r and ρ^r of the J-theory diagram are homotopic. Away from r, Adams [**3**] has constructed an H-map $\rho^r \colon BSO \simeq BSO_\otimes$ which has the cannibalistic class ρ^r as 2-connective cover, and the existence of such a map also follows directly from Lemma V.2.10. Clearly Adams' map and the simply connected cover of σ^r induce the same homomorphism on rational homology and are thus homotopic.

For clarify, we shall retain the now duplicative notations σ^r and ρ^r, since what is most important about σ^r is not its homotopy class but its location in the J-theory diagram. That diagram and the remarks above give the following result, which should be compared with Proposition V.2.2.

Proposition V.3.8. *Away from r, the following diagram is homotopy commutative:*

$$
\begin{array}{ccc}
BSpin & \xrightarrow{\quad g \quad} & B(SF; kO) \\
\downarrow & \searrow^{\rho^r} \quad \swarrow_{c(\psi^r)} & \\
BO & BSpin_\otimes & \\
\downarrow_{\psi^r-1} & \searrow^{\sigma^r} \quad \downarrow & \downarrow^{B\psi^r/1} \\
& BO_\otimes & \\
& \searrow_\tau & \\
BSpin & \xrightarrow{\quad g \quad} & B(SF; kO)
\end{array}
$$

Corollary V.3.9. *Let $x \in \pi_i BO[1/r]$, $i > 0$. If $i = 4j$, $\sigma^r x = 1 + \frac{1}{2}(r^{2j} - 1)\alpha_{2j}x$; if $i \equiv 1$ or $2 \bmod 8$, $\sigma^r x = 1$ for $r \equiv \pm 1 \bmod 8$ and $\sigma^r x = 1 + x$ for $r \equiv \pm 3 \bmod 8$.*

PROOF. For $i \geq 4$, this is immediate from Theorem V.3.6. For $i = 2$ it follows from Lemma V.2.9. For $i = 1$, it holds by our choice of γ^r and Corollary V.3.3. $\square$

4. Local analysis of the J-theory diagram

We shall analyze the localization of the J-theory diagram at each prime p, with r chosen so as to yield maximum information. Let $r(2) = 3$. For p odd, let $r(p)$ be any chosen prime power the image of which in the ring $\mathbb{Z}_{p^2}$ generates its group of units. This choice of $r(p)$ is motivated by the following facts [**3**, §2]. Let $\mathbb{Z}_{(p)}$ denote the localization of the integers at p.

Lemma V.4.1. *Let $r = r(p)$. In $\mathbb{Z}_{(p)}$, $\frac{1}{2}(r^{2j} - 1)\alpha_{2j}$ is a unit if $p = 2$ or if $p > 2$ and $2j \equiv 0 \bmod (p-1)$, while $r^{2j} - 1$ is a unit if $p > 2$ and $2j \not\equiv \bmod(p-1)$.*

Throughout this section, unless otherwise specified, all spaces in sight are assumed to be localized at p and r denotes $r(p)$. We write BO, BSO, and $BSpin$ interchangeably when p is odd. Recall from Adams [**6**, Lecture 4] or Peterson [**53**] that, at odd p, BO splits as an infinite loop space as $W \times W^\perp$, where $\pi_i W = 0$ unless $i = 2j(p-1)$ when $\pi_i W = \mathbb{Z}_{(p)}$. The letter W is chosen as a reminder that W carries the Wu classes $w_j = \Phi^{-1}P^j\Phi(1)$ in $H^*(BO; \mathbb{Z}_p)$ where Φ is the canonical mod p Thom isomorphism. Write ν and $\nu^\perp$ for splitting maps from W and $W^\perp$ to BO and write ω and $\omega^\perp$ for projections from BO to W and $W^\perp$.

Adams and Priddy [**8**], using Adams spectral sequence techniques, have recently proven the following characterization of BSO as an infinite loop space.

THEOREM V.4.2. *There exists one and, up to isomorphism in the stable category, only one connective spectrum the zeroth space of which is equivalent to the localization (or completion) of BSO (or of BSU) at any given prime.*

For p odd, we again write ν, $\nu^\perp$, ω, and $\omega^\perp$ for infinite loop maps which split X as $W \times W^\perp$ for any (p-local) infinite loop space X equivalent to BSO. (Examples include $BO_\otimes$ and F/Top.) The requisite splitting exists by the theorem and the splitting of BSO. We shall need the following immediate consequence of Lemmas V.2.6 and V.2.7.

Lemma V.4.3. *Let $p > 2$ and let $\theta \colon X \to Y$ be an H-map between H-spaces of the same homotopy type as BO. Then*

$$\theta\nu \simeq \nu\omega\theta\nu \colon W \longrightarrow Y \quad and \quad \theta\nu^\perp \simeq \nu^\perp\omega^\perp\theta\nu^\perp \colon W^\perp \longrightarrow Y.$$

The following basic result summarizes information contained in Theorems V.3.5 and V.3.6, Corollary V.3.9, and Lemma V.4.1.

THEOREM V.4.4. *At $p = 2$, $\sigma^3 \colon BO \to BO_\otimes$ and $\rho^3 \colon BSpin \to BSpin_\otimes$ are equivalences. At $p > 2$, the following composites are equivalences:*

$$W \xrightarrow{\nu} BO \xrightarrow{\sigma^r} BO_\otimes \xrightarrow{\omega} W, \quad W^\perp \xrightarrow{\nu^\perp} BO \xrightarrow{\psi^r - 1} BO' \xrightarrow{\omega^\perp} W^\perp$$

$$W \xrightarrow{\nu} BO \xrightarrow{\rho^r} BO_\otimes \xrightarrow{\omega} W, \quad W^\perp \xrightarrow{\nu^\perp} BO_\otimes \xrightarrow{\psi^r/1} BO_\otimes \xrightarrow{\omega^\perp} W^\perp$$

$$BO \xrightarrow{\Delta} BO \times BO \xrightarrow{\sigma^r \times (\psi^r - 1)} BO_\otimes \times BO \xrightarrow{\omega \times \omega^\perp} W \times W^\perp,$$

and

$$W \times W^\perp \xrightarrow{\nu \times \nu^\perp} BO \times BO_\otimes \xrightarrow{\rho^r \times (\psi^r/1)} BO_\otimes \times BO_\otimes \xrightarrow{\phi} BO_\otimes.$$

It is unusual to encounter pullbacks (as opposed to weak pullbacks, for which the uniqueness clause in the universal property is deleted) in the homotopy category. However, the equivalences of the theorem imply the following result.

Corollary V.4.5. *At p, the following diagram is a pullback in the homotopy category:*

$$
\begin{array}{ccc}
BO & \xrightarrow{\psi^r - 1} & BSpin \\
{\scriptstyle \sigma^r} \downarrow & & \downarrow {\scriptstyle \rho^r} \\
BO_\otimes & \xrightarrow{\psi^r/1} & BSpin_\otimes
\end{array}
$$

Write X_p for the localization at p of any space $X^{r(p)}$ which appears in the J-theory diagram and write f_p for the localization at p of $f^{r(p)}$ when f is one of the first five Greek letters. We thus have J_p, M_p, C_p, and BC_p.

Corollary V.4.6. *At p, the following composites are equivalences:*

$$J_p \xrightarrow{\alpha_p} SF \xrightarrow{\varepsilon_p} J_{\otimes p} \quad and \quad M_p \xrightarrow{\beta_p} (SF;kO)/Spin \xrightarrow{\delta_p} M_p.$$

PROOF. According to the J-theory diagram, these composites are maps of fibres induced from the pullback diagram of the previous corollary. It follows trivially that these composites induce monomorphisms, and therefore (by finiteness) isomorphisms, on homotopy groups.   □

Now elementary chases of the J-theory diagram yield the following interpretations of Adams' work, which are based on ideas and results of Sullivan [**72**, unpublished].

THEOREM V.4.7. *At $p = 2$, the following composites are equivalences:*

  (i) $BC_2 \times BSpin \xrightarrow{\pi \times g} B(SF;kO) \times B(SF;kO) \xrightarrow{\phi} B(SF;kO)$

  (ii) $C_2 \times Spin \xrightarrow{\Omega\pi \times \Omega g} (SF;kO) \times (SF;kO) \xrightarrow{\phi} (SF;kO)$

(iii) $C_2 \xrightarrow{\tau\Omega\pi} (SF; kO)/Spin$

(iv) $C_2 \times BO \xrightarrow{\pi\tau\Omega\pi\times\gamma_2} SF/Spin \times SF/Spin \xrightarrow{\phi} SF/Spin$

(v) $C_2 \times J_2 \xrightarrow{\Omega(q\pi)\times\alpha_2} SF \times SF \xrightarrow{\phi} SF$

THEOREM V.4.8. *At $p = 2$, the following composites are equivalences:*

(i) $BC_p \times W \times W^\perp \xrightarrow{1\times\nu\times\nu^\perp} BC_p \times BSpin \times BO_\otimes \xrightarrow{\pi\times g\times\tau} B(SF; kO)^3 \xrightarrow{\phi} B(SF; kO)$

(ii) $C_p \times \Omega W \times \Omega W^\perp \xrightarrow{1\times\Omega\nu\times\Omega\nu^\perp} C_p \times Spin \times SO_\otimes \xrightarrow{\Omega\pi\times\Omega g\times\Omega\tau} (SF; kO)^3 \xrightarrow{\phi} (SF; kO)$

(iii) $C_p \times M_p \xrightarrow{\tau\Omega\pi\times\beta_p} (SF; kO)/Spin \times (SF; kO)/Spin \xrightarrow{\phi} (SF; kO)/Spin$

(iv) $C_p \times BO \xrightarrow{\pi\tau\Omega\pi\times\gamma_p} SF/Spin \times SF/Spin \xrightarrow{\phi} SF/Spin$

(v) $C_p \times J_p \xrightarrow{\Omega(q\pi)\times\alpha_p} SF \times SF \xrightarrow{\phi} SF$

The behavior of $\tau\colon BO_\otimes \to B(SF; kO)$ and $g\colon BSpin \to B(SF; kO)$ on those parts of their domains which do not enter into the splitting of $B(SF; kO)$ is analyzed in the following immediate consequences of Lemma V.4.3 and Theorem V.4.4, and the J-theory diagram. We agree to write ζ^{-1} for any chosen homotopy inverse to a homotopy equivalence ζ.

Corollary V.4.9. *At 2, $\tau\colon BO_\otimes \to B(SF; kO)$ is homotopic to the composite*

$$BO_\otimes \xrightarrow{(\sigma^3)^{-1}} BO \xrightarrow{\psi^3-1} BSpin \xrightarrow{g} B(SF; kO).$$

Corollary V.4.10. *At $p > 2$, the composite $\tau\nu\colon W \to BO_\otimes \to B(SF; kO)$ is homotopic to both of the composites in the following diagram:*

$$W \xrightarrow{(\omega\sigma^r\nu)^{-1}} W \xrightarrow{\nu} BO \xrightarrow{\psi^r-1} BO \xrightarrow{g} B(SF; kO)$$
$$\downarrow{\nu} \qquad\qquad\qquad \uparrow{\nu}$$
$$BO \xrightarrow{\psi^r-1} BO \xrightarrow{\omega} W$$

Corollary V.4.11. *At $p > 2k$, the composite $g\nu^\perp\colon W^\perp \to BO \to B(SF; kO)$ is homotopic to both of the composites in the following diagram:*

$$W^\perp \xrightarrow{(\omega^\perp(\psi^r-1)\nu^\perp)^{-1}} W^\perp \xrightarrow{\nu^\perp} BO \xrightarrow{\sigma^r} BO \xrightarrow{\tau} B(SF; kO)$$
$$\downarrow{\nu^\perp} \qquad\qquad\qquad\qquad \uparrow{\nu^\perp}$$
$$BO \xrightarrow{\sigma^r} BO_\otimes \xrightarrow{\omega^\perp} W^\perp$$

Here the lower routes are relevant to the splitting of $B(SF; kO)$, while the upper routes are more readily interpreted bundle theoretically.

Define global spaces $C = \prod_p C_p$, $J = \prod_p J_p$, etc. The spaces J and C are often called $\operatorname{Im} J$ and $\operatorname{Coker} J$. J_2 is usually defined as the fibre of $\psi^3 - 1\colon BSO \to BSO$ localized at 2; this gives the same homotopy type as our J_2, but with a different H-space structure. In view of the key role played by g in the results above and in the study of the groups $JO(X)$, the present definition is preferable. It is also preferable on categorical grounds, as was noted by Tornehave [unpublished] and will be explained in Section VIII§3. In terms of stable stems, we choose to ignore the anomalous fact that η is in the image of J (anomalous because η^2 is not in the image of J), preferring instead to regard η as the first element of the periodic family not in the image of J in the $(8j + 1)$-stems (see Theorem V.5.3). Sullivan first defined the spaces C_p. Actually, his (unpublished) C_2 is the fibre of $f\colon F/O \to BSO_\otimes$ localized at 2. This definition is equivalent to ours by Theorem V.4.7 and Lemma V.3.4. The definition of BC_2 given here is new.

5. $J\,Spin\,X$ and the δ and ε invariants

In this section and the next, the base spaces X of bundles are to be connected finite CW-complexes. To derive global bundle theoretic consequences from the local results of the previous section, we shall rely on the following basic fact in the theory of localization [**23**, V§6,7] or [**48**, VII]. Let Y_0 denote the localization of a connected H-space Y at Q.

THEOREM V.5.1. *For any set of primes T, the natural map*

$$[X, Y_T] \longrightarrow \underset{p \in T}{\text{pullback}}([X, Y_p] \to [X, Y_0])$$

is an injection and is a bijection if $\pi_* Y$ is of finite type.

We shall generally be concerned with simply connected Y, when the brackets may be taken in the sense of unbased maps. When T is the set of all primes, $Y = Y_T$. The fact that $(BF)_0 \simeq *$ will allow us to ignore rational coherence below. We shall write f_p for the localization of a classifying map f at p and we shall use the same letter for an element of $KG(X)$ and for its classifying map; we drop the curly brackets used earlier to distinguish stable from unstable bundles or oriented bundles.

We use Corollary V.4.5 as a substitute for the main technical result of [**4**] in the following mixed local and global version of Adams' analysis [**2, 3, 4**] of the groups $JO(X)$. Note that an F-trivial stable O-bundle admits a reduction to $Spin$ (compare Lemma V.3.4).

THEOREM V.5.2. *The following are equivalent for a stable $Spin$-bundle ξ over X.*

(i) *ξ is trivial as a stable spherical fibration.*
(ii) *There exists a unit $\zeta \in KO(X)$ such that $\rho^r \xi = \psi^r \zeta / \zeta$ in $KSpin[1/r](X)$ for every integer $r \geq 2$.*
(iii) *For each prime p, there exists a unit $\zeta_p \in KO_p(X)$ such that $\rho^r \xi_p = \psi^r \zeta_p / \zeta_p$ in $KSpin_p(X)$, $r = r(p)$.*
(iv) *For each prime p, there exists an element $\eta_p \in KO_p(X)$ such that $\xi_p = \psi^r \eta_p - \eta_p$ in $KSpin_p(X)$, $r = r(p)$.*

PROOF. Regard ξ as a map $X \to BSpin$. If $Bj \circ \xi \simeq *$, there exists $\xi' \colon X \to SF/Spin$ such that $q\xi' \simeq \xi$. Define $\zeta = f\xi'$, $f \colon SF/Spin \to BO_\otimes$. Then (ii) holds by the J-theory diagram. (ii) trivially implies (iii) and (iii) implies (iv) by Corollary V.4.5. If (iv) holds, $Bj \circ \xi$ is null-homotopic because its localization at each prime is null homotopic. $\qquad\square$

Atiyah and Tall [**15**] gave a purely algebraic analog of the local equivalence (iii) $\Leftrightarrow$ (iv). Key points in their approach were the extension by continuity of the Adams operations ψ^r on p-adic γ-rings to p-adic integers r and use of the fact that $r(p)$ generates the topological group of units of the p-adic integers (if $p > 2$). These algebraic considerations gained geometric content with Sullivan's introduction of the p-adic completions of spaces [**72, 73**]. In particular, he pointed out that the p-adic completion of BO can be split by use of the obvious algebraic splitting [**15**, p. 284] of the functor to p-adic γ-rings obtained by p-adically completing $\widetilde{KO}(X)$ for X finite. Nevertheless, it does not seem to me that passage to p-adic completion would yield substantive additional information in the stable parts of Adams' and Sullivan's work discussed in this chapter.

We also prefer not to use the cocycle condition emphasized by Bott and Sullivan [**22, 72**] or the periodicity condition emphasized by Adams [**4**]. These conditions make sense only when one considers the cannibalistic classes ρ^r for general values of r, rather than just for the $r(p)$. These classes give no new information, and the cocycle and the periodicity conditions can be viewed as formalizations of why they give no new information.

The following three results analyze the kernels of the natural transformations represented by the maps $q \colon B(SF; kO) \to BSF$, $g \colon BSpin \to B(SF; kO)$, and $\tau \colon BO_\otimes \to B(SF; kO)$. Recall

that ψ^r was defined on the Abelian group $\widetilde{K}(SF; kO)[1/r](X)$ by its action on orientations. The difference $\psi^r - 1$ is represented by the map $B\psi^r/1$ on the classifying space $B(SF; kO)[1/r]$.

THEOREM V.5.3. *Let (ξ, μ) be a kO-oriented stable spherical fibration over X and consider the following statements.*

    (i) The underlying stable spherical fibration ξ is trivial.

    (ii)$_p$ For the prime p, there exists an element $(\eta_p, \nu_p) \in \widetilde{K}(SF; kO)_p(X)$ such that $(\xi_p, \mu_p) = \psi^r(\eta_p, \nu_p) - (\eta_p, \nu_p), \ r = r(p)$.

Statement (i) implies statement (ii)$_p$ for each odd prime p and, provided that $(\psi^3 - 1)\widetilde{K}Spin_2(X) = (\psi^3 - 1)\widetilde{K}O_2(X) \subset \widetilde{K}Spin_2(X)$, for the prime 2. Conversely, statements (ii)$_p$ for all primes p imply statement (i).

PROOF. Assume (i). Then (ξ, μ) is classified by $\tau_*(\zeta)$ for some map $\zeta \colon X \to BO_\otimes$. At $p = 2$, $\operatorname{Im} \tau_* \subset \operatorname{Im}(B\psi^3/1)_*$ under the stated proviso by Corollary V.4.9 and Proposition V.3.8; note that the proviso certainly holds if $H^2(X; \mathbb{Z}_2) = 0$. At $p > 2$, $\operatorname{Im}(\tau\nu)_* \subset \operatorname{Im}(B\psi^r/1)_*$ by Corollary V.4.10 and Proposition V.3.8 while $\operatorname{Im}(\tau\nu^\perp)_* \subset \operatorname{Im}(B\psi^r/1)_*$ by Lemma V.4.3, Theorem V.4.4, and Proposition V.2.2. For the converse, $q(B\psi^r/1) \simeq *$ since $q \circ B\psi^r = q$ and therefore (ii)$_p$ for all p implies (i) by Theorem V.5.1. $\square$

Proposition V.5.4. *Let ξ be a stable Spin-bundle over X. Then $(\xi, \mu(g))$ is trivial as a kO-oriented stable spherical fibration if and only if $\rho^{r(p)}\xi_p = 1 \in KSpin_p(X)$ for each prime p.*

PROOF. This is immediate from Theorems V.4.4 and V.5.1 and Corollary V.4.11, and a chose of the J-theory diagram. $\square$

Proposition V.5.5. *Let μ be a (special) unit in $KO(X)$. Then (ε, μ) is trivial as a kO-oriented stable spherical fibration if and only if $\psi^{r(p)}\mu_p = \mu_p$ for all primes p.*

PROOF. This is immediate from Theorems V.4.4 and V.5.1 and Corollaries V.4.9 and V.4.10, and a chase of the J-theory diagram. $\square$

The analysis in the rest of this section elaborates and makes rigorous a speculative program proposed by Adams [**4**, §7] and amplified by Sullivan [**72**, §6]. It will emerge that the theoretical framework envisioned by Adams leads to new bundle theoretic analogs δ and ε of the d and e invariants used in his computations of stable stems in [**5**]. Of course, the results of these computations are visible in the J-theory diagram.

Remark V.5.6. $\pi_* J_p$ can be read off from Theorem V.3.5 and the homotopy exact sequence of the p-local fibration $J_p \to BO_p \xrightarrow{\psi^{r(p)} - 1} BSpin_p$. For $i > 2$, the image of $J \colon \pi_i SO = \pi_i Spin \to \pi_i SF$ can then be read off from the splitting of SF at p. The map $e_* \colon \pi_* SF \to \pi_* BO_\otimes$ detects elements $\mu_i \in \pi_i SF$ of order 2, where $i \equiv 1$ or $2 \bmod 8$ and $i > 0$, such that μ_i comes from an element of $\pi_i J_2$ which is not in the image of $\pi_i Spin$.

Clearly e_* corresponds via adjunction to Adams' d-invariant (which assigns the induced homomorphism of reduced real K-theory to a map $S^{n+k} \to S^n$). Delooping the map e (which is an infinite loop map) and generalizing to arbitrary X, we can reinterpret this invariant as follows (compare Definition IV.3.2).

Definition V.5.7. For $\xi \in \widetilde{K}SF(X)$, define $\delta(\xi) = w(\xi; kO) \in BO^1_\otimes(X^+)$ to be the obstruction to the kO-orientability of ξ; equivalently, for $\xi \in [X^+, BSF]$, $\delta(\xi) = Be \circ \xi \in [X^+, BBO_\otimes]$.

Adams' e-invariant is defined on (a subgroup of) the kernel of d, and our ε-invariant will be defined on the kernel of δ. Of course, the latter kernel is just the group $Q(SF; kO)(X)$ of kO-orientable stable SF-bundles. Before defining ε, we note that Theorems V.5.1 and V.5.2, together with Lemmas V.3.1 and V.3.4, imply the following result.

Corollary V.5.8. *For $G = Spin$, SO, or O, $JG(X)$ is naturally isomorphic to*

$$\prod_p ([X^+, BG_p]/(\psi^{r(p)} - 1)_*[X^+, BO_p]).$$

This suggests the following definition.

Definition V.5.9. For $G = Spin$, SO, or O, define $JG_\otimes(X)$ to be the group

$$\prod_p ([X^+, (BG_\otimes)_p]/(\psi^{r(p)} - 1)_*[X^+, (BO_\otimes)_p]).$$

Of course, the groups $JG(X)$ and $JG_\otimes(X)$ are abstractly isomorphic. In the case of $Spin$, the J-theory diagram yields a geometrically significant choice of isomorphism.

Definition V.5.10. Define $\varepsilon\colon Q(SF; kO)(X) \to JSpin_\otimes(X)$ as follows. Given a kO-orientable stable SF-bundle ξ, choose a kO-orientation μ, localize at p, and apply the cannibalistic class $c(\psi^{r(p)})$. The image of this in the p-component of $JSpin_\otimes(X)$ is independent of the choice of μ, and $\varepsilon(\xi) = \prod_p c(\psi^{r(p)})(\xi, \mu)$. Equivalently, for $\xi\colon X \to BSF$ such that $Be \circ \xi \simeq *$, choose $\overline{\xi}\colon X \to B(SF; kO)$ such that $q\overline{\xi} \simeq \xi$. If also $q\overline{\xi}' \simeq \xi$, then $\overline{\xi}'/\overline{\xi} \simeq \tau\zeta$ for some $\zeta\colon X \to BO_\otimes$ and thus, at each prime p,

$$c(\psi^r)\overline{\xi}'_p/c(\psi^r)\overline{\xi}_p \simeq c(\psi^r)(\overline{\xi}'_p/\overline{\xi}_p) \simeq (\psi^r/1)\zeta_p, \quad r = r(p).$$

Therefore $\varepsilon(\xi) = \prod_p c(\psi^{r(p)})\overline{\xi}_p$ is a well-defined element of $JSpin_\otimes(X)$.

We need one more definition.

Definition V.5.11. Let $C(X)$ denote the set of stable kO-oriented SF-bundles (ξ, μ) over X with local cannibalistic classes $c(\psi^{r(p)}) = 1 \in KSpni_p(X)$ for each prime p. Equivalently, if (ξ, μ) is classified by $\overline{\alpha}$, it is required that $c(\psi^{r(p)}) \circ \overline{\alpha}_p\colon X \to (BSpin_\otimes)_p$ be null homotopic for each prime p.

It is immediate from the form of the splittings of the $B(SF; kO)_p$ that $C(X)$ is classified by the space BC.

THEOREM V.5.12. *The composite $JSpin(X) \subset Q(SF; kO)(X) \xrightarrow{\varepsilon} JSpin_\otimes(X)$ is an isomorphism, $C(X)$ maps monomorphically onto the kernel of ε under neglect of orientation, and therefore*

$$Q(SF; kO)(X) = JSpin(X) \oplus C(X).$$

PROOF. The first clause holds by comparison of Corollary V.5.8 and Definition V.5.9 with the equivalences of Theorem V.4.4. The second clause holds since $(q\pi)_*\colon [X, BC] \to [X, BSF]$ is a monomorphism by the splitting of $B(SF; kO)_p$ and Corollaries V.4.9 and V.4.10 (which show that $\pi_*[X, BC_p]$ intersects $t_*[X, (BO_\otimes)_p]$ trivially) and since $C(X)$ clearly maps onto the kernel of ε. □

Comparison of Theorems V.5.3 and V.5.12 may be illuminating, particularly at the prime 2.

We discuss the relationship between Adams' e-invariant and our ε-invariant in the following remark.

Remark V.5.13. Let $\varepsilon = \prod_p \varepsilon_p\colon SF \to J_\otimes = \prod_p J_{\otimes p}$. A straightforward chase of the J-theory diagram allows us to identify the ε-invariant on $\mathrm{Ker}(Be)_* \subset \pi_* BSF$ with the induced homomorphism

$$\varepsilon_*\colon \mathrm{Ker}(\pi_* SF \xrightarrow{e_*} \pi_* BO_\otimes) \longrightarrow \mathrm{Ker}(\pi_* J_\otimes \to \pi_* J_{\otimes 2} \xrightarrow{\pi_*} \pi_* BO_2)$$

(the kernels being taken to avoid the elements μ_i of Remark V.5.6 and their images in the 2-component of $\pi_* J_\otimes$). On the other hand, an inspection of [**5**, §7 and §9] will convince the reader that Adams' real e-invariant (denoted $e'_\mathbb{R}$ or $e_\mathbb{R}$ in [**5**]) admits precisely the same description. Indeed, it can be seen in retrospect that Adams' calculation of the e-invariant on the image of J in [**5**, §10] amounts to a direct geometric comparison between the two invariants.

We complete this section with the development of a more conceptual description of the functor $C(X)$ than that given by Definition V.5.11. We require some preliminaries.

Notation V.5.14. Define *bo*, *bso*, and *bspin* to be the 0-connected, 1-connected, and 2-connected covers of the spectrum kO; similarly, define *bu* and *bsu* to be the 0-connected and 2-connected covers of kU. In each case, the zero$^{\text{th}}$ space is the one suggested by the notation. (Warning *bo* and *bu* are usually taken as our kO and kU, this being a pointless waste of a useful notation.)

Lemma V.5.15. *Fix $r \geq 2$ and localize all spectra away from r. Then $\psi^r - 1 \colon kO \to kO$ lifts uniquely to $\psi^r - 1 \colon kO \to bspin$.*

PROOF. $\psi^r - 1$ obviously lifts uniquely to a map into *bo*. Recall that Bott periodicity implies that if $\eta \colon Q_\infty S^1 \to Q_\infty S^0$ is the unique non-trivial map, then

$$1 \wedge \eta \colon \Sigma KO = KO \wedge S^1 \cong KO \wedge Q_\infty S^1 \longrightarrow KO \wedge Q_\infty S^0 \cong KO$$

is equivalent to the fibre of $c \colon KO \to KU$ (where KO and KU denote the periodic Bott spectra). Passage to associated connective spectra (Section II§2) yields a map $\overline{\eta} \colon \Sigma kO \to kO$ with the same behavior on homotopy groups in non-negative degrees as $1 \wedge \eta$, and $\overline{\eta}$ obviously lifts uniquely to a map $\overline{\eta} \colon \Sigma kO \to bo$. Its adjoint $\widetilde{\eta} \colon kO \to \Omega bo$ clearly maps $\pi_0 kO$ onto $\pi_0 \Omega bo = \mathbb{Z}_2$. Write H^* for cohomology with coefficients in $\mathbb{Z}_2$ and recall that $H^* kO = A/ASq^1 + ASq^2$, where A denotes the mod 2 Steenrod algebra (e.g., by [**7**, p. 336]). In particular $H^0 kO = \mathbb{Z}_2$ and $H^1 kO = 0 = H^2 kO$. The fibration $bso \to bo \to \mathscr{K}(\mathbb{Z}_2, 1)$ gives an exact sequence

$$[kO, \Omega bo] \longrightarrow H^0 kO \longrightarrow [kO, bso] \longrightarrow [kO, bo] \longrightarrow 0$$

in which the first map is surjective by the properties of $\widetilde{\eta}$. The fibration $bspin \to bso \to \mathscr{K}(\mathbb{Z}_2, 2)$ gives an isomorphism $[kO, bspin] \to [kO, bso]$ and the conclusion follows.   □

Definition V.5.16. Define $j_p \in H\mathscr{S}$ t o be the completion at p of the fibre of $\psi^{r(p)} - 1 \colon kO \to bspin$ and define $j = \prod_p j_p$. Also, for use in Chapter VIII, define jO_2 to be the completion at 2 of the fibre of $\psi^3 - 1 \colon kO \to bso$ and define JO_2 to be the fibre of $\psi^3 - 1 \colon BO \to BSO$ at 2.

The use of completions is innocuous here (since the homotopy groups are finite in positive degrees) and serves to ensure that j is a ring spectrum. Indeed, we shall prove in Theorem VIII.3.2 that j_p is a ring spectrum such that the natural map $\kappa \colon j_p \to kO$ (completed at p) is a map of ring spectra.

THEOREM V.5.17. *The spaces BC and $B(SF; j)$ are equivalent. Therefore*

$$C(X) = \widetilde{K}(SF; j)(X)$$

is the group of j-oriented stable spherical fibrations.

PROOF. By Section II§2, the zero$^{\text{th}}$ space of j_p is equivalent to $J_p \times \widehat{\mathbb{Z}}_{(p)}$; its 1-component is $J_{\otimes p}$, as an H-space, by Section VIII§3. The 1-component of the zero$^{\text{th}}$ space of j is $J_\otimes = \prod_p J_{\otimes p}$ and the projections give a homotopy commutative diagram

$$
\begin{array}{ccccc}
J_\otimes & \xrightarrow{\ \tau\ } & B(SF; j) & \xrightarrow{\ q\ } & BSF \\
\| & & \downarrow & & \downarrow{\scriptstyle \simeq} \\
\prod_p J_{\otimes p} & \xrightarrow{\Pi_p \tau} & \prod_p B(SF; j_p)_p & \xrightarrow{\Pi_p q} & \prod_p (BSF)_p
\end{array}
$$

We conclude that $B(SF; j)$ is equivalent to $\prod_p B(SF; j_p)_p$; compare Lemma V.2.4. Fix $r = r(p)$ and complete all spaces at p. A j_p-oriented stable spherical fibration clearly lies in $C(X)$ when

regarded via $\kappa\colon j_p \to kO$ as a kO-oriented stable spherical fibration. Since the homotopy groups of $B(SF; j_p)$ are obviously finite, there are no $\varprojlim^1$ problems and we conclude that the composite

$$B(SF; j_p) \xrightarrow{B\kappa} B(SF; kO) \xrightarrow{c(\psi^r)} BSpin_{\otimes}$$

is null-homotopic. There results a lift $\zeta\colon B(SF; j_p) \to BC_p$. Consider the following diagram, in which $(SF; j_p) = \Omega B(SF; j_p)$:

$$
\begin{array}{ccccc}
C_p & \xrightarrow{\Omega\pi} & (SF; kO) & \xrightarrow{\Omega c(\psi^r)} & Spin_{\otimes} \\
\Omega\zeta \uparrow & \overset{\Omega B\kappa}{\nearrow} & \downarrow{\scriptstyle \Omega q} & & \\
\Omega J_{\otimes p} \xrightarrow{\Omega\tau} (SF; j_p) & \xrightarrow{\Omega q} & SF & \xrightarrow{\quad e \quad} & J_{\otimes p}
\end{array}
$$

We have that $e_*\colon \pi_* SF \to \pi_* J_{\otimes p}$ is a split epimorphism by Corollary V.4.6. By Theorems V.4.7 and V.4.8, $(\Omega q \Omega \pi)_*$ maps $\pi_* C_p$ monomorphically onto $\operatorname{Ker} e_*$. In the bottom row, $(\Omega q)_*$ maps $\pi_*(SF; j_p)$ monomorphically onto $\operatorname{Ker} e_*$. Therefore $(\Omega\zeta)_*$ is a monomorphism and thus an isomorphism. Delooping, we conclude that $\zeta_*\colon \pi_* B(SF; j_p) \to \pi_* BC_p$ is an isomorphism.   □

6. Sullivan's analysis of topological bundle theory

The following basic theorem is due to Sullivan.

THEOREM V.6.1. *There exists a $kO[1/2]$-orientation $\overline{g}$ of $STop$. The localization away from 2 of the H-map $\overline{f}\colon F/Top \to BO_{\otimes}[1/2]$ associated to $\overline{g}\colon BSTop \to B(SF; kO[1/2])$ is an equivalence.*

The first statement is proven in [**72**, §6]. Actually, the proof that $\overline{g}$ is an H-map is omitted there.* It is easy to see that $\overline{g}$ is multiplicative modulo torsion, however, and this suffices for the discussion of $\overline{f}$ as an H-map. Note that, by Lemma V.2.5, $\overline{f}$ is the unique H-map such that the square

$$
\begin{array}{ccc}
F/Top & \xrightarrow{\quad q \quad} & BSTop \\
{\scriptstyle \overline{f}}\downarrow & & \downarrow{\scriptstyle \overline{g}} \\
BO_{\otimes}[1/2] & \xrightarrow{\quad \tau \quad} & B(SF; kO[1/2])
\end{array}
$$

is homotopy commutative. The second statement of the theorem is proven in [**71**] and Tsuchiya [**78**, §6] has given brief summary of the passage from the earlier theory of Sullivan to the construction of $\overline{g}$.

Proposition V.2.3 and Lemma V.2.4 yield Sullivan's characterization of stable topological bundle theory away from 2 as kO-oriented spherical fibration theory.

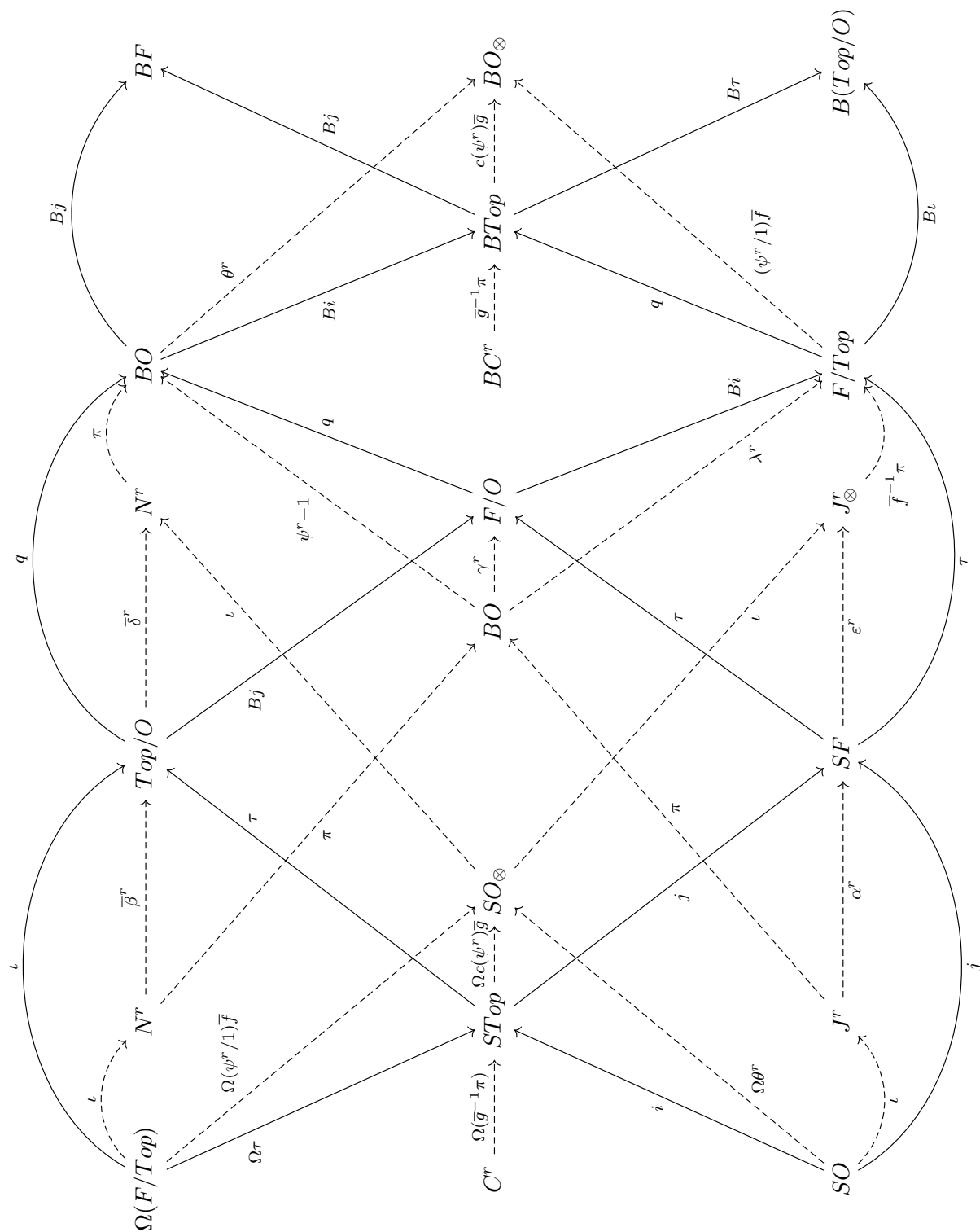**Corollary V.6.2.** $\overline{g}\colon BSTop[1/2] \to B(SF; kO)[1/2]$ *is an equivalence.*

In the analog of the J-theory diagram on the following page, the solid arrow diagram exists globally and the entire diagram exists and is homotopy commutative away from 2 and r. θ^r is defined to be $c(\psi^r)\overline{g}Bi$, and Sullivan [**72**, 6.81–6.82] obtained the following calculation by comparing the cannibalistic classes θ^r and ρ^r; explicitly, $\theta^r(x) = \rho^r(\psi^2(2x) - \psi^4(x))$.

THEOREM V.6.3. *Let $x \in \pi_{4j} BO[1/2, 1/r]$, $j > 0$. Then $\theta^r x = 1 + 2^{2j}(r^{2j} - 1)(1 - 2^{2j-1})\alpha_{2j} x$.*

λ^r is defined to be $Bi \circ \gamma^r$. By Proposition V.3.8, $\overline{f}\lambda^r$ is homotopic to θ^r and the fibres of λ^r and of θ^r are obtained by Barratt–Puppe sequence arguments, and the remaining maps already appear in the J-theory diagram. Proposition V.2.3 and the diagram imply the following analog of Proposition V.3.8.

---

*A proof will be given in Theorem V.7.16 below.

BF

BO

$BO_\otimes$

$B(Top/O)$

$BTop$

Bj

$c(\psi^r)\overline{g}$

$B\tau$

Bj

θ^r

Bi

$(\psi^r/1)\overline{f}$

$B\iota$

q

$BC^r \xrightarrow{\ \overline{g}^{-1}\pi\ } BTop$

q

F/Top

π

N^r

q

F/O

Bi

λ^r

$\overline{f}^{-1}\pi$

$\overline{\delta}^r$

ψ^r-1

$BO \dashrightarrow^{\gamma^r} F/O$

τ

$J^r_\otimes$

q

Top/O

ι

Bj

τ

ι

ε^r

τ

ι

$\overline{\beta}^r$

π

$SO_\otimes$

π

SF

N^r

τ

j

ι

$\Omega(\psi^r/1)\overline{f}$

$STop$

α^r

$\Omega(F/Top)$

$C^r \xrightarrow{\ \Omega(\overline{g}^{-1}\pi)\ } STop$

$\Omega c(\psi^r)\overline{g}$

J^r

j

$\Omega\tau$

i

$\Omega\theta^r$

ι

SO

Proposition V.6.4. *Away from* 2 *and* r, *the following diagram is homotopy commutative:*

$$
\begin{array}{ccccc}
BO & \xrightarrow{\;Bi\;} & BTop & \xrightarrow{\;\overline{g}\;} & B(SF;kO) \\
\end{array}
$$

$$
\begin{array}{ccc}
BO & \xrightarrow{Bi} & BTop & \xrightarrow{\overline{g}} & B(SF;kO)
\end{array}
$$

(diagram with λ^r, ψ^r-1, F/Top, θ^r, $\overline{f}$, q, $BO_\otimes$, $c(\psi^r)$, τ, $B\psi^r/1$)

Since $1-2^{2j-1}$ is a unit in $\mathbb{Z}_{(p)}$ if $2j \equiv 0 \bmod (p-1)$, the following local theorems and corollaries, in which $r = r(p)$, result from exactly the same calculations and diagram chases that were used to prove their analogs in Section V§4.

THEOREM V.6.5. *At* $p > 2$, *the following composites are equivalences:*

$$
W \xrightarrow{\nu} BO \xrightarrow{\theta^r} BO_\otimes \xrightarrow{\omega} W, \quad W \xrightarrow{\nu} BO \xrightarrow{\lambda^r} F/Top \xrightarrow{\omega} W,
$$

$$
BO \xrightarrow{\Delta} BO \times BO \xrightarrow{\lambda^r \times (\psi^r-1)} F/Top \times BO \xrightarrow{\omega \times \omega^\perp} W \times W^\perp
$$

and

$$
W \times W^\perp \xrightarrow{\nu \times \nu^\perp} BO \times BO_\otimes \xrightarrow{\theta^r \times (\psi^r/1)\overline{f}} BO_\otimes \times BO_\otimes \xrightarrow{\phi} BO_\otimes.
$$

Corollary V.6.6. *At* $p > 2$, *the following diagram is a pullback in the homotopy category:*

$$
\begin{array}{ccc}
BO & \xrightarrow{\psi^r-1} & BO \\
{\scriptstyle \lambda^r}\downarrow & & \downarrow{\scriptstyle \theta^r} \\
F/Top & \xrightarrow{(\psi^r/1)\overline{f}} & BO_\otimes
\end{array}
$$

Define N_p to be the localization of $N^{r(p)}$ at p and define $\overline{\beta}_p$ and $\overline{\delta}_p$ to be the localizations of $\overline{\beta}^{r(p)}$ and $\overline{\delta}^{r(p)}$ at p.

Corollary V.6.7. *At* $p > 2$, *the following composite is an equivalence:*

$$
N_p \xrightarrow{\overline{\beta}_p} Top/O \xrightarrow{\overline{\delta}_p} N_p.
$$

THEOREM V.6.8. *At* $p > 2$, *the following composites are equivalences.*

(i) $BC_p \times W \times W^\perp \xrightarrow{1 \times \nu \times \nu^\perp} BC_p \times BO \times F/Top \xrightarrow{\overline{g}^{-1}\pi \times Bi \times q} (BTop)^3 \xrightarrow{\phi} BTop.$

(ii) $C_p \times \Omega W \times \Omega W^\perp \xrightarrow{1 \times \Omega\nu \times \Omega\nu^\perp} C_p \times SO \times \Omega(F/Top) \xrightarrow{\Omega(\overline{g}^{-1}\pi) \times i \times \Omega q} (STop)^3 \xrightarrow{\phi} STop$

(iii) $C_p \times N_p \xrightarrow{\tau\Omega(g^{-1}\pi) \times \overline{\beta}_p} Top/O \times Top/O \xrightarrow{\phi} Top/O.$

The odd primary parts of Brumfiel's calculations [**24**] of $\pi_* BTop$ and $\pi_* Top/O$ can be read off from the theorem and the diagram.

Corollary V.6.9. *At* $p > 2$, *the composite* $q\nu \colon W \to F/Top \to BTop$ *is homotopic to both of the composites in the following diagram:*

$$
\begin{array}{ccccccccc}
W & \xrightarrow{(\omega\lambda^r\nu)^{-1}} & W & \xrightarrow{\nu} & BO & \xrightarrow{\psi^r-1} & BO & \xrightarrow{Bi} & BTop \\
& & \downarrow{\scriptstyle \nu} & & & & \uparrow{\scriptstyle \nu} & & \\
& & BO & \xrightarrow{\psi^r-1} & BO & \xrightarrow{\omega} & W & &
\end{array}
$$

Corollary V.6.10. *At $p > 2$, the composite $Bi \circ \nu^{\perp} \colon W^{\perp} \to BO \to BTop$ is homotopic to both of the composites in the following diagram:*

$$
\begin{array}{ccccccccc}
W^{\perp} & \xrightarrow{(\omega^{\perp}(\psi^r - 1)\nu^{\perp})^{-1}} & W^{\perp} & \xrightarrow{\ \nu^{\perp}\ } & BO & \xrightarrow{\ \lambda^r\ } & F/Top & \xrightarrow{\ q\ } & BTop \\
& & {\scriptstyle \nu^{\perp}}\big\downarrow & & & & {\scriptstyle \nu^{\perp}}\big\uparrow & & \\
& & BO & \xrightarrow{\ \lambda^r\ } & F/Top & \xrightarrow{\ \omega^{\perp}\ } & W^{\perp} & &
\end{array}
$$

The bundle theoretic interpretations of the results above are evident from the diagram and the arguments of the previous section. Consider all bundle theories in sight as localized away from 2. Corollary V.6.2 asserts that every $kO[1/2]$-oriented stable F-bundle has the form $(\xi, \mu(\overline{g}))$ for some Top-bundle ξ and that two stable Top-bundles ξ and ξ' are equal if $(\xi, \mu(\overline{g}))$ and $(\xi', \mu(\overline{g}))$ are equal as $kO[1/2]$-oriented stable F-bundles. (Here, away from 2, we may write F and Top but think in terms of the integrally oriented case.) The Adams operation ψ^r acts on $\widetilde{K}Top(X)$ via its action on the Sullivan orientation (away from 2 and r). More precisely, the action of ψ^r on $\widetilde{K}(SF; kO)(X)$ is to be transported to $\widetilde{K}Top(X)$ along the equivalence $\overline{g}_*$. Similarly, the action of ψ^r on the group of (special) units $[X^+, BO_\otimes]$ in $KO(X)$ may be transported to $[X^+, F/Top]$ along $\overline{f}_*$. Then, with ψ^r action trivially on $\widetilde{K}F(X)$, the transformations induced by the maps $Bj \colon BTop \to BF$, $Bi \colon BO \to BTop$, and $q \colon F/Top \to BTop$ commute with the ψ^r (by Propositions V.2.2, V.2.3 and V.6.4 in the last two cases), The following three results analyze the kernels of these transformations. Again, the proofs are the same as for the analogous results of Section V§5.

THEOREM V.6.11. *Away from 2, the following are equivalent for a stable Top-bundle ξ over X.*

(i) *ξ is trivial as a stable spherical fibration.*

(ii) *For each odd prime p, there exists an element $\eta_p \in \widetilde{K}Top_p(X)$ such that $\xi_p = \rho^r \eta_p - \eta_p$, $r = r(p)$.*

Proposition V.6.12. *Let ξ be a stable O-bundle over X. Then, away from 2, ξ is trivial as a stable Top-bundle if and only if $\theta^{r(p)}\xi_p = 1 \in KO_p(X)$ for each odd prime p.*

Proposition V.6.13. *Let (ξ, τ) be an F-trivialized stable Top-bundle over X. Then, away from 2, ξ is trivial as a stable Top-bundle if and only if $\psi^{r(p)}(\xi, \tau) = (\xi, \tau)$ for each odd prime p (or, equivalently, $\psi^{r(p)}\zeta_p = \zeta_p$ for each odd prime p, where ζ is that unit of $KO[1/2](X)$ such that the Sullivan orientation of ξ is the cup product of ζ and the orientation induced by τ from the canonical orientation of the trivial stable F-bundle).*

Away from 2, the δ-invariant may be interpreted as the obstruction to the existence of a topological structure (that is, a reduction of the structural monoid to Top) on a stable F-bundle. $Q(SF; kO)(X)$ may be interpreted as $JTop(X)$, and the ε-invariant may thus be regarded as defined on $JTop(X)$. $C(X)$ may be interpreted as the set of stable Top-bundles over X all of whose local cannibalistic classes $c(\psi)^{r(p)}\overline{g}_p$ are trivial. Theorem V.5.12 may then be interpreted as follows.

THEOREM V.6.14. *Away from 2, the composite $JO(X) \subset JTop(X) \xrightarrow{\varepsilon} JO_\otimes(X)$ is an isomorphism, $C(X)$ maps monomorphically onto the kernel of ε under passage to fibre homotopy equivalence, and therefore*

$$JTop(X) = JO(X) \oplus C(X).$$

Remark V.6.15. For what it is worth, we note that there is a precise analog to Theorem V.6.14 in which $B(Top/O)$ plays the role of BF. The proof is slightly more complicated, because rational coherence must be taken into account, but the conclusion is again that, away from 2, the image of

$$(B\iota)_* \colon [X^+, F/Top] \longrightarrow [X^+, B(Top/O)]$$

is a direct summand of the image of

$$(B\tau)_* \colon [X^+, BTop] \longrightarrow [X^+, B(Top/O)]$$

with complementary summand $C(X)$. This remark also has an analog in the J-theory case since the results of the next section imply that the J-theory diagram admits a lower right-hand corner.

Remark V.6.16. We have used Top and F instead of PL and G since the former theories fit naturally into our general context. Stably and away from 2, there is no distinction. Unstably, Sullivan's $\bar{g}$ is the limit of $kO[1/2]$-orientations of $SPL(n)$-block bundles $\bar{g}(n) \colon BSPL(n) \to B(SG(n); kO[1/2])$, where the classifying space on the right can be constructed either by the methods of [**47**] or by use of Brown's theorem. Haefliger and Wall's result [**32**] that $G(n)/PL(n) \to G/PL$ is an equivalence for $n \geq 3$, together with an unstable comparison diagram obtained by Barratt–Puppe sequence arguments and use of Lemmas V.2.6 and V.2.7, show that $\bar{g}(n)[1/2]$ is an equivalence for every $n \geq 3$. Note, however, that the block bundle version of $PL(n)$ used in this remark is not the one most relevant to geometric work in piecewise-linear topology.

7. Infinite loop analysis of the main diagrams

In order to determine which of the various splittings we have obtained are actually splittings of infinite loop spaces, we must determine which maps in our main diagrams are infinite loop maps. It turns out that homotopy theoretic arguments, which can be thought of as ultimately based on how tightly Bott periodicity ties together the p-local k-invariants of BO and BU, coupled with representation theoretical calculations, yield a great deal of information about this question. The relevant arguments are due to Madsen, Snaith, and Tornehave [**42**] and will be outlined here. To begin with, these authors have proven the following analog for maps of the Adams–Priddy unique deloopability of spaces result, Theorem V.4.2. Their proof is based on the fact that $[K(\widehat{\mathbb{Z}}_p, n), \widehat{BU}_p] = 0$ for $n \geq 3$. An alternative proof based on the techniques of Adams and Priddy [**8**] is possible. Let T be any set of primes. In all of these results of this section, our Theorems II.2.13 and II.2.14 show that the result for localizations at T follows immediately from the result for completions at p for $p \in T$.

THEOREM V.7.1. *Let D and E be T-local or T-complete connective spectra the zeroth spaces of which have completions at each $p \in T$ equivalent to those of either BU or BSO. Then the natural homomorphism $[D, E] \to [D_0, E_0]$ is a monomorphism.*

The proof of Madsen, Snaith, and Tornehave (MST henceforward) equally well yields the following complement except in the real case at $p = 2$ where the result is due to Ligaard [**38**].

THEOREM V.7.2. *Let D and E be T-local or T-complete connective spectra the zeroth spaces of which have completions at each $p \in T$ equivalent, respectively, to those of U and BU or SO and BO or $Spin$ and BSO. Then $[D, E] = 0$.*

MST then proceed to analysis of which Hmaps $f \colon \widehat{BU}_p \to \widehat{BU}_p$ or $f \colon \widehat{BSO}_p \to \widehat{BSO}_p$ are in fact infinite loop maps. Such a map can be unique written in the form $f = f_1 + f_2\psi^p$, where f_1 and f_2 are H-maps and f_1 is prime to ψ^p (in a suitable sense), and their basic observation is that f is an infinite loop map if and only if f_2 (since f is then essentially a linear combination of the ψ^r with r prime to p). This assertion has the following consequence.

THEOREM V.7.3. *Let X, Y, and Z be T-local or T-complete infinite loop spaces whose completions at each $p \in T$ are equivalent to those of BSO. Let $f \colon X \to Y$ and $g \colon Y \to Z$ be H-maps such that gf is an infinite loop map and either f or g is both an infinite loop map and a rational equivalence. Then the remaining map g or f is an infinite loop map.*

PROOF. By Theorems II.2.13, II.2.14 and V.4.2, it suffices to show this when X, Y, and Z are $\widehat{BSO}_p$ as infinite loop spaces. For definiteness, let f be an infinite loop map and a rational equivalence. Then $f = f_1$, $g = g_1 + g_2\psi^p$, and $fg = fg_1 + fg_2\psi^p = fg_1$. But $fg_2 = 0$ implies $g_2 = 0$ by Lemma V.2.8. □

The criterion above for determining whether an H-map $\widehat{BSO}_p \to \widehat{BSO}_p$ is an infinite loop map will be interpreted representation theoretically at the end of Section VIII§1. MST prove the following pair of results by representation theoretical calculations based on [**3**] for ρ^r; the result for $\theta^r \simeq \rho^r(\psi^2 \circ 2 + \psi^4)$ follows.

Lemma V.7.4. $\rho^r \colon BSO[1/r] \to BSO_\otimes[1/r]$ *is an infinite loop map.*

Lemma V.7.5. $\theta^r \colon BO[1/2, 1/r] \to BO_\otimes[1/2, 1/r]$ *is an infinite loop map.*

The following analog is simpler, requiring only Theorem V.4.2.

Lemma V.7.6. $\psi^r \colon BSO_\otimes[1/r] \to BSO_\otimes[1/r]$ *is an infinite loop map.*

PROOF. At p prime to r, Theorem V.4.2 gives an equivalence $\xi \colon BSO \to BSO_\otimes$ of infinite loop spaces. By the argument in Remark V.3.7, ψ^r is homotopic to the infinite loop map $\xi\psi^r\xi^{-1}$. □

At this point, it is convenient to insert a remark relevant only at the prime 2.

Remark V.7.7. Recall from Anderson and Hodgkin [**12**] that $\widetilde{KO}^*(K(\pi, n)) = 0$ for $n \geq 2$ and all finite Abelian groups π. By use of Proposition II.3.2 and Lemma II.2.10 and the non-splittable fibrations

$$bso \to bo \to \mathscr{K}(\mathbb{Z}_2, 1) \quad \text{and} \quad bspin \to bso \to \mathscr{K}(\mathbb{Z}_2, 2),$$

it is easy to deduce that

$$[\mathscr{K}(\mathbb{Z}_2, n), kO] = 0 \text{ for } n \geq 0, \quad [\mathscr{K}(\mathbb{Z}_2, n), bo] = 0 \text{ for } n \geq 0,$$
$$[\mathscr{K}(\mathbb{Z}_2, n), bso] = 0 \text{ for } n \geq 1, \quad [\mathscr{K}(\mathbb{Z}_2, n), bspin] = 0 \text{ for } n \geq 2,$$
$$[\mathscr{K}(\mathbb{Z}_2, 0), bso] = \mathbb{Z}_2 \quad \text{and} \quad [\mathscr{K}(\mathbb{Z}_2, 1), bspin] = \mathbb{Z}_2.$$

In the last two cases, both maps of spectra induce the trivial map

$$K(\mathbb{Z}_2, 0) \longrightarrow BSO \quad \text{and} \quad K(\mathbb{Z}_2, 1) \longrightarrow BSpin \text{ on the zero}^{\text{th}} \text{ spaces.}$$

In Section VIII§3, we shall prove that $c(\psi^r) \colon B(SF; kO) \to BSpin_\otimes$ is an infinite loop map at p, where $r = r(p)$. Thus the fibre BC_p of $c(\psi^r)$ and its loop space C_p are infinite loop spaces. Define $\psi^r/1 = c(\psi^r)\tau \colon BO_\otimes \to BSpin_\otimes$ as an infinite loop map at p. On $BSO_\otimes$, this delooping is the one coming from Lemma V.7.6 in view of Theorem V.7.1. When $p = 2$, this definition fixes a choice of delooping of the trivial map $BO(1) \to BSpin_\otimes$, and in fact gives the non-trivial delooping. Define $J_{\otimes p}$ as an infinite loop space to be the fibre of $\psi^r/1$. We shall construct an infinite loop fibration $C_p \to SF \to J_{\otimes p}$ in Section VIII§3 and will show that it splits when $p > 2$ in Section VIII§4. By the following basic result of Hodgkin and Snaith [**33**], [**70**, §9], this shows that, to the eyes of K-theory, SF is equivalent to $J_{\otimes p}$.

THEOREM V.7.8. $\widetilde{K}^*(C_p) = 0$ *and* $\widetilde{KO}^*(C_p) = 0$; *there are no non-trivial maps* $C_p \to BSO$ *on either the space of the spectrum level.*

We can now prove the following analog of Theorems V.7.1 and V.7.2, which is due to MST at $p > 2$ and to Ligaard at $p = 2$. Their proofs are somewhat more difficult, but give more precise information.

THEOREM V.7.9. *Let X be a T-local or T-complete infinite loop space whose completion at each $p \in T$ is equivalent to that of $BO_\otimes$. Then an H-map $f \colon J_\otimes \to X$ or $g \colon SF \to X$ is the zero$^{\text{th}}$ map of at most one map of spectra.*

PROOF. We may work at p, with X replaced by BSO, in view of Theorems II.2.13, II.2.14 and V.4.2 and Lemma V.3.1. Clearly the result for g will follow immediately from the result for f. With $r = r(p)$, consider the following diagram, the rows of which are infinite loop fibrations:

$$
\begin{array}{ccccccc}
Spin & \longrightarrow & \widetilde{J}_p & \xrightarrow{\pi} & BSO & \xrightarrow{\psi^r - 1} & BSpin \\
\Omega\xi \downarrow & & \downarrow \xi' & & \downarrow \xi & & \downarrow \xi \\
Spin_\otimes & \longrightarrow & \widetilde{J}_{\otimes p} & \xrightarrow{\pi} & BSO_\otimes & \xrightarrow{\psi^r/1} & BSpin_\otimes \\
\| & & \downarrow \alpha' & & \downarrow \alpha & & \| \\
Spin_\otimes & \longrightarrow & J_{\otimes p} & \xrightarrow[f]{\pi} & BO_\otimes & \xrightarrow{\psi^r/1} & BSpin_\otimes \\
& & & & \downarrow \overline{f} & & \downarrow \beta \\
& & & & BSO & \xleftarrow{\widetilde{f}} & BSO_\otimes
\end{array}
$$

The maps ξ are infinite loop equivalences coming from Theorem V.4.2, α, β, and the π are the natural infinite loop maps, and ξ' and α' are infinite loop maps coming from Barratt–Puppe sequence arguments (e.g. [**48**, I]). Clearly ξ' is an equivalence. By Hodgkin and Snaith [**33**, 4.7], the top fibration yields an exact sequence

$$
0 \longrightarrow PKO(BSpin) \xrightarrow{(\psi^r - 1)^*} PKO(BSO) \xrightarrow{\pi^*} PKO(\widetilde{J}_p) \longrightarrow 0
$$

of primitive elements in p-complete K-theory. (They deal with KU, but the result for KO follows.) Let $f: J_{\otimes p} \to BSO$ be an infinite loop map which is trivial as a map of spaces. We must show that f is trivial as an infinite loop map. By Theorem V.7.2, $f = \overline{f}\pi$ as an infinite loop map for some infinite loop map $\overline{f}: BO_\otimes \to BSO$ (because $[spin, bso] = 0$ and fibrations are negatives of cofibrations in the stable category [**48**, XI]). By the exact sequence of primitive elements (i.e., H-maps) and, when $p = 2$, Lemma V.2.10, there is an H-map $\widetilde{f}: BSO_\otimes \to BSO$ such that $\widetilde{f}\beta\xi(\psi^r - 1) \simeq \overline{f}\alpha\xi$. By Theorem V.7.3, $\widetilde{f}$ is an infinite loop map. By Theorem V.7.1, we conclude that $f\alpha'\xi' = \widetilde{f}\beta\xi(\psi^r - 1)\pi$ is the trivial infinite loop map. Since $(\alpha')^*: [j_{\otimes 2}, bso] \to [\widetilde{j}_{\otimes 2}, bso]$ is a monomorphism by Remark V.7.7, f is also the trivial infinite loop map.   □

These results allow infinite loop analysis of the comparison diagram parts of the J-theory diagram and of its analog for topological bundle theory. The following result was noted by Madsen, Snaith, and Tornehave [**42**].

Proposition V.7.10. $f: SF/Spin \to BSO_\otimes$ is (globally) an infinite loop map.

PROOF. By Theorems II.2.13 and II.2.14, it suffices to prove the result with all spaces completed at p. By Lemmas V.3.1 and V.3.4, it suffices to consider $f: F/O \to BSO_\otimes$ (even at $p = 2$). Let B be the zero$^\text{th}$ space of the cofibre in $H\mathscr{S}$ of the composite infinite loop map $\iota: C_p \to (SF; kO) \to SF \to F/O$ and let $\zeta_p: F/O \to BO$ be the natural map. By Theorems V.4.2, V.4.7 and V.4.8, B is equivalent to BSO as an infinite loop space. Consider the following part of the J-theory diagram with $r = r(p)$

$$
\begin{array}{ccccccc}
& & & & BSO & & \\
& & & \nearrow^{q} & & \searrow^{\rho^r} & \\
BSO & \xrightarrow{\gamma_p} & F/O & \xrightarrow{\zeta_p} & B & \dashrightarrow^{\eta} & BSO_\otimes \\
& & & \searrow_{f} & & \nearrow_{\psi^r/1} & \\
& & \searrow_{\sigma^r} & & BSO_\otimes & & \\
\end{array}
$$

By Theorems V.4.2 and V.7.8, $\rho^r q\iota: C_p \to BSO_\otimes$ is the trivial infinite loop map, hence there is an infinite loop map $\eta: B \to BSO_\otimes$ such that $\eta\zeta_p = \rho^r q$ as infinite loop maps. By Remark V.3.7, $\sigma^r \simeq \rho^r$. Thus $\eta(\zeta_p\gamma_p) \simeq (\psi^r/1)\rho^r$. By Theorem V.7.3 and Lemmas V.7.4 and V.7.6, $\zeta_p\gamma_p$ is

an infinite loop map. Clearly $\zeta_p \gamma_p$ is a homotopy equivalence, and $f\gamma_p \simeq \rho^r(\zeta_p \gamma_p)^{-1}\zeta_p \gamma_p$ while f and $\rho^r(\zeta_p \gamma_p)^{-1}\zeta_p$ both restrict to the trivial map on C_p. It follows from the splitting of F/O in Theorems V.4.7 and V.4.8 that f is homotopic to the infinite loop map $\rho^r(\zeta_p \gamma_p)^{-1}\zeta_p$. □

THEOREM V.7.11. *The following is (globally) a commutative diagram of infinite loop spaces and maps:*

$$
\begin{array}{ccccccc}
SF & \xrightarrow{\tau} & SF/Spin & \xrightarrow{q} & BSpin & \xrightarrow{Bj} & BSF \\
{\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \| \\
SF & \xrightarrow{e} & BO_\otimes & \xrightarrow{\tau} & B(SF;kO) & \xrightarrow{q} & BSF
\end{array}
$$

PROOF. By Theorem V.7.9 and the previous proposition, the left square is a commutative diagram of infinite loop spaces and maps. As pointed out in Section V§2, a Barratt–Puppe sequence argument on the spectrum level gives an infinite loop map $g': BSpin \to B(SF;kO)$ which makes the right two squares commute on the infinite loop space level. On the space level, $g - g' = \tau h$ for some $h: BSpin \to BO_\otimes$. However, commutation of the middle square implies that $g_* = (g')_*$ on rational homology and thus that $h_* = 0$. Therefore $h \simeq 0$ by Lemmas V.2.8, V.2.10 and V.3.1 and Theorem V.4.2. Thus $g \simeq g'$. □

Corollary V.7.12. *At $p = 2$, the following composite is an equivalence of infinite loop spaces:*

$$BC_2 \times BSpin \xrightarrow{\pi \times g} B(SF;kO) \times B(SF;kO) \xrightarrow{\phi} B(SF;kO).$$

Corollary V.7.13. *At $p > 2$, the following composite is an equivalence of infinite loop spaces:*

$$BC_p \times W \times W^\perp \xrightarrow{1 \times \nu \times \nu^\perp} BC_p \times BSpin \times BO_\otimes \xrightarrow{\pi \times g \times \tau} B(SF;kO)^3 \xrightarrow{\phi} B(SF;kO).$$

Those parts of the J-theory diagram related to the universal cannibalistic class $c(\psi^r)$ will be analyzed on the infinite loop level in Section VIII§3.

All remaining parts of the J-theory diagram depends on the Adams conjecture and thus on $\gamma^r: BO \to SF/Spin$. Madsen [**41**] has shown that γ^3 cannot be so chosen that its localization at 2 or that of α^3, is even an H-map. (See also [**26**, II.12.2]). Nevertheless, it seems likely that, away from 2 and r, γ^r can be chosen as an infinite loop map. The following conjecture is even a bit stronger.

Claim V.7.14. [2] *The complex Adams conjecture holds on the infinite loop space level. That is, for each r, the composite*

$$BU \xrightarrow{\psi^r - 1} BU \xrightarrow{Bj} BSF$$

is trivial as an infinite loop map when localized away from r.

By Theorem II.2.13, it suffices to work one prime at a time. The proof in Section VIII§4 that SF splits as $J_p \times C_p$ as an infinite loop space at each odd prime p will give explicit splitting maps $J_p \to SF$, but it is not known whether or not these maps are homotopic to (some choices of) α_p in the J-theory diagram.

Turning to the analysis of $BTop$ away from 2, we have the following analog of Proposition V.7.10, which was also noted by Madsen, Snaith, and Tornehave [**42**].

Proposition V.7.15. *$\overline{f}: F/Top[1/2] \to BO_\otimes[1/2]$ is an infinite loop map.*

---

[2] See the discussion following Remark VIII.4.6.

PROOF. Again, by Theorems II.2.13 and II.2.14, we may work on the p-complete level, $p > 2$. With $\zeta_p \colon F/O \to B$ as in the proof of Proposition V.7.10, consider the following part of the main diagram in Section V§6, $r = r(p)$:
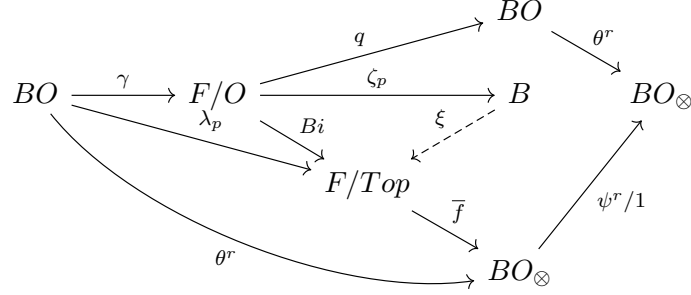
By Theorems V.4.2 and V.7.8, $Bi \circ \iota \colon C_p \to F/Top$ is the trivial infinite loop map, hence there is an infinite loop map $\xi \colon B \to F/Top$ such that $\xi\zeta_p = Bi$ as infinite loop maps. Now $\lambda_p \simeq \xi \circ (\zeta_p\gamma_p)$ is an infinite loop map, since $\zeta_p\gamma_p$ is an infinite loop map by the proof of Proposition V.7.10, and thus $\overline{f}$ is an infinite loop map by Lemma V.7.5 and Theorem V.7.3.   □

THEOREM V.7.16. *Away from 2, the following is a commutative diagram of infinite loop spaces and maps:*

$$
\begin{array}{ccccccc}
SF & \xrightarrow{\tau} & F/Top & \xrightarrow{q} & BSTop & \xrightarrow{Bj} & BSF \\
\Big\downarrow{\scriptstyle\chi} & & \Big\downarrow{\scriptstyle\overline{f}} & & \Big\downarrow{\scriptstyle\overline{g}} & & \Big\| \\
SF & \xrightarrow{e} & BO_{\otimes} & \xrightarrow{\tau} & B(SF;kO) & \xrightarrow{q} & BSF
\end{array}
$$

Therefore $\overline{f}$ and $\overline{g}$ are equivalences of infinite loop spaces.

PROOF. Again, the left square commutes on the infinite loop level by Theorem V.7.9, hence Barratt–Puppe sequence arguments give a map $\overline{g}' \colon BSTop \to B(SF;kO)$ which makes the right two squares commute on the infinite loop space level. $\overline{g} - \overline{g}' = \tau h$ for some map $h \colon BSTop \to BO_{\otimes}$, and the rationalization of h is null homotopic. It follows by use of the splitting of $BSTop$, the fact that $[BC_p, BO_{\otimes}] = 0$, and Lemma V.2.8 that $h \simeq 0$ and $\overline{g} \simeq \overline{g}'$. Note that, despite the role played by the product on $BSTop$ in its splitting, this argument does not depend on h being an H-map (again, because $[BC_p, BO_{\otimes}] = 0$) and therefore more than suffices to prove Sullivan's unpublished assertion that $\overline{g}$ is an H-map (a result which has nowhere been used in our work above).   □

By Corollary V.7.13 and Theorem V.7.16, $BSTop$ splits at p as $BC_p \times W \times W^{\perp}$ as an infinite loop space. This fact, together with the firm grasp on BC_p as an infinite loop space given by Section VIII§3, has been used to obtain precise information on the characteristic classes (at $p > 2$) for stable topological bundles in [**26**, II].

CHAPTER VI

E_∞ ring spaces and bipermutative categories

An E_∞ space is, essentially, an H-space which is commutative, associative, and unital up to all possible higher coherence homotopies. An E_∞ ring space is, essentially, an E_∞ space with respect to two products, one additive and the other multiplicative, such that the distributive laws are satisfied up to all possible higher coherence homotopies. The precise definition will be given in Section VI§1. Some consequences of the definition, and elementary examples, will be given in Section VI§2.

A symmetric monoidal category is a category with a product which is commutative, associative, and unital up to coherent natural isomorphism. It determines an equivalent permutative category, the classifying of which is an E_∞ space. A symmetric bimonoidal category is a symmetric monoidal category with respect to two products, one additive and the other multiplicative, such that the distributive laws are satisfied up to coherent natural isomorphism. It determines an equivalent bipermutative category, the classifying space of which is an E_∞ ring space. The precise definitions, and proofs, will be given in Sections VI§3 and VI§4, and many examples of bipermutative categories will be displayed in Section VI§5.

The relationship between E_∞ ring spaces and E_∞ ring spectra will be determined in Chapter VII and applications will be given in Chapter VIII. The homology of E_∞ ring spaces has been studied in [**26**, II].

1. The definition of E_∞ ring spaces

As will be made precise below, an operad $\mathscr{C}$ is a collection of suitably interrelated spaces $\mathscr{C}(j)$ with actions by the symmetric groups Σ_j. $\mathscr{C}$ is an E_∞ operad if the Σ_j actions are free and the spaces $\mathscr{C}(j)$ are contractible, so that the orbit spaces $\mathscr{C}(j)/\Sigma_j$ are $K(\Sigma_j, 1)$'s. An action θ of $\mathscr{C}$ on a space X in $\mathscr{T}$ (our ground category of based spaces) is a morphism of operads $\mathscr{C} \to \mathscr{E}_X$, where $\mathscr{E}_X(j)$ is the function space $F(X^j, X)$ of based maps and $\mathscr{E}_X$ is given an operad structure in the evident way [**45**, 1.2]. An E_∞ space (X, θ) is a space X together with a given action θ by some E_∞ operad $\mathscr{C}$. With product given by $\theta_2(c)\colon X^2 \to X$ for any $c \in \mathscr{C}(2)$, X indeed an H-space which is commutative, associative, and unital up to all possible higher coherence homotopies [**45**, p. 4].

An E_∞ ring space will be an E_∞ space with respect to actions by two interrelated E_∞ operads. These actions will satisfy the distributive laws up to all possible higher coherence homotopies, although these implied homotopies fortunately need not be made explicit.

Thus assume given two operads $\mathscr{C}$ and $\mathscr{G}$. Actions by $\mathscr{C}$ will be thought of as additive and will always be denoted by θ. The corresponding basepoint will be denoted by zero and wedges and smash products will be taken with respect to this basepoint. Actions by $\mathscr{G}$ will be thought of as multiplicative and will always be denoted by ξ. The corresponding basepoint will be denoted by one. As in Section IV§1, it is convenient to insist that $\mathscr{G}$-spaces have zeros. Recall that $\mathscr{T}_e$ denotes the category of spaces X together with cofibrations $e\colon S^0 \to X$ and that $\mathscr{G}[\mathscr{T}_e]$ denotes the category of $\mathscr{G}_0$-spaces (X, ξ). It is important to observe that, for non-triviality, zero and one must lie in different components of X. Indeed, in the contrary case, the left translations $\xi_2(g)(0, x)$ and $\xi_2(g)(1, x)$ by zero and one would be homotopic (for any fixed $g \in \mathscr{G}(2)$). Since the first map is trivial and, if $\mathscr{G}(1)$ is connected (as is always the case in practice), the second map is homotopic to the identity [**45**, p. 4], it follows that X would be contractible.

As explained in [**45**, §2], an operad $\mathscr{C}$ determines a monad (C, μ, η) in $\mathscr{T}$ such that the notion of a $\mathscr{C}$-space is equivalent to that of a C-algebra in $\mathscr{T}$. We shall define a notion of an action of an operad $\mathscr{G}$ on an operad $\mathscr{C}$ in such a way that C restricts to a monad in the category $\mathscr{G}[\mathscr{T}_e]$. Thus for a $\mathscr{G}_0$-space X, the space CX will inherit a structure of $\mathscr{G}_0$-space such that the unit $\eta\colon X \to CX$ and product $\mu\colon CCX \to CX$ will be morphisms of $\mathscr{G}_0$-spaces. We shall then define a $(\mathscr{C}, \mathscr{G})$-space to be an algebra over the monad C in $\mathscr{G}[\mathscr{T}_e]$. The requirement that the additive action $\theta\colon CX \to X$ be a morphism of $\mathscr{G}_0$-spaces will succinctly encode the distributive laws. An E_∞ ring space will be a $(\mathscr{C}, \mathscr{G})$-space where $\mathscr{C}$ and $\mathscr{G}$ are E_∞ operads.

It is useful to think of the passage from $\mathscr{C}$-spaces to $(\mathscr{C}, \mathscr{G})$-spaces as resulting from a change of ground category from $\mathscr{T}$ to $\mathscr{G}[\mathscr{T}_e]$.

The details of the definitions are necessary for rigor and useful in the study of homology operations [**26**] and homotopy operations (work in progress). For our present theoretical purposes, it is the conceptual outline above that is crucial. We first recall the definitions of operads and actions by operads.

Notation VI.1.1. Let $j_r \geq 0$, $1 \leq r \leq k$, and let $j = j_1 + \cdots + j_k$. For $\sigma \in \Sigma_k$, define $\sigma(j_1, \ldots, j_k) \in \Sigma_j$ to be that permutation of j letters which permutes the k blocks of letters determined by the given partition of j as σ permutes k letters. For $\tau_r \in \Sigma_{j_r}$, define $\tau_1 \oplus \cdots \oplus \tau_k \in \Sigma_j$ to be the image of $(\tau_1, \ldots, \tau_k)$ under the evident inclusion of $\Sigma_{j_1} \times \cdots \times \Sigma_{j_k}$ in Σ_j.

Definition VI.1.2. An operad $\mathscr{C}$ is a collection of spaces $\mathscr{C}(j)$ for $j \geq 0$, with $\mathscr{C}(0)$ a single point $*$, together with maps

$$\gamma\colon \mathscr{C}(k) \times \mathscr{C}(j_1) \times \cdots \times \mathscr{C}(j_k) \longrightarrow \mathscr{C}(j_1 + \cdots + j_k)$$

for all $k \geq 0$ and $j_r \geq 0$, an unit element $1 \in \mathscr{C}(1)$, and right actions of Σ_j on $\mathscr{C}(j)$, all subject to the following formulas.

(a) If $c \in \mathscr{C}(k)$, $d_r \in \mathscr{C}(j_r)$ for $1 \leq r \leq k$, and $e_s \in \mathscr{C}(i_s)$ for $1 \leq s \leq j_1 + \cdots + j_k$, then

$$\gamma(\gamma(c; d_1, \ldots, d_k); e_1, \ldots, e_{j_1 + \cdots + j_k}) = \gamma(c; f_1, \ldots, f_k),$$

where $f_r = \gamma(d_r; e_{j_1 + \cdots + j_{r-1} + 1}, \ldots, e_{j_1 + \cdots + j_r})$ (or $*$ if $j_r = 0$).

(b) If $c \in \mathscr{C}(k)$, then $g(c; 1^k) = c$; if $d \in \mathscr{C}(j)$, then $\gamma(1; d) = d$.

(c) If $c \in \mathscr{C}(k)$, $d_r \in \mathscr{C}(j_r)$, $\sigma \in \Sigma_k$, and $\tau_r \in \Sigma_{j_r}$, then

$$\gamma(c\sigma; d_1, \ldots, d_k) = \gamma(c; d_{\sigma^{-1}(1)}, \ldots, d_{\sigma^{-1}(k)})\sigma(j_1, \ldots, j_k)$$

and

$$\gamma(c; d_1\tau_1, \ldots, d_k\tau_k) = \gamma(c; d_1, \ldots, d_k)(\tau_1 \oplus \cdots \oplus \tau_k).$$

Definition VI.1.3. An action θ of an operad $\mathscr{C}$ on a space X consists of Σ_j-equivariant maps $\theta_j\colon \mathscr{C}(j) \times X^j \to X$ such that $\theta_0(*)$ is the basepoint $* \in X$, $\theta_1(1; x) = x$, and, if $c \in \mathscr{C}(k)$, $d_r \in \mathscr{C}(j_r)$ for $1 \leq r \leq k$, and $x_s \in X$ for $1 \leq s \leq j_1 + \cdots + j_k = j$,

$$\theta_j(\gamma(c; d_1, \ldots, d_k); x_1, \ldots, x_j) = \theta_k(c; y_1, \ldots, y_k),$$

where

$$y_r = \theta_{j_r}(d_r; x_{j_1 + \cdots + j_{r-1} + 1}, \ldots, x_{j_1 + \cdots + j_r}) \text{ (or } * \text{ if } j_r = 0).$$

We require the multiplicative analog of Notation VI.1.1, and distributivity permutations, to define actions of operads on operads and of operad pairs on spaces.

Notation VI.1.4. For $j_r \geq 1$, let $S(j_1, \ldots, j_k)$ denote the set of all sequences $I = \{i_1, \ldots, i_k\}$ such that $1 \leq i_r \leq j_r$, and order $S(j_1, \ldots, j_k)$ lexicographically. This fixes an action of Σ_j on $S(j_1, \ldots, j_k)$, where $j = j_1 \cdots j_k$. For $\sigma \in \Sigma_k$, define

$$\sigma\langle j_1, \ldots, j_k \rangle\colon S(j_1, \ldots, j_k) \longrightarrow S(j_{\sigma^{-1}(1)}, \ldots, j_{\sigma^{-1}(k)})$$

by
$$\sigma\langle j_1, \ldots, j_k\rangle\{i_1, \ldots, i_k\} = \{i_{\sigma^{-1}(1)}, \ldots, i_{\sigma^{-1}(k)}\}.$$
Via the given isomorphisms of $S(j_1, \ldots, j_k)$ and $S(j_{\sigma^{-1}(1)}, \ldots, j_{\sigma^{-1}(k)})$ with $\{1, 2, \ldots, j\}$, $\sigma\langle j_1, \ldots, j_k\rangle$ may be regarded as an element of Σ_j. For $\tau \in \Sigma_{j_r}$, define $\tau_1 \otimes \cdots \otimes \tau_k \in \Sigma_j$ by
$$(\tau_1 \otimes \cdots \otimes \tau_k)\{i_1, \ldots, i_k\} = \{\tau_1 i_1, \ldots, \tau_k i_k\}.$$

Notation VI.1.5. Given non-negative integers k, j_r for $1 \leq r \leq k$, and h_{ri} for $1 \leq r \leq k$ and $1 \leq i \leq j_r$, define $\nu = \nu\{k, j_r, h_{ri}\}$ to be that permutation of the set of

$$\sum_{I \in S(j_1, \ldots, j_k)} \left(\prod_{r=1}^{k} h_{r i_r} \right) = \prod_{r=1}^{k} \left(\sum_{i=1}^{j_r} h_{ri} \right)$$

letters which corresponds to the comparison of the two ordered sets (where II denotes the ordered disjoint union)

$$\coprod_{I \in S(j_1, \ldots, j_k)} S(h_{1 i_1}, \ldots, h_{k i_k}) \quad \text{and} \quad S\left(\sum_{i=1}^{j_1} h_{1i}, \ldots, \sum_{i=1}^{j_k} h_{ki} \right)$$

obtained by sending an element $\{a_1, \ldots, a_k\}$ with $1 \leq a_r \leq h_{r i_r}$ of the Ith summand of the first set to that element $\{b_1, \ldots, b_k\}$ of the second set such that $b_r = h_{r1} + \cdots + h_{r i_r - 1} + a_r$.

Definition VI.1.6. An action λ of an operad $\mathscr{G}$ on an operad $\mathscr{C}$ consists of maps
$$\lambda \colon \mathscr{G}(k) \times \mathscr{C}(j_1) \times \cdots \times \mathscr{C}(j_k) \longrightarrow \mathscr{C}(j_1 \cdots j_k)$$
for all $k \geq 0$ and $j_r \geq 0$ subject to the following formulas:

(a) If $g \in \mathscr{G}(k)$, $g_r \in \mathscr{G}(j_r)$ for $1 \leq r \leq k$, and $c_s \in \mathscr{C}(i_s)$ for $1 \leq s \leq j_1 + \cdots + j_k$, then
$$\lambda(\gamma(g; g_1, \ldots, g_k); c_1, \ldots, c_{j_1 + \cdots + j_k}) = \lambda(g; d_1, \ldots, d_k),$$
where
$$d_r = \lambda(g_r; c_{j_1 + \cdots + j_{r-1} + 1}, \ldots, c_{j_1 + \cdots + j_r}) \quad (\text{or } * \text{ if } j_r = 0).$$

(a′) If $g \in \mathscr{G}(k)$, $c_r \in \mathscr{C}(j_r)$ for $1 \leq r \leq k$, and $d_{ri} \in \mathscr{C}(h_{ri})$ for $1 \leq i \leq j_r$, then
$$\gamma\left(\lambda(g; c_1, \ldots, c_k); \prod_{I \in S(j_1, \ldots, j_k)} d_I \right) \nu = \lambda(g; e_1, \ldots, e_k),$$
where
$$d_I = (d_{1 i_1}, \ldots, d_{k i_k}) \quad \text{and} \quad e_r = \gamma(c_r; d_{ri}, \ldots, d_{r j_r}).$$

(b) If $c \in \mathscr{C}(j)$ and $1 \in \mathscr{G}(1)$ is the unit of $\mathscr{G}$, then $\lambda(1; c) = c$.

(b′) If $g \in \mathscr{G}(k)$ and $1 \in \mathscr{C}(1)$ is the unit of $\mathscr{C}$, then $\lambda(g; 1^k) = 1$.

(c) If $g \in \mathscr{G}(k)$, $c_r \in \mathscr{C}(j_r)$, and $\sigma \in \Sigma_k$, then
$$\lambda(g\sigma; c_1, \ldots, c_k) = \lambda(g; c_{\sigma^{-1}(1)}, \ldots, c_{\sigma^{-1}(k)})\sigma\langle j_1, \ldots, j_k\rangle.$$

(c′) If $g \in \mathscr{G}(k)$, $c_r \in \mathscr{C}(j_r)$, and $\tau_r \in \Sigma_{j_r}$, then
$$\lambda(g; c_1 \tau_1, \ldots, c_k \tau_k) = \lambda(g; c_1, \ldots, c_k)(\tau_1 \otimes \cdots \otimes \tau_k).$$

Formulas (a), (b), and (c) relate the λ to the internal structure of $\mathscr{G}$ and formulas (a′), (b′), and (c′) relate the λ to the internal structure of $\mathscr{C}$.

Definition VI.1.7. By an operad pair $(\mathscr{C}, \mathscr{G})$, we understand operads $\mathscr{C}$ and $\mathscr{G}$ together with a given action of $\mathscr{G}$ on $\mathscr{C}$. $(\mathscr{C}, \mathscr{G})$ is said to be an E_∞ pair if $\mathscr{C}$ and $\mathscr{G}$ are E_∞ operads. A morphism $(\mathscr{C}, \mathscr{G}) \to (\mathscr{C}', \mathscr{G}')$ of operad pairs is a pair of morphisms of operads $\mathscr{C} \to \mathscr{C}'$ and $\mathscr{G} \to \mathscr{G}'$ which commute with the given actions.

Henceforward, assume given an operad pair $(\mathscr{C}, \mathscr{G})$.

Definition VI.1.8. Let $(X, \xi) \in \mathscr{G}[\mathscr{T}_e]$. For $k \geq 0$ and $j_r \geq 0$ define maps

$$\xi_k \colon \mathscr{G}(k) \times \mathscr{C}(j_1) \times X^{j_1} \times \cdots \times \mathscr{C}(j_k) \times X^{j_k} \longrightarrow \mathscr{C}(j_1 \cdots j_k) \times X^{j_1 \cdots j_k}$$

by the following formula for $g \in \mathscr{G}(k)$, $c_r \in \mathscr{C}(j_r)$, and $y_r \in X^{j_r}$:

$$\xi_k(g, c_1, y_1, \ldots, c_k, y_k) = \left(\lambda(g; c_1, \ldots, c_k); \prod_{I \in S(j_1, \ldots, j_k)} \xi_k(g; y_I) \right),$$

where, if $y_r = (x_{r1}, \ldots, x_{rj_r})$, then $y_I = (x_{1i_1}, \ldots, x_{ki_k}) \in X^k$.

Recall the construction of the monad C in $\mathscr{T}$ from [**45**, 2.4].

Proposition VI.1.9. *For $(X, \xi) \in \mathscr{G}[\mathscr{T}_e]$, the maps ξ_k induce an action of $\mathscr{G}$ on CX such that $\mu \colon CCX \to CX$ and $\eta \colon X \to CX$ are morphisms of $\mathscr{G}_0$-spaces. Therefore the monad C in $\mathscr{T}$ restricts to a monad in the category $\mathscr{G}[\mathscr{T}_e]$.*

PROOF. By Definition VI.1.6(c′) and (a′) (applied to the degeneracies σ_i of [**45**, 2.3]), the ξ_k respect the equivalence relation used to define CX. The resulting maps $\xi_k \colon \mathscr{G}(k) \times (CX)^k \to CX$ satisfy the associativity, unit, and equivariance conditions required of an action by Definition VI.1.6(a), (b), and (c). The map $e \colon S^0 \to CX$ is the composite of $e \colon S^0 \to X$ and $\eta \colon X \to CX$. Now $CCX \in \mathscr{G}[\mathscr{T}_e]$ by iteration, and the maps μ and η commute with the actions ξ by Definition VI.1.6(a′) and (b′). $\square$

Definition VI.1.10. A $(\mathscr{C}, \mathscr{G})$-space (X, θ, ξ) is a $\mathscr{C}$-space (X, θ) and a $\mathscr{G}_0$-space (X, ξ) such that $\theta \colon CX \to X$ is a morphism of $\mathscr{G}_0$-spaces. Equivalently, it is required that the diagrams

$$
\begin{array}{ccc}
\mathscr{G}(k) \times \mathscr{C}(j_1) \times X^{j_1} \times \cdots \times \mathscr{C}(j_k) \times X^{j_k} & \xrightarrow{1 \times \theta_{j_1} \times \cdots \times \theta_{j_k}} & \mathscr{G}(k) \times X^k \\
{\scriptstyle \xi_k} \downarrow & & \downarrow {\scriptstyle \xi_k} \\
\mathscr{C}(j_1 \cdots j_k) \times X^{j_1 \cdots j_k} & \xrightarrow{\theta_{j_1 \cdots j_k}} & X
\end{array}
$$

commute. A map of $(\mathscr{C}, \mathscr{G})$-spaces is a map which is both a map of $\mathscr{C}$-spaces and a map of $\mathscr{G}_0$-spaces.

Since the notions of $\mathscr{C}$-space and of C-algebra in $\mathscr{T}$ are equivalent [**45**, 2.8], it follows immediately that the notions of $(\mathscr{C}, \mathscr{G})$-space and of C-algebra in $\mathscr{G}[\mathscr{T}_0]$ are equivalent.

Definition VI.1.11. An E_∞ ring space is a $(\mathscr{C}, \mathscr{G})$-space where $(\mathscr{C}, \mathscr{G})$ is some E_∞ operad pair.

We have not defined and do not need any notion of a morphism between E_∞ ring spaces over different E_∞ operad pairs.

2. Units; examples of operad pairs

We here point out a few consequences of the definitions and display some examples of actions by one operad on another; these should help to motivate our general theory.

For a space X, CX is the free $\mathscr{C}$-space generated by X. For the same categorical reason [**45**, 2.9], if X is a $\mathscr{G}_0$-space, then CX is the free $(\mathscr{C}, \mathscr{G})$-space generated by X. In other words, if $(\mathscr{C}, \mathscr{G})[\mathscr{T}_e]$ denotes the category of $(\mathscr{C}, \mathscr{G})$-spaces, then the function

$$\mathrm{Hom}_{\mathscr{G}[\mathscr{T}_e]}((X, \xi), (Y, \xi)) \longrightarrow \mathrm{Hom}_{(\mathscr{C}, \mathscr{G})[\mathscr{T}_e]}((CX, \mu, \xi), (Y, \theta, \xi))$$

obtained by sending $f \colon X \to Y$ to the composite

$$CX \xrightarrow{Cf} CY \xrightarrow{\theta} Y$$

is a natural isomorphism with inverse $g \to g \circ \eta$ for $g \colon CX \to Y$.

Definition VI.2.1. As observed in Lemma IV.1.5, (S^0, ξ) is a $\mathscr{G}_0$-space with

$$\xi_j(g)(\varepsilon_1, \ldots, \varepsilon_j) = \varepsilon_1 \cdots \varepsilon_j$$

for $g \in \mathscr{G}(j)$ and $\varepsilon_i = 0$ or 1 (where the product is the ordinary one), and $e \colon S^0 \to X$ is a morphism of $\mathscr{G}_0$-spaces for any $\mathscr{G}_0$-space X. For a $(\mathscr{C}, \mathscr{G})$-space X, let e also denote the induced map

$$(CS^0, \mu, \xi) \longrightarrow (\xi, \theta, \xi)$$

of $(\mathscr{C}, \mathscr{G})$-spaces; e is called the unit of X.

Of course, e is analogous to the unit $\mathbb{Z} \to R$ of a ring R.

If G denotes the monad associated to $\mathscr{G}$, then GX is the free $\mathscr{G}$-space generated by a space X. When $\mathscr{C}$ and $\mathscr{G}$ are E_∞ operads, the homology isomorphisms $CX \leftarrow (C \times G)(X) \to GX$ of [**45**, 3.10] and [**46**, A.2] show that C and G can be used interchangeably and thus both be thought of as additive. The $\mathscr{G}$-action on a $(\mathscr{C}, \mathscr{G})$-space gives rise to curious and useful exponential units.

Definition VI.2.2. Let (X, ξ) be a $\mathscr{G}$-space. For any given element $r \in \pi_0 X$, define a map $e_r \colon S^0 \to X$ by sending 0 to 1 and 1 to any chosen point in the component r. With S^0 regarded as a based space with basepoint 0, form GS^0 and let e_r also denote the induced map

$$(GS^0, \mu) \longrightarrow (X, \xi)$$

of $\mathscr{G}$-spaces; e_r is called an exponential unit of X. Up to homotopy through $\mathscr{G}$-maps, e_r is independent of the choice of the point in the component r.

Think of the set $\mathbb{Z}^+$ of non-negative integers as the free monoid (under addition) generated by the element 1. The maps e_r for a $(\mathscr{C}, \mathscr{G})$-space are analogous to the maps of monoids from $\mathbb{Z}^+$ into the underlying multiplicative monoid of a ring R.

There are several algebraic examples of operad pairs, for which the component spaces $\mathscr{C}(j)$ and $\mathscr{G}(j)$ are discrete. Let m and n denote the operads of [**45**, 3.1] such that an m-space is a topological monoid and an n-space is a commutative topological monoid. Recall that $m(j) = \Sigma_j$ with identity element e_j and that $n(j) = \{f_j\}$.

Lemma VI.2.3. (m, m) is an operad pair with respect to the maps $\lambda \colon \Sigma_k \times \Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \to \Sigma_{j_1 \cdots j_k}$ determined by

$$\lambda(e_k; e_{j_1}, \ldots, e_{j_k}) = e_{j_1 \cdots j_k}$$

and the equivariance formulas (c) and (c′) of Definition VI.1.3.

Indeed, the equivariance formulas are dictated by this lemma, which asserts that these formulas are compatible with the distributivity and unit formulas of Definition VI.1.6. An (m, m)-space is a topological "pseudo semi-ring," "pseudo" meaning that the addition need not be commutative and "semi" meaning that there need not be additive inverses.

Lemma VI.2.4. For any operad $\mathscr{G}$, $(n, \mathscr{G})$ is an operad pair with respect to the only possible maps λ, namely

$$\lambda(g; f_{j_1}, \ldots, f_{j_k}) = f_{j_1 \cdots j_k} \text{ for all } g \in \mathscr{G}(k).$$

An (n, m)-space is a topological semi-ring and an (n, n)-space is a commutative topological semi-ring. The unique maps $\mathscr{C} \to n$ and $\mathscr{G} \to n$ define a morphism $(\mathscr{C}, \mathscr{G}) \to (n, n)$ of operad pairs for any operad pair $(\mathscr{C}, \mathscr{G})$. Thus any commutative topological semi-ring R is a $(\mathscr{C}, \mathscr{G})$-space. As soon as we verify that E_∞ operad pairs exist, it will follow that any such R is an E_∞ ring space. Conversely, the set $\pi_0 X$ of components of an E_∞ ring space X will be a commutative semi-ring and the discretization map $X \to \pi_0 X$ will be a map of E_∞ ring spaces. Evidently, E_∞ semi-ring space would be a technically more accurate term than E_∞ ring space.

Recall that, algebraically, a commutative semi-ring can be completed to a commutative ring by formation of the Grothendieck group with respect to addition, the multiplication being carried

along automatically. Analogously, we shall wee in Chapter VII that an E_∞ ring space can be completed to an E_∞ ring space which is grouplike with respect to addition, the multiplicative E_∞ structure being carried along automatically. Indeed, this topological completion will induce the algebraic completion on π_0.

The definition of an E_∞ ring space implies the existence of considerable structure on the higher homotopy groups of such a space. The following lemma displays the most obvious bit of structure.

Lemma VI.2.5. *Let (X, θ, ξ) be a $(\mathscr{C}, \mathscr{G})$-space, where $\mathscr{C}$ and $\mathscr{G}$ are locally connected operads. Then the homotopy groups $\pi_* X$ defined with respect to the basepoint zero form a commutative graded semi-ring; if $\pi_0 X$ is a group under addition, then $\pi_* X$ is a ring.*

PROOF. Addition in $\pi_* X$ is induced by the additive product $\theta_2(c)$ for any $c \in \mathscr{C}(2)$. Since the multiplicative product $\xi_2(g)$ for any $g \in \mathscr{G}(2)$ factors through the smash product $X \wedge X$, it induces pairings $\pi_i X \otimes \pi_j X \to \pi_{i+j} X$ for all $i, j \geq 0$ by letting the product of $\alpha \colon S^i \to X$ and $\beta \colon S^j \to X$ be the composite

$$S^{i+j} = S^i \wedge S^j \xrightarrow{\alpha \wedge \beta} X \wedge X \xrightarrow{\xi_2(g)} X.$$

The axioms for a commutative semi-ring follow directly from Definitions VI.1.6 and VI.1.8 and the assumption that the $\mathscr{C}(j)$ and $\mathscr{G}(j)$ are connected (compare [**45**, p. 4]).   □

The simplest E_∞ operad pair, and the one suited to categorical applications, is derived from (m, m) by application of the product-preserving functor $|D_*(?)|$ from spaces to contractible spaces given by [**45**, 10.2 and 11.1]. Recall from [**45**, 15.1 and 10.3] that application of this functor to m gives an E_∞ operad $\mathcal{Q}$ such that $\mathcal{Q}(j)$ is just the normalized version of Milnor's universal Σ_j-bundle.

Lemma VI.2.6. *$(\mathcal{Q}, \mathcal{Q})$ is an E_∞ operad pair; the action λ of $\mathcal{Q}$ on itself is obtained by application of the functor $|D_*(?)|$ to the action of m on itself.*

PROOF. The formulas of Definition VI.1.6 can be written out as commutative diagrams, hence, by functoriality, these formulas hold for $\mathcal{Q}$ since they do so for m.   □

A categorical description of $(\mathcal{Q}, \mathcal{Q})$ will be given in Section VI§4. The following remarks will be needed in Section VII§3.

Remark VI.2.7.   (i) $\Sigma_j = D_0(\Sigma_j)$, and there results an inclusion of operads $m \subset \mathcal{Q}$ and an inclusion of operad pairs $(m, m) \subset (\mathcal{Q}, \mathcal{Q})$. Thus a $\mathcal{Q}$-space is also a topological monoid (with product $\oplus$) and a $(\mathcal{Q}, \mathcal{Q})$-space is also a topological pseudo semi-ring (with second product $\otimes$). The products $\oplus$ and $\otimes$ coincide with those given in terms of the actions as $\theta_2(e_2)$ and $\xi_2(e_2)$.

(ii) Ligaard and Madsen [**39**, 2.2] have verified that any $\mathcal{Q}$-space Y is a strongly homotopy commutative H-space with respect to the product $\oplus$. Therefore the classifying space BY is an H-space and the natural map $\zeta \colon Y \to \Omega BY$ is a group completion in the sense of [**46**, 2.1] (e.g., by [**47**, 15.1]).

(iii) It is sometimes convenient to replace general E_∞ spaces by equivalent $\mathcal{Q}$-spaces. This can be done as follows. Given the $\mathscr{C}$-space (X, θ), where $\mathscr{C}$ is an E_∞ operad, construct the maps

$$X \xleftarrow{\varepsilon(\theta\pi_1)} B(C \times D, C \times D, X) \xrightarrow{B(\pi_2, 1, 1)} B(D, C \times D, X).$$

Here the bar constructions are specified by [**45**, 9.6 and 1.11 (see p. 126)], $B(D, C \times D, X)$ is a $\mathcal{Q}$-space and both maps are morphisms of $\mathscr{C} \times \mathcal{Q}$-spaces by [**45**, 12.2], $\varepsilon(\theta\pi_1)$ is a strong deformation retraction with right inverse $\tau(\eta)$ by [**45**, 9.8 and 11.10], and $B(\pi_2, 1, 1)$ is a homotopy equivalence by [**46**, A.2(ii) and A.4(ii)].

(iv) For X as in (iii), let $GX = \Omega BB(D, C \times D, X)$ and let $g = \zeta \circ B(\pi_2, 1, 1) \circ \tau(\eta) \colon X \to GX$. By (ii), g is a natural group completion of the $\mathscr{C}$-space X. The existence of such a construction was asserted in [**46**, 2.1], but the argument given there was incomplete.

(v) In the second result labelled Theorem 3.7 in [**46**], I claimed that $\oplus$ on DX was a morphism of $\mathcal{Q}$-spaces. That assertion is clearly false, as it would imply that $\oplus$ is actually commutative. The mistake occurs in the formula for Y [**46**, p. 76], from which a factor $\sigma(j_1, \ldots, j_k)$ was omitted (compare Section VI§4).

(vi) Aside from use of (iv) in the proof of Theorem VII.3.1, we shall ignore the classifying spaces which result from the monoid structures on $\mathcal{Q}$-spaces and $(\mathcal{Q}, \mathcal{Q})$-spaces in favor of the deloopings constructed by application of the machinery of Chapter VII. The latter have numerous special properties essential to our theory, and I have not proven that the two are equivalent.

3. Symmetric bimonoidal and bipermutative categories

Categories with appropriate internal structure provide a very rich source of E_∞ spaces and E_∞ ring spaces. Here all categories with internal structure are to be small and topological, and all functors and natural transformations are to be continuous. For a category $\mathcal{a}$, $\mathcal{O}\mathcal{a}$ and $\mathcal{M}\mathcal{a}$ denote the spaces of objects and morphisms of $\mathcal{a}$, and S, T, I, and C denote the source, target, identity, and composition functions, all of which are required to be continuous. If no topology is in sight, we can always impose the discrete topology.

Recall that a symmetric monoidal category is a category $\mathcal{a}$ together with a functor $\square \colon \mathcal{a} \times \mathcal{a} \to \mathcal{a}$ and an object $*$ such that $\square$ is associative, (right) unital, and commutative up to coherent natural isomorphisms a, b, and c [**40**, VII§1 and §7]. $\mathcal{a}$ is permutative if $\square$ is strictly associative and unital, with no isomorphisms required. Coherence with the remaining piece of structure, the commutativity isomorphism c, is then guaranteed by commutativity of the following diagrams for $A, B, C \in \mathcal{O}\mathcal{a}$:

$$\begin{array}{ccc} A \square B \xrightarrow{\ I\ } A \square B & A = {*} \square A & A \square B \square C \xrightarrow{\ c\ } C \square A \square B \\ {}^{c}\searrow \quad \nearrow^{c} & {}_{I}\downarrow \quad \downarrow^{c} & {}_{I\square c}\searrow \quad \nearrow^{c\square I} \\ B \square A & A = A \square {*} & A \square C \square B \end{array}$$

, , and

Symmetric monoidal categories can be replaced functorially by naturally equivalent permutative categories, but the relevant notions of morphism require explanation. This is particularly so since the usual categorical definition of a coherent functor between symmetric monoidal categories would allow examples like the forgetful functor from modules over a commutative ring R under $\otimes_R$ to Abelian groups under $\otimes_\mathbb{Z}$ and is too lax for our purposes.

Definition VI.3.1. A morphism $\mathcal{a} \to \mathcal{a}'$ of symmetric monoidal categories is a functor $F \colon \mathcal{a} \to \mathcal{a}'$ such that $F{*} = {*}$ together with a natural isomorphism $\phi \colon FA \square FB \to F(A \square B)$ such that the following diagrams are commutative:

$$\begin{array}{ccccc} (FA \square FB) \square FC & \xrightarrow{\ a\ } & FA \square (FB \square FC) & \xrightarrow{1\square\phi} & FA \square F(B \square C) \\ {}_{\phi\square 1}\downarrow & & & & \downarrow^{\phi} \\ F(A \square B) \square FC & \xrightarrow{\ \phi\ } & F((A \square B) \square C) & \xrightarrow{Fa} & F(A \square (B \square C)) \end{array}$$

$$\begin{array}{ccc} FA \xrightarrow{\ b\ } FA \square F{*} & & FA \square FB \xrightarrow{\ c\ } FB \square FA \\ {}_{I}\downarrow \quad \downarrow^{\phi} & \text{and} & {}_{\phi}\downarrow \quad \downarrow^{\phi} \\ FA \xrightarrow{Fb} F(A \square {*}) & & F(A \square B) \xrightarrow{Fc} F(B \square A) \end{array}$$

A morphism $\mathcal{a} \to \mathcal{a}'$ of permutative categories is a functor $F \colon \mathcal{a} \to \mathcal{a}'$ such that $F{*} = {*}$, $FA \square FB = F(A \square B)$, and $c = Fc$ on $FA \square FB = F(A \square B)$. Note that a morphism of symmetric monoidal categories between permutative categories need not be a morphism of permutative categories.

A slight elaboration of the proof of [**46**, 4.2] gives the following more precise result.

Proposition VI.3.2. *There is a functor Φ from the category of symmetric monoidal categories to the category of permutative categories and a natural equivalence $\pi\colon \Phi\mathcal{a} \to \mathcal{a}$ of symmetric monoidal categories. If $\mathcal{a}$ is permutative, then π is a morphism of permutative categories.*

One often encounters categories with two symmetric monoidal structures, one additive and one multiplicative, which satisfy the (right) distributive and nullity of zero laws up to coherent natural isomorphisms d and n. We shall say that such a category is symmetric bimonoidal. Laplaza [**35**] has made careful study of such categories. In particular, he has given a list of diagrams the commutativity of which ensures that all further coherence diagrams which can reasonably be expected to commute do in fact commute. Comparison of his list with the notion of an E_∞ ring space leads to the following definition.

Definition VI.3.3. A bipermutative category $(\mathcal{a}, \oplus, 0, c, \otimes, 1, \widetilde{c})$ is a pair of permutative categories $(\mathcal{a}, \oplus, 0, c)$ and $(\mathcal{a}, \otimes, 1, \widetilde{c})$ such that the following three conditions are satisfied.

(i) $0 \otimes A = 0 = A \otimes 0$ for $A \in \mathcal{O}\mathcal{a}$ and $I(0) \otimes f = I(0) = f \otimes I(0)$ for $f \in \mathcal{M}\mathcal{a}$; that is, 0 is a strict two-sided zero object for $\otimes$.

(ii) The right distributivity law is strictly satisfied by objects and morphisms, and the following diagram commutes for $A, B, C \in \mathcal{O}\mathcal{a}$:

$$\begin{array}{ccc}
(A \oplus B) \otimes C & = & (A \otimes C) \oplus (B \otimes C) \\
{\scriptstyle c \otimes 1}\downarrow & & \downarrow{\scriptstyle c} \\
(B \oplus C) \otimes C & = & (B \otimes C) \oplus (A \otimes C)
\end{array} \quad .$$

(iii) Define a natural left distributivity isomorphism ℓ as the following composite

$$A \otimes (B \oplus C) \xrightarrow{\widetilde{c}} (B \oplus C) \otimes A = (A \otimes A) \oplus (C \otimes A) \xrightarrow{\widetilde{c} \times \widetilde{c}} (A \otimes B) \oplus (A \otimes C);$$

then the following diagram commutes for $A, B, C, D \in \mathcal{O}\mathcal{a}$:

$$\begin{array}{ccc}
(A \oplus B) \otimes (C \oplus D) & =\!=\!=\!=\!=\!=\!=\!=\!=\!=\!= & (A \otimes (C \oplus D)) \oplus (B \otimes (C \oplus D)) \\
{\scriptstyle \ell}\downarrow & & \downarrow{\scriptstyle \ell \oplus \ell} \\
((A \oplus B) \otimes C) & & (A \otimes C) \oplus (A \otimes D) \\
\oplus ((A \oplus B) \otimes D) & & \oplus (B \otimes C) \oplus (B \otimes D) \\
& \diagdown\diagdown \qquad \swarrow {\scriptstyle I \oplus c \oplus I} & \\
& (A \otimes C) \oplus (B \otimes C) & \\
& \oplus (A \otimes D) \oplus (B \otimes D) &
\end{array}$$

Laplaza's work [**35**, p. 40] implies that a bipermutative category is symmetric bimonoidal. In the absence of strict commutativity, it is clearly un reasonable to demand that both distributive laws hold strictly. The choice of which law to make strict is logically arbitrary, but our choice is dictated by consistency with the lexicographic ordering used in Notation VI.1.4.

Definition VI.3.4. A morphism $\mathcal{a} \to \mathcal{a}'$ of symmetric bimonoidal categories is a functor $F\colon \mathcal{a} \to \mathcal{a}'$ such that $F0 = 0$ and $F1 = 1$ together with natural isomorphisms $\phi\colon FA \oplus FB \to F(A \oplus B)$ and $\psi\colon FA \otimes FB \to F(A \otimes B)$ such that (F, ϕ) and (F, ψ) are morphisms of symmetric monoidal categories and the following diagrams are commutative:

$$\begin{array}{ccc}
F0 & \xrightarrow{n} & FA \otimes F0 \\
{\scriptstyle I}\downarrow & & \downarrow{\scriptstyle \psi} \\
F0 & \xrightarrow{Fn} & F(A \otimes 0)
\end{array} \quad \text{and} \quad
\begin{array}{ccc}
(FA \oplus FB) \otimes FC & \xrightarrow{d} & (FA \otimes FC) \oplus (FB \otimes FC) \\
{\scriptstyle \phi \otimes 1}\downarrow & & \downarrow{\scriptstyle \psi \oplus \psi} \\
F(A \oplus B) \otimes FC & & F(A \otimes C) \oplus F(B \otimes C) \\
{\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \phi} \\
F((A \oplus B) \otimes C) & \xrightarrow{Fd} & F((A \otimes C) \oplus (B \otimes C))
\end{array}$$

A morphism $\mathscr{a} \to \mathscr{a}'$ of bipermutative categories is a functor $F \colon \mathscr{a} \to \mathscr{a}'$ which is a morphism of permutative categories with respect to both the additive and the multiplicative structures. Again, a morphism of symmetric bimonoidal categories between bipermutative categories need not be a morphism of bipermutative categories.

Proposition VI.3.5. *There is a functor Φ from the category of symmetric bimonoidal categories to the category of bipermutative categories and a natural equivalence $\pi \colon \Phi\mathscr{a} \to \mathscr{a}$ of symmetric bimonoidal categories. If $\mathscr{a}$ is bipermutative, then π is a morphism of bipermutative categories.*

PROOF. To avoid technical topological difficulties, assume either that $\mathscr{O}\mathscr{a}$ is discrete (which is the case in practice) or that 0 and 1 are non-degenerate basepoints such that 0 is a strict unit for $\oplus$ and 1 is a strict unit for $\otimes$. The latter condition can always be arranged by growing whiskers on the given 0 and 1 (by adjoining copies of the category $\mathscr{I}$ with two objects and one non-identity morphism) so as to obtain a new 0 and 1 as required. We construct $\mathscr{b} = \Phi\mathscr{a}$ as follows. Let $(\mathscr{O}\mathscr{a})'$ be the free topological monoid, with product denoted by $\boxtimes$, generated by $\mathscr{O}\mathscr{a}$ modulo the relations $e = 1$ and $0 \boxtimes A = 0 = A \boxtimes 0$ for all $A \in \mathscr{O}\mathscr{a}$. Let $\mathscr{O}\mathscr{b}$ be the free topological monoid, with product denoted by $\boxplus$, generated by $(\mathscr{O}\mathscr{a})'$ modulo the relation $e = 0$. Extend the product $\boxtimes$ from $(\mathscr{O}\mathscr{a})'$ to all of $\mathscr{O}\mathscr{b}$ by the formula

$$(A_1 \boxplus \cdots \boxplus A_m) \boxtimes (B_1 \boxplus \cdots \boxplus B_n)$$
$$= (A_1 \boxtimes B_1) \boxplus \cdots \boxplus (A_1 \boxtimes B_n) \boxplus \cdots \boxplus (A_m \boxtimes B_1) \boxplus \cdots \boxplus (A_m \boxtimes B_n)$$

for $A_i, B_j \in (\mathscr{O}\mathscr{a})'$. Both product on $\mathscr{O}\mathscr{b}$ are associative, 0 is a strict unit for $\boxplus$ and zero for $\boxtimes$, 1 is a strict unit for $\boxtimes$, and the right distributive law holds. Let $\eta \colon \mathscr{O}\mathscr{a} \to \mathscr{O}\mathscr{b}$ denote the evident inclusion. Define $\pi \colon \mathscr{O}\mathscr{b} \to \mathscr{O}\mathscr{a}$ by $\pi(0) = 0$, $\pi(1) = 1$,

$$\pi(A_1 \boxtimes \cdots \boxtimes A_n) = A_1 \otimes (A_2 \otimes (A_3 \otimes \cdots (A_{n-1} \otimes A_n) \cdots))$$

for $A_i \in \mathscr{O}\mathscr{a}$, $A_i \neq 0$ and $A_i \neq 1$, and

$$\pi(A_1' \boxtimes \cdots \boxtimes A_n') = \pi A_1' \oplus (\pi A_2' \oplus (\pi A_3' \oplus \cdots (\pi A_{n-1}' \oplus \pi A_n') \cdots))$$

for $A_i' \in (\mathscr{O}\mathscr{a})'$, $A_i' \neq 0$. Define $\mathscr{M}\mathscr{B}$ by

$$\mathscr{b}(B, B') = \{B\} \times \mathscr{a}(\pi B, \pi B') \times \{B'\}.$$

The singleton sets determine S and T for $\mathscr{B}$, and I and C are induced from the corresponding functions for $\mathscr{a}$. $\mathscr{M}\mathscr{B}$ is topologized as a subspace of $\mathscr{O}\mathscr{b} \times \mathscr{M}\mathscr{a} \times \mathscr{O}\mathscr{b}$. The products $\boxplus$ and $\boxtimes$ on $\mathscr{M}\mathscr{B}$ and the symmetries c and $\widetilde{c}$ of $\mathscr{b}$ are determined by the following arrows of $\mathscr{a}$:

$$\pi(B \boxplus C) \xrightarrow{\cong} \pi B \oplus \pi C \xrightarrow{f \oplus g} \pi B' \oplus \pi C' \xrightarrow{\cong} \pi(B' \boxplus C')$$

$$\pi(B \boxtimes C) \xrightarrow{\cong} \pi B \otimes \pi C \xrightarrow{f \otimes g} \pi B' \otimes \pi C' \xrightarrow{\cong} \pi(B' \boxtimes C')$$

for morphisms (B, f, B') and (C, g, C') of $\mathscr{b}$ and

$$\pi(B \boxplus C) \xrightarrow{\cong} \pi B \oplus \pi C \xrightarrow{c} \pi C \oplus \pi B \xrightarrow{\cong} \pi(C \boxplus B)$$

$$\pi(B \boxtimes C) \xrightarrow{\cong} \pi B \otimes \pi C \xrightarrow{\widetilde{c}} \pi C \otimes \pi B \xrightarrow{\cong} \pi(C \boxtimes B)$$

for objects B and C of $\mathscr{b}$; the unlabelled isomorphisms are uniquely determined by the monoidal structures of $\mathscr{a}$. Define $\eta \colon \mathscr{M}\mathscr{a} \to \mathscr{M}\mathscr{B}$ by $\eta(f) = (A, f, A')$ for $f \colon A \to A'$ and define $\pi \colon \mathscr{M}\mathscr{B} \to \mathscr{M}\mathscr{a}$ by $\pi(B, g, B') = g$ for $g \colon \pi B \to \pi B'$. Then η and π are functors, $\pi\eta$ is the identity functor, and the morphisms $(B, I\pi B, \eta\pi B)$ of $\mathscr{b}$ define a natural isomorphism between $\eta\pi$ and the identity functor of $\mathscr{b}$. The remaining verifications are equally straightforward. $\qquad\square$

4. Bipermutative categories and E_∞ ring spaces

We here describe the E_∞ operad pair $(\mathcal{Q}, \mathcal{Q})$ categorically, review the passage from permutative categories to $\mathcal{Q}$-spaces obtained in [46, §4], and construct a functor from the category of bipermutative categories to the category of $(\mathcal{Q}, \mathcal{Q})$-spaces.

Recall tat the translation category $\widetilde{G}$ of a monoid G has objects the elements of G and morphisms from g' to g'' those elements $g \in G$ such that $g'g = g''$. When G is a group, g is unique and a functor with range $\widetilde{G}$ is therefore uniquely determined by its object function. Note that G acts from the right on $\widetilde{G}$ via the product of G.

Let $\widetilde{\gamma} \colon \widetilde{\Sigma}_k \times \widetilde{\Sigma}_{j_1} \times \cdots \times \widetilde{\Sigma}_{j_k} \to \widetilde{\Sigma}_{j_1 + \cdots + j_k}$ be the functor defined on objects by the formula

$$(\text{VI.1}) \qquad \widetilde{\gamma}(\sigma; \tau_1, \ldots, \tau_k) = (\tau_{\sigma^{-1}(1)} \oplus \cdots \oplus \tau_{\sigma^{-1}(k)}) \sigma(j_1, \ldots, j_k).$$

(The factor $\sigma(j_1, \ldots, j_k)$ was inadvertently omitted from the definition of $\widetilde{\gamma}$ given in [46, p. 82].)

Let $\widetilde{\lambda} \colon \widetilde{\Sigma}_k \times \widetilde{\Sigma}_{j_1} \times \cdots \times \widetilde{\Sigma}_{j_k} \to \widetilde{\Sigma}_{j_1 \cdots j_k}$ be the functor defined on objects by the formula

$$(\text{VI.2}) \qquad \widetilde{\lambda}(\sigma; \tau_1, \ldots, \tau_k) = (\tau_{\sigma^{-1}(1)} \otimes \cdots \otimes \tau_{\sigma^{-1}(k)}) \sigma \langle j_1, \ldots, j_k \rangle.$$

Let $B\mathcal{a}$ denote the classifying space of a (small, topological) category $\mathcal{a}$ and recall that B is a product-preserving functor from categories to spaces (e.g. [46, 4.6]). As observed in [46, 4.7], $B\widetilde{G}$ coincides with $|D_* G|$ for any topological group G. By comparison of (VI.1) to the equivariance formulas in Definition VI.1.2, the structural maps γ of the E_∞ operad $\mathcal{Q}$ coincides with the maps

$$B\widetilde{\gamma} \colon B\widetilde{\Sigma}_k \times B\widetilde{\Sigma}_{j_1} \times \cdots \times B\widetilde{\Sigma}_{j_k} \longrightarrow B\widetilde{\Sigma}_{j_1 + \cdots + j_k}.$$

By comparison of (VI.2) to the equivariance formulas in Definition VI.1.6, the maps λ which give the action of $\mathcal{Q}$ on itself coincide with the maps

$$B\widetilde{\Lambda} \colon B\widetilde{\Sigma}_k \times B\widetilde{\Sigma}_{j_1} \times \cdots \times B\widetilde{\Sigma}_{j_k} \longrightarrow B\widetilde{\Sigma}_{j_1 \cdots j_k}.$$

Alternatively, this description can be used to define the E_∞ operad pair $(\mathcal{Q}, \mathcal{Q})$.

Let $(\mathcal{a}, \square, *, c)$ be a permutative category. As pointed out in [46, p. 81], c determines Σ_j-equivariant functors

$$c_j \colon \widetilde{\Sigma}_j \times \mathcal{a}^j \longrightarrow \mathcal{a}$$

such that c_j restricts to the j-fold iterate of $\square$ on $\{e_j\} \times \mathcal{a}^j = \mathcal{a}^j$. The coherence diagrams of the previous section imply the following result. Indeed, by the very meaning of coherence, we need only observe that the diagram of the lemma makes sense on objects.

Lemma VI.4.1. *The following diagram is commutative for all $j \geq 0$, $k \geq 0$, and $j_i \geq 0$ such that $j_1 + \cdots + j_k = j$:*

where μ is the evident shuffle isomorphism of categories.

Comparison of the lemma to Definition VI.1.3 gives the following consequence.

Proposition VI.4.2. *Define $\theta_j = Bc_j \colon \mathcal{Q}(j) \times (B\mathcal{a})^j = B(\widetilde{\Sigma}_j \times \mathcal{a}^j) \to B\mathcal{a}$. Then the θ_j define an action θ of $\mathcal{Q}$ on $B\mathcal{a}$, and B restricts to a functor from permutative categories to $\mathcal{Q}$-spaces.*

Now let $(\mathcal{a}, \oplus, 0, c, \otimes, 1, \widetilde{c})$ be a bipermutative category. Then coherence implies the following analog of Lemma VI.4.1.

Lemma VI.4.3. *The following diagram is commutative for all $j \geq 0$, $k \geq 0$, and $j_i \geq 0$ such that $j_1 \cdots j_k = j$:*

$$
\begin{array}{ccc}
\widetilde{\Sigma}_k \times \widetilde{\Sigma}_{j_1} \times a^{j_1} \times \cdots \times \widetilde{\Sigma}_{j_k} \times a^{j_k} & \xrightarrow{1 \times c_{j_1} \times \cdots \times c_{j_k}} & \widetilde{\Sigma}_k \times a^k \\
\Big\downarrow{\scriptstyle\omega} & & \\
\widetilde{\Sigma}_k \times \widetilde{\Sigma}_{j_1} \times \cdots \times \widetilde{\Sigma}_{j_k} \times (\widetilde{\Sigma}_k \times a^k)^j & \xrightarrow{\widetilde{\lambda} \times \widetilde{c}_k^j} & \widetilde{\Sigma}_j \times a^j
\end{array}
\xrightarrow[\widetilde{c}_j]{\widetilde{c}_k} a \; ,
$$

where ω is defined on objects and morphisms by the formula

$$
\omega(\sigma, \tau_1, y_1, \ldots, \tau_k, y_k) = \left(\sigma, \tau_1, \ldots, \tau_k, \prod_{I \in S(j_1, \ldots, j_k)} (\sigma, y_I) \right).
$$

Comparison of the lemma to Definitions VI.1.6 and VI.1.7 gives the following consequence.

Proposition VI.4.4. *The actions θ and ξ of $\mathcal{Q}$ on Ba specified by $\theta_j = Bc_j$ and $\xi_j = B\widetilde{c}_j$ give an action of the E_∞ operad pair $(\mathcal{Q}, \mathcal{Q})$ on Ba, and B restricts to a functor from bipermutative categories to $(\mathcal{Q}, \mathcal{Q})$-spaces.*

The following addendum is sometimes useful.

Remark VI.4.5. For $i \in \mathcal{O}a$, let a_i denote the subcategory of a which contains the unique object i and all morphisms from i to i; Ba_i is the ordinary classifying space if this monoid of morphisms. Clearly (Ba_0, θ) is a sub $\mathcal{Q}$-space of (Ba, θ) and (Ba_1, ξ) is a sub $\mathcal{Q}$-space of (Ba, ξ).

There is a more general way of looking at the constructions above. One can think of $\{ D_* \Sigma_j \mid j \geq 0 \}$ as specifying an operad $\mathcal{Q}_*$ in the category of simplicial spaces (or sets, since $D_q \Sigma_j$ are discrete). The actions of $\mathcal{Q}$ on Ba result by passage to geometric realization [**45**, §11] from actions of $\mathcal{Q}_*$ on the simplicial spaces $B_* a$ [**46**, 4.6]. Although no such examples will be studied in this volume, there exist simplicial spaces with actions by $\mathcal{Q}_*$ or by the pair $(\mathcal{Q}_*, \mathcal{Q}_*)$ which are not of the form $B_* a$ for any a; clearly our theory will apply to their realizations.

5. Examples of bipermutative categories

The seminal example, which will map into all others, is the following one.

Example VI.5.1. Let $\mathcal{E}$ denote the category of finite sets $n \geq 0$ and their isomorphisms. Think n as $\{1, 2, \ldots, n\}$ and identify $\mathcal{E}(n, n)$ with the symmetric group Σ_n. Then $(\mathcal{E}, \oplus, 0, c, \otimes, 1, \widetilde{c})$ is a bipermutative category, where $\oplus$ and $\otimes$ are defined on objects and morphisms by

$$
m \oplus n = m + n \quad \text{and} \quad (\sigma \oplus \tau)(i) = \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq m \\ m + \tau(i - m) & \text{if } m < i \leq m + n \end{cases}
$$

and

$$
m \otimes n = mn \quad \text{and} \quad (\sigma \otimes \tau)((i-1)n + j) = ((\sigma(i) - 1)n + \tau(j)), \; 1 \leq i \leq m \text{ and } 1 \leq j \leq n,
$$

and where $c = c(m, n) \in \Sigma_{m+n}$ and $\widetilde{c} = \widetilde{c}(m, n) \in \Sigma_{mn}$ are defined by

$$
c(i) = \begin{cases} n + i & \text{if } 1 \leq i \leq m \\ i - m & \text{if } m < i \leq m + n \end{cases}
$$

and

$$
\widetilde{c}((i-1)n + j) = (j-1)m + i, \; 1 \leq i \leq m \text{ and } 1 \leq j \leq n.
$$

For $\otimes$, mn should be thought of as $\{(1,1), \ldots, (1,n), \ldots, (m,1), \ldots, (m,n)\}$ and it is this choice of order (required for consistency with Notation VI.1.4) which leads to the strict right, rather than left,

distributive law. The space DS^0 is the disjoint union of the orbit spaces $\mathcal{Q}(j)/\Sigma_j$ [**45**, 8.11], and the unit $e\colon DS^0 \to B\mathscr{E}$ of Definition VI.2.1 coincides with the disjoint union of the homeomorphisms

$$|D_*\Sigma_j|/\Sigma_j \longrightarrow E\Sigma_j/\Sigma_j = B\Sigma_j$$

specified in [**45**, 10.3]. (Alternatively, use $E\Sigma_j = B\widetilde{\Sigma}_j$.) We therefore regard e as an identification and conclude that $B\mathscr{E}$ is the free $(\mathcal{Q}, \mathcal{Q})$-space generated by S^0.

For a topological ring A, the groups $K_i A$ for $i > 0$ can be defined in terms of permutative categories $\mathscr{P}A$ or $\mathscr{F}A$ of finitely generated projective or free left A-modules (as will be discussed in Section VIII§1). When A is commutative, $\mathscr{P}A$ and $\mathscr{F}A$ can be taken as bipermutative categories. In case of $\mathscr{F}A$, we can be more explicit.

Example VI.5.2. Define $\mathscr{GL}A$ as follows. The objects of $\mathscr{GL}A$ are the non-negative integers, each though of as A^n together with its standard ordered basis $\{e_1, \ldots, e_n\}$. The morphisms of $\mathscr{GL}A$ are the isomorphisms $A^n \to A^n$. Thus $\mathscr{GL}A(m, n)$ is empty if $m \neq n$ and $\mathscr{GL}A(n, n) = GL(n, A)$. Define a functor $e\colon \mathscr{E} \to \mathscr{GL}A$ by $e(n) = n$ on objects and $e(\sigma)(e_i) = e_{\sigma^{-1}(i)}$ on morphisms $\sigma \in \Sigma_n$. Then $(\mathscr{GL}A, \oplus, 0, c)$ is a permutative category and, if A is commutative, $(\mathscr{GL}A, \oplus, 0, c, \otimes, 1, \widetilde{c})$ is a bipermutative category, where $\oplus$, $\otimes$, c, and $\widetilde{c}$ are specified by the requirements that $\oplus$ and $\otimes$ have their usual meanings (with respect to the isomorphisms $A^m \oplus A^n \to A^{m+n}$ and $A^m \otimes A^n \to A^{mn}$ defined as usual on ordered bases) and that $e\colon \mathscr{E} \to \mathscr{GL}A$ be a morphism of permutative and, if A is commutative, bipermutative categories. Note that e factors through $\mathscr{GL}\mathbb{Z}$ for any A. By the naturality of the unit of $(\mathcal{Q}, \mathcal{Q})$-spaces, $Be\colon B\mathscr{E} \to B\mathscr{GL}A$ coincides (under the identification $DS^0 = B\mathscr{E}$) with the unit of $B\mathscr{GL}A$. We identify morphisms of $\mathscr{GL}A$ with their matrices with respect to the standard bases. For a morphism of rings $\alpha\colon A \to A'$, define $\mathscr{GL}\alpha\colon \mathscr{GL}A \to \mathscr{GL}A'$ by applying α to all entries of matrices. Then $\mathscr{GL}$ is a functor from rings to permutative categories and from commutative rings to bipermutative categories.

Example VI.5.3. If A is commutative, define $\mathcal{O}A$ to be the sub bipermutative category of $\mathscr{GL}A$ whose morphisms are the orthogonal matrices ($MM^t = I$). Then $\mathcal{O}$ is also a functor from commutative rings to bipermutative categories. Although we cannot simply restrict morphisms to matrices of determinant one, since the permutation matrices required to define c and $\widetilde{c}$ would no longer be present, we can obtain a sub permutative category $\mathscr{SGL}$ of $\mathscr{GL}$ under $\oplus$ by restricting objects to the even numbers and restricting morphisms to the elements of $SGL(2n, A)$. Similarly, define $\mathcal{SO} = \mathcal{O} \cap \mathscr{SGL}$.

In the examples above, the set of $n \times n$-matrices with entries in A is to be given the obvious product topology and $GL(n, A)$ and $O(n, A)$ are to be given the subspace topologies. We have insisted that rings be topologized in order to treat algebraic and topological K-theory simultaneously.

Example VI.5.4. Let $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ be the (topologized) real numbers, complex numbers, and quaternions. Define subcategories $\mathcal{O} = \mathcal{O}\mathbb{R}$, $\mathscr{U}$ and $\mathscr{Sp}$ of $\mathscr{GL}\mathbb{R}$, $\mathscr{GL}\mathbb{C}$, and $\mathscr{GL}\mathbb{H}$ by restricting to orthogonal, unitary and symplectic linear transformations, respectively. Then $\mathcal{O}$ and $\mathscr{U}$ are bipermutative categories, and complexification $\mathcal{O} \to \mathscr{U}$ are bipermutative categories. $\mathscr{Sp}$ is an (additive) permutative category and symplectification $\mathscr{U} \to \mathscr{Sp}$ is a morphism of permutative categories. When appropriately specified on bases, the forgetful functors $\mathscr{Sp} \to \mathscr{U}$ and $\mathscr{U} \to \mathcal{O}$ are morphisms of (additive) permutative categories, with object functions which send n to $2n$.

The following three examples, whose significance was first understood by Quillen [**58**, **59**] and Tornehave [**75**, **77**], are central to the interplay between algebraic and topological K-theory to be discussed in Chapter VIII.

Example VI.5.5. For a perfect field k of characteristic $q \neq 0$, let

$$\phi^q\colon \mathscr{GL}k \longrightarrow \mathscr{GL}k \quad \text{and} \quad \phi^q\colon \mathcal{O}k \longrightarrow \mathcal{O}k$$

denote the morphisms of bipermutative categories derived from the Frobenius automorphism $x \to x^q$ of k. For $r = q^a$, let ϕ^r be the a-fold iterate of ϕ^q. This example is most interesting when k is the algebraic closure of the field of q elements.

Example VI.5.6. Let $r = q^a$ where q is a prime and $a \geq 1$. Let k_r be the field with r elements. Define a forgetful functor $f \colon \mathscr{GL}k_r \to \mathscr{E}$ by letting $f(n) = r^n$ on objects and letting $f(\tau)$ be τ regarded as a permutation of the set k_r^n of r^n letters on morphisms $\tau \in GL(n, k_r)$. Of course, f depends on the chosen isomorphism of sets from k_r^n to $1, 2, \ldots, r^n$. With the obvious lexicographic choice, f gives an exponential morphism of permutative categories

$$f \colon (\mathscr{GL}k_r, \oplus, 0, c) \longrightarrow (\mathscr{E}, \otimes, 1, \widetilde{c}).$$

Moreover, the composite morphism of $\mathcal{Q}$-spaces

$$(DS^0, \mu) = (B\xi, \theta) \xrightarrow{Be} (B\mathscr{GL}k_r, \theta) \xrightarrow{Bf} (B\mathscr{E}, \xi) = (DS^0, \xi)$$

coincides with the exponential unit e_r defined in Definition VI.2.2 since $B(fe)$ sends 0 to 1 and 1 to a point in the component $\mathcal{Q}(r)/\Sigma_r$ of DS^0. This works equally well with $\mathscr{GL}k_r$ replaced by $\mathcal{O}k_r$.

Example VI.5.7. Let k be a field of characteristic $\neq 0$. $O(n, k)$ consists of the isometries with respect to the bilinear form B associated to the standard quadratic map $Q \colon k^n \to K$, $Q(x_1, \ldots, x_n) = \sum x_i^2$ [**51**, p. 84]. Recall from [**51**, p. 137] that the spinor form $\nu \colon O(n, k) \to \mathring{k}/\mathring{k}^2$ is defined by

$$\nu(\tau) = Q(z_1) \cdots Q(z_r) \text{ if } \tau = \tau_{z_1} \cdots \tau_{z_r} \in O(n, k).$$

Here $\tau_y(x) = x - [2B(x, y)/Q(y)]y$ for $x, y \in k^n$ with $y \neq 0$. Every τ is a product of such symmetries [**51**, p. 102] and, modulo squares, $\nu(\tau)$ is independent of the choice of factorization. If $y = e_i - e_j$, then τ_y permutes e_i and e_j and $\nu(\tau_y) = 2$. Now specialize to $k = k_3$. Then ν takes values in $\mathbb{Z}_2$ and $\nu(\sigma) \det(\sigma) = 1$ for $\sigma \in \Sigma_n \subset O(n, k_3)$. The subcategory $\boldsymbol{n}k_3$ of $\mathcal{O}k_3$ whose morphisms $n \to n$ are those $\tau \in O(n, k_3)$ such that $\nu(\tau) \det(\tau) = 1$ is a sub bipermutative category since ν and det are given by formulas of the same form on direct sums and tensor products. Again, Example VI.5.6 works equally well with $\mathscr{GL}k_3$ replaced by $\boldsymbol{n}k_3$.

We have only listed examples to which we shall refer in Chapter VIII. As pointed out by Swan [unpublished], all of our examples, and many others, can be subsumed within a general framework of systems of groups $G(n)$ for $n \geq 0$ together with homomorphisms $\Sigma_n \to G(n)$, $G(m) \times G(n) \to G(m+n)$, and, for the bipermutative case, $G(m) \times G(n) \to G(mn)$ subject to the appropriate axioms. The following remarks, which apply to any such example, describe the action maps

$$\theta_p \colon \mathcal{Q}(p) \times_{\Sigma_p} BG(n)^p \longrightarrow BG(pn) \quad \text{and} \quad \xi_p \colon \mathcal{Q}(p) \times_{\Sigma_p} BG(n)^p \longrightarrow BG(n^p)$$

solely in terms of homomorphisms of groups. When p is a prime, the induced maps of mod p homology determine operations on $H_* \left(\coprod_{n \geq 0} BG(n); \mathbb{Z}_p \right)$ [**26**, I], and the computation of these operations is thus reduced to the homological analysis of appropriate representations.

Remark VI.5.8. Let $(\boldsymbol{a}, \oplus, 0, c)$ be a permutative category with objects $\{n \mid n \geq 0\}$ and with morphisms from n to n forming a topological group $G(n)$ which contains Σ_n. Recall that the wreath product $\Sigma_p \wr G(n)$ is the semi-direct product of Σ_p and $G(n)^p$ determined by the evident action of Σ_p on $G(n)^p$. If we regard $\Sigma_p \wr G(n)^p$ and $G(n)$ as categories with a single object, then $\Sigma_p \wr G(n)$ is the orbit category $\widetilde{\Sigma}_p \times_{\Sigma_p} G(n)^p$ of $\widetilde{\Sigma}_p \times G(n)^p$. The functor $c_p \colon \widetilde{\Sigma}_p \times G(n)^p \to G(pn)$ factors through the homomorphism $\Sigma_p \wr G(n) \to G(pn)$ specified by

$$(\sigma; g_1, \ldots, g_p) \longrightarrow \sigma(n, \ldots, n)(g_1 \oplus \cdots \oplus g_p) = (g_{\sigma^{-1}(1)} \oplus \cdots \oplus g_{\sigma^{-1}(p)})\sigma(n, \ldots, n).$$

Application of the classifying space functor B thus gives the commutative diagram

$$
\begin{array}{ccc}
\mathcal{Q}(p) \times BG(n)^p & \cong & B(\widetilde{\Sigma}_p \times G(n)^p) \\
\downarrow & & \downarrow \\
\mathcal{Q}(p) \times_{\Sigma_p} BG(n)^p & \cong & B(\Sigma_p \wr G(n))
\end{array}
$$

with maps Bc_p and θ_p to $BG(pn)$.

If, further, $(\mathscr{a}, \oplus, 0, c, \otimes, 1, \widetilde{c})$ is bipermutative, then the functor $\widetilde{c}_p \colon \widetilde{\Sigma}_p \times G(n)^p \to G(n)^p$ factors through the homomorphism $\Sigma_p \wr G(n) \to G(n^p)$ specified by

$$(\sigma, g_1, \ldots, g_p) \to \sigma \langle n, \ldots, n \rangle (g_1 \otimes \cdots \otimes g_p)$$

$$= (g_{\sigma^{-1}(1)} \otimes \cdots \otimes g_{\sigma^{-1}(p)}) \sigma \langle n, \ldots, n \rangle$$

and application of B gives the commutative diagram

$$
\begin{array}{ccc}
\mathcal{Q}(p) \times BG(n)^p & \cong & B(\widetilde{\Sigma}_p \times G(n)^p) \\
\downarrow & & \downarrow \\
\mathcal{Q}(p) \times_{\Sigma_p} BG(n)^p & \cong & B(\Sigma_p \wr G(n))
\end{array}
$$

with maps $B\widetilde{c}_p$ and ξ_p to $BG(n^p)$.

We use these remarks to determine $B\mathcal{O}\mathbb{Z}$. The following observations are due to Z. Fiedorowicz.

Remark VI.5.9. A moment's reflection will convince the reader that, for an integer valued matrix M, $MM^t = I$ if and only if each row column of M has precisely one non-zero entry and that entry is ± 1. Indeed, the natural homomorphism $\Sigma_p \wr O(1, \mathbb{Z}) \to O(p, \mathbb{Z})$ is an isomorphism for all p. Abbreviate $\pi = O(1, \mathbb{Z})$ and regard $B\pi^+$, the union of $B\pi \simeq \mathbb{R}\mathrm{P}^\infty$ and a disjoint basepoint 0, to be the sub $\mathcal{Q}_0$-space $BO(0, \mathbb{Z}) \amalg BO(1, \mathbb{Z})$ of $B\mathcal{O}\mathbb{Z}$. $D(B\pi^+)$ is the free $(\mathcal{Q}, \mathcal{Q})$-space generated by $B\pi^+$, and there results a map $D(B\pi^+) \to B\mathcal{O}\mathbb{Z}$ of $(\mathcal{Q}, \mathcal{Q})$-spaces. In fact, this map is just the identification (compare [**45**, 2.4] or [**26**, I.5.7])

$$\coprod \theta_p \colon D(B\pi^+) \cong \coprod \mathcal{Q}(p) \times_{\Sigma_p} BO(1, \mathbb{Z})^p \longrightarrow \coprod BO(p, \mathbb{Z}) = B\mathcal{O}\mathbb{Z}.$$

The functor $e \colon \mathscr{E} \to \mathcal{O}\mathbb{Z}$ gives rise under B to the injection $DS^0 \to D(B\pi^+)$ determined by the points 0 and 1 of $B\pi^+$.

The recognition principle for E_∞ ring spaces

The zero$^{\text{th}}$ space of a spectrum is an E_∞ space, and an E_∞ space determines a spectrum and therefore a cohomology theory. If ΓX denotes the zero$^{\text{th}}$ space of the spectrum associated to an E_∞ space X, then there is a map $\iota\colon X \to \Gamma X$ which respects the E_∞ structure and is a group completion, in the sense that $\iota_*\colon H_*(X;\Bbbk) \to H_*(\Gamma X;\Bbbk)$ is a localization of the (Pontryagin) ring $H_*(X;\Bbbk)$ at its submonoid $\pi_0 X$ for every commutative coefficient ring $\Bbbk$. (See [**46**, §1] for a discussion of this definition; the letter Γ is chosen as a reminder of the group completion property and has nothing to do with the use of this letter in the other theories of infinite loop spaces.) Here spectra are to the understood in the coordinate-free sense introduced in Chapter II, and the results of [**45**, **46**] just summarized will be recast in terms of such spectra in Section VII§3.

In Chapter IV, Frank Quinn, Nigel Ray, and I introduced the notion of an E_∞ ring spectrum. In Section VII§2, the zero$^{\text{th}}$ space of such a spectrum will be shown to be an E_∞ ring space. The proof requires use of the little convex bodies operads $\mathscr{K}_V$ introduced in Section VII§1; the essential feature of $\mathscr{K}_V$ is that the orthogonal group OV acts on it. In Section VII§4, the spectrum determined by the additive E_∞ structure of an E_∞ ring space will be shown to be an E_∞ ring spectrum and it will be proven that, for an E_∞ ring space X, $\iota\colon X \to \Gamma X$ respects both E_∞ space structures. In effect, this means that the multiplicative E_∞ structure is preserved on passage from the additive E_∞ structure to its associated spectrum. As a special case of more general results, we shall see that if X is the disjoint union $\coprod K(\Sigma_j, 1)$, then ΓX is equivalent as an E_∞ ring space to QS^0. This result is a multiplicative elaboration of the Barratt–Quillen theorem [**16**, **68**] [**46**, §3].

The component $\Gamma_1 X$ of the identity element of ΓX is a multiplicative E_∞ space. We prove in Section VII§5 that, under mild hypotheses, the localization of $\Gamma_1 X$ at any submonoid M of the positive integers is equivalent as an infinite loop space to the component of the identity element of the zero$^{\text{th}}$ space of the spectrum derived from the multiplicative E_∞ space structure on a certain subspace X_M of X. In other words, although $\Gamma_1 X$ is constructed by use of the additive E_∞ space structure on X, its localizations depend only on the multiplicative E_∞ space structure. In earlier approaches to multiplicative structures, only the localized infinite loop spaces were visible because there was no way to handle the additive and multiplicative structures in combination. In particular, the main result of Tornehave's paper [**76**], which describes localizations of SF in terms of the symmetric groups, will drop out as a special case by use of our version of the Barratt–Quillen theorem.

The essential results of this chapter were obtained in 1972 and presented in lectures during the winter of 1973. I mention this since at least one other author has since announced his intention of developing a similar theory.

1. The little convex bodies operads

The little cubes operad $\mathscr{C}_n$ of [**45**, §4] played a canonical role in the passage from E_∞ spaces to spectra. Indeed, as explained in [**45**, p. 153–155], the geometry given by the action of any E_∞ operad on a space was automatically transformed into the little cubes geometry on the derived infinite loop space. We need a canonical E_∞ operad pair in order to obtain the analogous (but considerably more delicate) passage from E_∞ ring spaces to E_∞ ring spectra. From the definition of E_∞ ring spectra in Chapter IV, it is clear that the linear isometries operad $\mathscr{L}$ of Definition I.1.2

must be chosen as the canonical operad for the multiplicative structure. We require an operad $\mathscr{K}_\infty$ on which $\mathscr{L}$ acts and which can be used interchangeably with $\mathscr{C}_\infty$ in the additive, or one operad, theory.

Recall the definitions of $\mathscr{I}$ and $\mathscr{I}_*$ from Definitions I.1.1 and I.1.8. Let $\mathscr{I}^+$ and $\mathscr{I}_*^+$ denote their respective full subcategories of positive dimensional real inner product spaces. Ideally, we would like to construct a functor $\mathscr{K}$ from $\mathscr{I}^+$ to the category of operads such that application of $\mathscr{K}$ to $\mathbb{R}^n$, $1 \leq n \leq \infty$, yields an operad $\mathscr{K}_n$ equivalent to $\mathscr{C}_n$. In fact, we shall have to settle a good deal less. While $\mathscr{K}_\infty$ will be (weakly) equivalent to $\mathscr{C}_\infty$, the $\mathscr{K}_n$ for $n < \infty$ will not be (or at least will not be proven to be) equivalent to the $\mathscr{C}_n$. Moreover, in order to construct the functor $\mathscr{K}$ at all, we shall have to weaken the notion of operad and shall have to carefully examine the resulting geometric structures in order to make sure that the machinery of [**45, 46**] still applies.

The difficulties can be explained quite simply. To carry out our original program, we would have to construct a space $\mathscr{E}V$ of embeddings $V \to V$ for each finite dimensional real inner product space V such that the following properties were satisfied:

(1) $fcf^{-1} \in \mathscr{E}W$ if $c \in \mathscr{E}V$ and $f \in \mathscr{I}(V, W)$, $\dim V = \dim W$.
(2) $c \times d \in \mathscr{E}(V \oplus W)$ if $c \in \mathscr{E}V$ and $d \in \mathscr{E}W$.
(3) $c \circ c' \in \mathscr{E}V$ if $c, c' \in \mathscr{E}V$.
(4) The space of j-tuples of elements of $\mathscr{E}V$ with pairwise disjoint images has the Σ_j-equivariant homotopy type of the configuration space $F(V, j)$ of j-tuples of distinct points of V.

I have been unable to construct such spaces $\mathscr{E}V$, and there is reason to believe that no such spaces of embeddings exist. We shall be forced to replace (4) by the following weaker condition

(4′) $\mathscr{E}V$ is a contractible space.

Even then, the closure conditions (1), (2), and (3) appear to be incompatible, and we shall be forced to drop (3) altogether. However, we shall have to have spaces of composable sequences of embeddings which satisfy analogs of (1), (2), and (4′). With these considerations in mind, we proceed to our basic definitions.

Definition VII.1.1. Let V be a finite dimensional real inner product space. A little convex body in V is a topological embedding $c: V \to V$ such that the maps $c_t: V \to V$ specified by

$$c_t(x) = tx + (1 - t)c(x) \text{ for } x \in V$$

are also embeddings for all $t \in I$. Since $c_{t,s} = c_{s+t-st}$, it follows that each c_t is again a little convex body. A sequence $(c_1, \ldots, c_q)$ of little convex bodies is said to be composable if $q = 1$ or, inductively, if $q > 1$ and

$$(c_{1,t}, \ldots, c_{i-1,t}, c_{i,t} \circ c_{i+1,t}, c_{i+2,t}, \ldots, c_{q,t}), \ 1 \leq i < q,$$

is a composable sequence of little convex bodies for all $t \in I$. It follows that each $(c_{1,t}, \ldots, c_{q,t})$ is again composable and, by inductive use of the case $t = 0$, that all sequences obtained by composing some of the maps $c_{i,t}$ (in ordered blocks, with t fixed) are composable.

Although the definition evolved from convexity considerations, the term convex body is a misnomer: the image of a little convex body need not be convex.

Example VII.1.2.   (i) If $c: \mathbb{R} \to \mathbb{R}$ is an embedding which is an increasing function, then so is each c_t. Any sequence $(c_1, \ldots, c_r)$ of increasing embeddings $\mathbb{R} \to \mathbb{R}$ is composable.
(ii) If $c: V \to V$ is a little convex body and $f \in \mathscr{I}(V, W)$, $\dim V = \dim W$, then $(fcf^{-1})_t = fc_tf^{-1}$. If $(c_1, \ldots, c_q)$ is a composable sequence of little convex bodies in V, then $(fc_1f^{-1}, \ldots, fc_qf^{-1})$ is a composable sequence of little convex bodies in W.
(iii) If $c: V \to V$ and $d: W \to W$ are little convex bodies, then $(c \times d)_t = c_t \times d_t$. If $(c_1, \ldots, c_q)$ and $(d_1, \ldots, d_q)$ are composable sequences of little convex bodies in V and in W, then $(c_1 \times d_1, \ldots, c_q \times d_q)$ is a composable sequence of little convex bodies in $V \oplus W$.

For rigor, we should at this point define the notion of a "partial operad." However, to avoid excess verbiage, we prefer to be informal. We agree to continue to use the term operad for structures specified as in Definition VI.1.2, but with the structural maps γ defined only on specified subsequences of the spaces $\mathscr{C}(k) \times \mathscr{C}(j_1) \times \cdots \times \mathscr{C}(j_k)$, with the two-fold iterates of the γ (as in Definition VI.1.2(a)) defined only on specified subspaces of the spaces

$$\mathscr{C}(k) \times \mathscr{C}(j_1) \times \cdots \times \mathscr{C}(j_k) \times \mathscr{C}(i_1) \times \cdots \times \mathscr{C}(i_{j_1+\cdots+j_k}),$$

and so forth. The only examples will be the little convex bodies operads (and their products with honest operads), where the γ will be obtained by composition and their domains will be specified by allowing only composable sequences (in the sense of Definition VII.1.1) to be composed.

Definition VII.1.3. Let V be a finite dimensional real inner product space. Define the little convex bodies operad $\mathscr{K}_V$ of V as follows. Let $\mathscr{K}_V(j)$ be the set of those j-tuples $\langle c_1, \ldots, c_j \rangle$ of little convex bodies such that the images of the c_r are pairwise disjoint. Let ${}^jV$ denote the disjoint union of j copies of V, regard $\langle c_1, \ldots, c_j \rangle$ as a map ${}^jV \to V$, and topologize $\mathscr{K}_V(j)$ as a subspace of the space of all continuous functions ${}^jV \to V$. (Regard $\mathscr{K}_V(0) = \langle \rangle$ as the unique "embedding" of the empty set in V.) The requisite data are specified by

(a) $\gamma(c; d_1, \ldots, d_k) = c \circ (d_1 + \cdots + d_k) \colon {}^{j_1+\cdots+j_k}V = {}^{j_1}V + \cdots + {}^{j_k}V \to V$ for those $c = \langle c_1, \ldots, c_k \rangle \in \mathscr{K}_V(k)$ and $d_r = \langle d_{r,1}, \ldots, d_{r,j_r} \rangle \in \mathscr{K}_V(j_r)$ such that each pair $(c_r, d_{r,s})$ is a composable sequence;

(b) $1 \in \mathscr{K}_V(1)$ is the identity function; and

(c) $\langle c_1, \ldots, c_j \rangle \sigma = \langle c_{\sigma(1)}, \ldots, c_{\sigma(j)} \rangle$ for $\sigma \in \Sigma_j$.

Clearly the action of Σ_j on $\mathscr{K}_V(i)$ is free. For $f \in \mathscr{I}(V, W)$, $\dim V = \dim W$, define a morphism of operads $\mathscr{K}_f \colon \mathscr{K}_V \to \mathscr{K}_W$ by $c \to fcf^{-1}$ on little convex bodies. Then $\mathscr{K}$ is a functor from $\mathscr{I}_*^+$ to the category of (partial) operads. For finite dimensional inner product spaces V and W, define a morphism of operads $\sigma \colon \mathscr{K}_V \to \mathscr{K}_{V \oplus W}$ by $c \to c \times 1$ on little convex bodies. By passage to limits, precisely as in the proof of Proposition I.1.9, $\mathscr{K}$ extends to a functor from $\mathscr{I}^+$ to the category of operads. For $1 \leq n \leq \infty$, define $\mathscr{K}_n = \mathscr{K}_V$ where $V = \mathbb{R}^n$.

Recall from [**45**, §4] that a little n-cube is a map $c \colon J^n \to J^n$, where $J = (0, 1)$, which is a linear embedding with parallel axes (i.e., a product of n increasing linear embeddings $J \to J$). The little n-cubes operad $\mathscr{C}_n$ is defined in precisely the same way that the $\mathscr{K}_V$ where, but here the γ are everywhere defined. Example VII.1.2 imply the following pair of results.

Lemma VII.1.4. *Let $\alpha \colon J \to \mathbb{R}$ be an increasing homeomorphism and let $f \in \mathscr{I}(\mathbb{R}^n, V)$, $\dim V = n$. T hen the maps $\mathscr{C}_n(j) \to \mathscr{K}_V(j)$ specified by sending a little n-cube c to the little convex body $f\alpha^n c(\alpha^{-1})^n f^{-1}$ define a morphism of operads $\mathscr{C}_n \to \mathscr{K}_V$. Thus $\mathscr{K}_V$ contains a copy of $\mathscr{C}_n$ for each such pair (α, f).*

Lemma VII.1.5. *Fix an increasing homeomorphism $\alpha \colon J \to \mathbb{R}$ and let $i_n \colon \mathscr{C}_n \to \mathscr{K}_n$ be the morphism of operads specified by $c \to \alpha^n c(\alpha^{-1})^n$ on little n-cubes. then $i_{n+1} \circ \sigma = \sigma \circ i_n \colon \mathscr{C}_n \to \mathscr{K}_{n+1}$ and the i_n induce a morphism of operads $i_\infty \colon \mathscr{C}_\infty \to \mathscr{K}_\infty$ by passage to limits.*

Assume that $\alpha(1/2) = 0$ and let $g \colon \mathscr{C}_n(j) \to F(J^n, j)$ and $f \colon \mathscr{K}_V(j) \to F(\mathbb{R}^n, j)$ be the maps specified by sending a little cube c to its center point $c(1/2, \ldots, 1/2)$ and a little convex body c to its center point $c(0)$. Define a homeomorphism $j_n \colon F(J^n, j) \to F(\mathbb{R}^n, j)$ of configuration spaces by $x \to \alpha^n(x)$ on points $x \in J^n$. Then the following Σ_j-equivariant diagram is commutative:

$$
\begin{array}{ccc}
\mathscr{C}_n(j) & \xrightarrow{\;i_n\;} & \mathscr{K}_n(j) \\
\;\downarrow{\scriptstyle g} & & \;\downarrow{\scriptstyle f} \\
F(J^n, j) & \xrightarrow{\;j_n\;} & F(\mathbb{R}^n, j)
\end{array}
$$

By [**45**, 4.8], g is a Σ_j-equivariant homotopy equivalence. Thus, up to homotopy, $\mathscr{C}_n(j)$ is a Σ_j-equivariant retract of $\mathscr{K}_n(j)$. I have not been able to prove that i_n is actually a Σ_j-equivariant homotopy equivalence (although this could perhaps be arranged at the price of a more complicated notion of little convex body).

By the very definition of composable sequences of little convex bodies in V, the deformation specified by $h_t(c) = c_t$ contracts the space of such sequences of length q to the identity sequence $(1, \ldots, 1)$. In particular, it follows that $\sigma \colon \mathscr{K}_V(j) \to \mathscr{K}_{V \oplus W}(j)$ is null homotopic for all V and W. Indeed, the requisite deformation k is specified by choosing any point $\langle d_1, \ldots, d_j \rangle \in \mathscr{K}_W(j)$ and defining

$$k_t(c_1 \times 1, \ldots, c_j \times 1) = \begin{cases} \langle c_1 \times d_{1,1-2t}, \ldots, c_j \times d_{j,1-2t} \rangle, & 0 \le t \le 1/2 \\ \langle c_{1,2t} \times d_1, \ldots, c_{j,2t} \times d_j \rangle, & 1/2 \le t \le 1. \end{cases}$$

The point is that the disjoint image requirement if satisfied on the first half of the deformation because $\langle c_1, \ldots, c_j \rangle \in \mathscr{K}_V(j)$ and on the second half because $\langle d_1, \ldots, d_j \rangle \in \mathscr{K}_W(j)$. Therefore $\pi_* \mathscr{K}_\infty(j) = 0$. (In the case of little cubes, this argument is due to Boardman and Vogt [**20**, p. 65].) By the same argument, the product of $\sigma \colon \mathscr{K}_V(k) \to \mathscr{K}_{V \oplus W}(k)$ and the $\sigma \colon \mathscr{K}_V(j_r) \to \mathscr{K}_{V \oplus W}(j_r)$ for $1 \le r \le k$ restricts to a null homotopic map from the domain of γ (for $\mathscr{K}_V$) to the domain of γ (for $\mathscr{K}_{V \oplus W}$), hence the domain of γ for $\mathscr{K}_\infty$ has trivial homotopy groups, and similarly for the domains of all iterates of the maps γ. It would follow that $\mathscr{K}_\infty(j)$ was contractible if we knew either that $\mathscr{K}_\infty(j)$ had the homotopy type of a CW-complex or that each $\sigma \colon \mathscr{K}_n(j) \to \mathscr{K}_{n+1}(j)$ was a cofibration (and similarly for the domains of the maps γ and their iterates). I have not verified either assertion. However, we need not let this difficulty detain us since the conclusions we wish to derive from E_∞ space structures can be phrased homologically, so that it is harmless to require of a partial E_∞ operad $\mathscr{C}$ only that the $\mathscr{C}(j)$ and the domains of the γ and their iterates have trivial homology groups. Thus $i_\infty \colon \mathscr{C}_\infty \to \mathscr{K}_\infty$ may be regarded as a weak equivalence of (partial) E_∞ operads.

2. The canonical E_∞ operad pair

We first show that just as $\mathscr{C}_n$ acts naturally on n-fold loop spaces [**45**, 5.1] so $\mathscr{K}_V$ acts naturally on "V-fold" loop spaces. We then discuss the "partial monads" K_V to which the partial operads $\mathscr{K}_V$ give rise. Finally, we show that $\mathscr{L}$ acts on $\mathscr{K}_\infty$, so that K_∞ restricts to a (partial) monad in $\mathscr{L}[\mathscr{T}_e]$ (as in Proposition VI.1.9), and prove that the zero[th] space of any E_∞ ring spectrum is an E_∞ ring space.

We should begin by defining the notion of an action by a (partial) operad on a space, but shall instead leave the requisite modification of Definition VI.1.3 to the reader. We continue to use the term E_∞ space for a space with an action by a (partial) E_∞ operad. With the appropriate definition, the following result is trivial to verify. We agree to fix tV as the one-point compactification of V, so that

$$\Omega^V X = F(tV, X) \quad \text{and} \quad \Sigma^V X = X \wedge tV.$$

We agree to identify $t\mathbb{R}^n$ with $tJ^n = I^n / \partial I^n$ via the homeomorphism $t(\alpha^{-1})^n$, where $\alpha \colon J \to \mathbb{R}$ is as in Lemma VII.1.5.

THEOREM VII.2.1. *For $X \in \mathscr{T}$ and $V \in \mathscr{I}_*^+$, let $\theta_{V,j} \colon \mathscr{K}_V(j) \times (\Omega^V X)^j \to \Omega^V X$ be the map specified by*

$$\theta_{V,j}(\langle c_1, \ldots, c_j \rangle, y_1, \ldots, y_j)(b) = \begin{cases} y_r(a) & \text{if } c_r(a) = b \in V \\ * & \text{if } b \notin \bigcup_{r=1}^j \operatorname{Im} c_r \end{cases}$$

Then the $\theta_{V,j}$ define an action θ_V of $\mathscr{K}_V$ on $\Omega^V X$. Under the natural identification $\Omega^V \Omega^W X = \Omega^{V \oplus W} X$, the action θ_V coincides with the composite of $\theta_{V \oplus W}$ and $\sigma \colon \mathscr{K}_V \to \mathscr{K}_{V \oplus W}$. For $E \in \mathscr{S}$, the actions θ_V of $\mathscr{K}_V$ on $E_0 \cong \Omega^V EV$ for $V \subset \mathbb{R}^\infty$ define an action θ_∞ of $\mathscr{K}_\infty$ on E_0. Under the

identification $\Omega^n X = \Omega^V X$, $V = \mathbb{R}^n$, the action of $\mathscr{C}_n$ on $\Omega^n X$ coincides with the composite of $i_n \colon \mathscr{C}_n \to \mathscr{K}_n$ and the action θ_n of $\mathscr{K}_n$ on $\Omega^n X$, and similarly for the action of $\mathscr{C}_\infty$ on E_0.

Define a space $K_V X$ for each space X by letting $K_V X = \coprod \mathscr{K}_V(j) \times_{\Sigma_j} X^j / \approx$, where

$$(\langle c_1, \ldots, c_j \rangle, x_1, \ldots, x_j) \approx (\langle c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_j \rangle, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_j)$$

whenever $x_i = *$ (compare [**45**, 2.4]). Define $\eta \colon X \to K_V X$ by $\eta(x) = [\langle 1 \rangle, x]$. We would like to define $\mu \colon K_V K_V X \to K_V X$ by

$$\mu[c; [d_1, y_1], \ldots, [d_k, y_k]] = [\gamma(c; d_1, \ldots, d_k), y_1, \ldots, y_k]$$

for $c \in \mathscr{K}_V(k)$, $d_r \in \mathscr{K}_v(j_r)$, and $y_r \in X^{j_r}$. We cannot do so since γ not defined on all of $\mathscr{K}_V(k) \times \mathscr{K}_v(j_1) \times \cdots \times \mathscr{K}_V(j_r)$. We should therefore define the notion of a partial monad C, with structural maps $\mu \colon C^{(2)} X \to CX$ for a suitable subspace $C^{(2)} X$ of CCX. The various $(n-1)$-fold iterates of μ would have to be defined (and equal) on specified subspaces $C^{(n)} X$ of $C^n X$. (More precisely, $C^{(n)}$ would be required to be a subfunctor of C^n.) With the proper formal definitions, a partial operad $\mathscr{C}$ gives rise to a partial monad C in such a way that the notions of an action by $\mathscr{C}$ and of an action by C are equivalent. Here again, we leave the pedantic details to the reader and continue to use the terms monad and C-space for the more general concepts. The only examples will be the monads C derived from operads of the form $\mathscr{C}' \times \mathscr{K}_V$ (where $\mathscr{C}'$ is an honest operad). Here the subspaces $C^{(n)} X$ of $C^n X$ are obvious: all n-fold sequence of little convex bodies which would be composed under the iterates of μ are required to be composable in the sense of Definition VII.1.1.

We have an action $\theta_V \colon K_V \Omega^V X \to \Omega^V X$ for any space X.

THEOREM VII.2.2. *For $X \in \mathscr{T}$ and $V \in \mathscr{I}_*^+$, define $\alpha_V \colon K_V X \to \Omega^V \Sigma^V X$ to be the composite*

$$K_V X \xrightarrow{K_V \eta} K_V \Omega^V \Sigma^V X \xrightarrow{\theta_V} \Omega^V \Sigma^V X,$$

where $\eta \colon X \to \Omega^V \Sigma^V X$ is the natural inclusion, and define $\alpha_\infty \colon K_\infty X \to QX$ by passage to limits over $V \subset \mathbb{R}^\infty$. Then $\alpha_V \colon K_V \to \Omega^V \Sigma^V$ and $\alpha_\infty \colon K_\infty \to Q$ are morphisms of monads in $\mathscr{T}$ and the actions of K_V on $\Omega^V X$ and of K_∞ on E_0 for $E \in \mathscr{S}$ are induced by pullback along α_V and α_∞ from the actions of $\Omega^V \Sigma^V$ and of Q. Moreover, α_∞ is a weak homotopy equivalence if X is connected and is a group completion in general.

PROOF. The monads $\Omega^V \Sigma^V$ and Q are defined as in [**45**, p. 17 and 46], and the first statement holds by slight elaborations of the purely formal diagram chases in the proof of [**45**, 5.2]; moreover, with $\alpha_n = \alpha_V$ for $V = \mathbb{R}^n$, $\alpha_n i_n \colon C_n \to \Omega^n \Sigma^n$ coincides with the morphism of monads derived in the cited result. By [**46**, A.2(i)], $i_\infty \colon C_\infty X \to K_\infty X$ is a homology isomorphism for any space X. Now the last statement for X connected is given by [**45**, 6.1], while the general case is proven by explicit homological calculation in [**26**, I§5]. $\square$

We next exploit the fact that $\mathscr{K}_\infty$ is obtained by use of the functor $\mathscr{K}$ (from $\mathscr{I}^+$ to operads) to specify an action of $\mathscr{L}$ on $\mathscr{K}_\infty$. Yet again, we can and must first generalize all of the definitions and results Definitions VI.1.6 to VI.1.8, VI.1.10 and VI.1.11 and Proposition VI.1.9 so as to allow the additive operad of an operad pair to be a partial operad. The multiplicative operad will be required to be honest. We omit the details, and we continue to use the term E_∞ ring space for a space with an action by a (partial) E_∞ operad pair.

Lemma VII.2.3. *$(\mathscr{K}_\infty, \mathscr{L})$ is an E_∞ operad pair with respect to the action maps λ specified on $g \in \mathscr{L}(k)$ and $c_r \in \mathscr{K}_\infty(j_r)$ by*

$$\lambda(g; c_1, \ldots, c_k) = \left\langle \prod_{I \in S(j_1, \ldots, j_k)} \mathscr{K}_g(c_I) \right\rangle \in \mathscr{K}_\infty(j_1 \cdots j_k)$$

where, if $c_r = \langle c_{r1}, \ldots, c_{rj_r} \rangle$ and $I = \{i_1, \ldots, i_k\}$, then c_I is the little convex body $c_{1i_1} \times \cdots \times c_{ki_k}$ in $(\mathbb{R}^\infty)^k$.

Here $\mathscr{K}_g(c_I)$ is $gc_I g^{-1}$ on finite dimensional subspaces gW of $\mathbb{R}^\infty$ and is the identity on the orthogonal complement of gW whenever c_I is the identity on the orthogonal complement of $W \subset (\mathbb{R}^\infty)^k$. The verifications of the identities specified in Definition VI.1.6 are tedious, but elementary.

If $p \colon \mathscr{G} \to \mathscr{L}$ is a morphism of E_∞ operads, then, by Lemma IV.1.9, Q is a monad in $\mathscr{G}[\mathscr{T}_e]$. As explained in Section VI§1, if $\mathscr{G}$ acts on $\mathscr{C}$, then C is a (partial) monad in $\mathscr{G}[\mathscr{T}_e]$. We have the following consistency statement, which implies that the zero[th] space of any E_∞ ring spectrum is an E_∞ ring space.

THEOREM VII.2.4. *Let $(\pi, \rho) \colon (\mathscr{C}, \mathscr{G}) \to (\mathscr{K}_\infty, \mathscr{L})$ be a morphism of E_∞ operad pairs. Then the morphisms $\pi \colon C \to K_\infty$ and $\alpha_\infty \colon K_\infty \to Q$ of monads in $\mathscr{T}$ restrict to morphisms of monads in $\mathscr{G}[\mathscr{T}_e]$. If E is a $\mathscr{G}$-spectrum, then its zero[th] space E_0 is a $(\mathscr{C}, \mathscr{G})$-space by pullback of its Q-action $QE_0 \to E_0$ along $\alpha_\infty \pi$.*

PROOF. By Lemma IV.1.9, the second statement will follow from the first. Clearly $\mathscr{G}$ acts on $\mathscr{K}_\infty$ via
$$\lambda(g; c_1, \ldots, c_k) = \lambda(\rho g; c_1, \ldots, c_k)$$
and $(\pi, 1) \colon (\mathscr{C}, \mathscr{G}) \to (\mathscr{K}_\infty, \mathscr{G})$ is a morphism of operad pairs. Let $X \in \mathscr{G}(\mathscr{T}_e)$. $\pi \colon CX \to K_\infty X$ is a morphism of $\mathscr{G}_0$-spaces in view of Definition VI.1.8. Since $K_\infty \eta \colon K_\infty QX$ is a morphism of $\mathscr{G}_0$-spaces, because η is so, $\alpha_\infty \colon K_\infty X \to QX$ will be a morphism of $\mathscr{G}_0$-spaces provided that $\theta_\infty \colon K_\infty QX \to QX$ is so. Thus we must verify that the following diagrams commute:

$$
\begin{array}{ccc}
\mathscr{G}(k) \times \mathscr{K}(j_1) \times (QX)^{j_1} \times \cdots \times \mathscr{K}_\infty(j_k) \times (QX)^{j_k} & \xrightarrow{1 \times \theta_{\infty, j_1} \times \cdots \times \theta_{\infty, j_k}} & \mathscr{G}(k) \times (QX)^k \\
\xi_k \downarrow & & \downarrow \xi_k \\
\mathscr{K}_\infty(j_1 \cdots j_k) \times (QX)^{j_1 \cdots j_k} & \xrightarrow{\theta_{\infty, j_1 \cdots j_k}} & QX
\end{array}
$$

ξ_k on the left is specified in Definition VI.1.8. Think of points of QX as maps $S^\infty \to X \wedge S^\infty$, where $S^\infty = t\mathbb{R}^\infty$. For $c = \langle c_1, \ldots, c_j \rangle \in \mathscr{K}_\infty(j)$ and $y = (y_1, \ldots, y_j) \in (QX)^j$, $\theta_{\infty, j}(c, y)$ is then the composite
$$S^\infty \xrightarrow{\bar{c}} {}^j S^\infty \xrightarrow{y_1 \vee \cdots \vee y_j} {}^j(X \wedge S^\infty) \xrightarrow{p} X \wedge S^\infty,$$
where ${}^j Y$ denotes the wedge of j copies of Y, $\bar{c}$ is the pinch map specified by $\bar{c}(b) = *$ unless $b = c_r(a)$ for some r and a, when $\bar{c}(b) = a$ in the rth copy of S^∞, and p is the evident folding map. For $g \in \mathscr{G}(k)$ and $z = (z_1, \ldots, z_k) \in (QX)^k$, $\xi_k(g, z)$ is that map which makes the following diagram commute

$$
\begin{array}{ccc}
S^\infty & \xleftarrow{\rho g} S^\infty \wedge \cdots \wedge S^\infty \xrightarrow{z_1 \wedge \cdots \wedge z_k} & X \wedge S^\infty \wedge \cdots \wedge X \wedge S^\infty \\
\xi_k(g, z) \downarrow & & \| \wr \\
X \wedge S^\infty & \xleftarrow{1 \wedge \rho g} X \wedge S^\infty \wedge \cdots \wedge S^\infty \xleftarrow{\xi_k(g) \wedge 1} & X \wedge \cdots \wedge X \wedge S^\infty \wedge \cdots \wedge S^\infty
\end{array}
$$

and which, on the orthogonal complement of $(\rho g)t(V_1 \oplus \cdots \oplus V_k)$ where $V_r \subset \mathbb{R}^\infty$ is a finite-dimensional subspace such that z_r is in the image of $\Omega^{V_r} \Sigma^{V_r} X$, has constant coordinate in X and the identity map as coordinate in S^∞. Now the desired commutativity is easily verified by direct computation, the point being that the smash products used in the definition of ξ_k distribute over the wedge sums sued in the definition of the $\theta_{\infty j}$. $\square$

3. The one operad recognition principle

We translate the one operad, additive, recognition principle of [45, §14] and [46, §2] into the language of coordinate-free spectra. It simplifies slightly in the process since the construction of a prespectrum and the passage from a prespectrum to a spectrum were awkwardly combined in the earlier versions. It also complexifies slightly since we must take account of the distinction between partial and honest operads and their actions. However, the basic constructions and the bulk of the proofs remain unchanged and will not be repeated here.

Let $\mathscr{C}'$ be a locally contractible (honest) operad, for example n or an E_∞ operad. Define $\mathscr{C} = \mathscr{C}' \times \mathscr{K}_\infty$ [**45**, 3.8] and observe that $\mathscr{C}$ is a (partial) E_∞ operad (that is, Σ_j acts freely on $\mathscr{C}(j)$ and $\mathscr{C}'(j)$ and the domains of the structural maps γ and their iterates have trivial homology groups). Let $\pi\colon \mathscr{C} \to \mathscr{K}_\infty$ and $\psi\colon \mathscr{C} \to \mathscr{C}'$ be the projections.

Let (X, θ) be a $\mathscr{C}$-space. For technical reasons, we assume once and for all that X is one of the following three types (which certainly include all examples of any interest).

(1) The action of $\mathscr{C}$ on X is obtained by pullback along $\psi\colon \mathscr{C} \to \mathscr{C}'$ from an action of $\mathscr{C}'$ on X.

(2) X is E_0 regarded as a C-space by pullback of its Q-space structure along $\alpha_\infty \pi$, where $E \in \mathscr{S}$.

(3) X is CY regarded as a C-space via the structural map γ of the (partial) monad C, where $Y \in \mathscr{T}$.

In (1) and (2), the domain of θ is CX; in (3), the domain of θ is the domain $C^{(2)}Y$ of μ.

For each finite dimensional sub inner product space V of $\mathbb{R}^\infty$, define $\mathscr{C}_V = \mathscr{C}' \times \mathscr{K}_V$ and let C_V be the monad in $\mathscr{T}$ associated to $\mathscr{C}_V$. Recall, and generalize to the context of partial monads, the notion of a (right) action of a monad on a functor [**45**, 9.4]. By [**45**, 9.5], the adjoint of $\alpha_V\colon K_V \to \Omega^V \Sigma^V$ gives an action of K^V on the functor Σ^V. By pullback along $\pi\colon C_V \to K_V$, we obtain an action β_V of C_V on Σ^V.

The basic geometric construction of [**45**] is the two-sided bar construction of [**45**, 9.6 and 11.1]

$$B(F, C, X) = |B_*(F, C, X)|,$$

where C is a monad, X is a C-space, and F is a C-functor. Here $||$ denotes the geometric realization functor from simplicial spaces to spaces, and the space $B_q(F, C, X)$ of q-simplices is $FC^q X$ where C^q denotes the q-fold iterate of C. This construction generalizes readily to the context of partial monads and their actions. In practice, due to X being of one of the three types specified above and to the definition of C and the C_v in terms of $\mathscr{C}'$, $\mathscr{K}_\infty$, and the $\mathscr{K}_V$, there will always be obvious subspaces $B_q(F, C, X)$ of $FC^q X$ in sight so that the appropriate faces and degeneracies are defined. Indeed, this will simply amount to the requirement that precisely the composable sequences of little convex bodies (in the sense of Definition VII.1.1) are allowed to be composed.

We may thus define a space $(TX)(V)$ by

$$(VII.1) \qquad (TX)(V) = B(\Sigma^V, C_V, X) \equiv |B_*(\Sigma^V, C_V, X)|.$$

By convention, when $V = \{0\}$, Σ^V, Ω^V, K_V, and C_V are all the identity functor on $\mathscr{T}$, and the α_V and β_V are identity maps. Thus the zero$^{\text{th}}$ space $T_0 X = (TX)\{0\}$ is just X.

For an orthogonal pair of finite dimensional subspaces V and W of $\mathbb{R}^\infty$, the morphism of operads $\mathscr{K}_V \to \mathscr{K}_{V+W}$ induces a morphism of operads $\mathscr{C}_V \to \mathscr{C}_{V+W}$. With the first equality given by [**45**, 9.7 and 12.1], we therefore obtain an inclusion

$$(VII.2) \qquad \sigma\colon \Sigma^W B(\Sigma^V, C_V, X) = B(\Sigma^{V+W}, C_V, X) \longrightarrow B(\Sigma^{V+W}, C_{V+W}, X).$$

We would like to say that (TX, σ) gives a prespectrum, as defined in Definition II.1.1. For this, T must be appropriately defined on isometries $f\colon V \to V'$ for subspaces V and V' of $\mathbb{R}^\infty$ o the same finite dimension. It is easily verified by separate arguments in the cases (1), (2), and (3) that there are maps $\xi(f)\colon X \to X$ such that if $\Sigma^f\colon \Sigma^V \to \Sigma^{V'}$ is defined in the obvious way and if $\mathscr{C}_f = 1 \times \mathscr{K}_f\colon \mathscr{C}_V \to \mathscr{C}_{V'}$, then maps $(TX)(f)$ as required can be specified by

$$(VII.3) \qquad (TX)(f) = B(\Sigma^f, C_f, \xi(f))\colon B(\Sigma^V, C_V, X) \longrightarrow B(\Sigma^{V'}, C_{V'}, X).$$

We omit the details since the requisite maps $\xi(f)$ will appear most naturally in the two operad theory (as use of the letter ξ would suggest) and since, as explained in Remark II.1.10, the $(TX)(f)$ in any case play no essential role.

Thus (TX, σ) is a prespectrum. Consider the spectrum $\Omega^\infty TX$ and the natural map $\iota \colon TX \to \Omega^\infty TX$ of prespectra given by Definition II.1.4 and Lemma II.1.5. Let ΓX, or $\Gamma(X, \theta)$ when necessary for clarify, denote the zero$^{\text{th}}$ space of $\Omega^\infty TX$. The crux of the recognition principle is the analysis of the zero$^{\text{th}}$ map $\iota \colon X \to \Gamma X$.

THEOREM VII.3.1. *Consider the following diagram:*

$$
\begin{array}{ccc}
B(C, C, X) & \xrightarrow{\ B(\alpha_\infty \pi, 1, 1)\ } & B(Q, C, X) \\
{\scriptstyle \varepsilon(\theta)}\big\downarrow\big\uparrow{\scriptstyle \tau(\eta)} & & \big\downarrow{\scriptstyle \gamma^\infty} \\
X & \xrightarrow{\quad \iota \quad} & \Gamma X
\end{array}
$$

(i) $\epsilon(\theta)$ *is a strong deformation retraction with right inverse* $\tau(\eta)$;

(ii) $B(\alpha_\infty \pi, 1, 1)$ *is a group completion and is therefore a weak homotopy equivalence if X is grouplike (i.e., if $\pi_0 X$ is a group);*

(iii) γ^∞ *is a weak homotopy equivalence;*

(iv) $\iota = \gamma^\infty \circ B(\alpha_\infty \pi, 1, 1) \circ \tau(\eta)$, *hence ι is a group completion.*

Moreover, $\varepsilon(\theta)$, $B(\alpha_\infty \pi, 1, 1)$, and γ^∞ (but not $\tau(\eta)$ and ι) are maps of $\mathscr{C}$-spaces.

PROOF. Formal results from [**45**] apply equally well in the context of partial monads as in that of monads, and [**45**, 9.2, 9.8 and 11.10] imply (i). Results in [**45**] which apply to general simplicial spaces also apply equally well here, and [**45**, 12.3 and 14.4(iii)] imply (iii). The first part of (iv) is a trivial calculation (compare [**45**, 14.4(iv)]), and the last statement is proven by a slight elaboration, necessitated by our partial structures, of the proof of [**45**, 12.2 and 12.4]. It remains to prove (ii). Here we shall have to use the infinite little cubes operad $\mathscr{C}_\infty$, and we rewrite $C = C' \times K_\infty$ and let $C' \times C_\infty$ be the monad associated to $\mathscr{C}' \times \mathscr{C}_\infty$. By use of $i_\infty \colon \mathscr{C}_\infty \to \mathscr{K}_\infty$, we obtain the commutative diagram

$$
\begin{array}{ccc}
B(C' \times C_\infty, C' \times C_\infty, X) & \xrightarrow{\ B(\alpha_\infty i_\infty \pi, 1, 1)\ } & B(Q, C' \times C_\infty, X) \\
\ & & \ \\
\big\downarrow{\scriptstyle B(i \times i_\infty, 1 \times i_\infty, 1)} & & \big\downarrow{\scriptstyle B(1, 1 \times i_\infty, 1)} \\
B(C' \times K_\infty, C' \times K_\infty, X) & \xrightarrow{\ B(\alpha_\infty \pi, 1, 1)\ } & B(Q, C' \times K_\infty, X)
\end{array}
$$

with $X \overset{\varepsilon(\theta)}{\underset{\simeq}{\longleftarrow}}$ and $X \overset{\varepsilon(\theta)}{\underset{\simeq}{\longleftarrow}}$ to the two left spaces.

where X is regarded as a $C' \times C_\infty$-space by pullback along $1 \times i_\infty$. By the left triangle, $B(1 \times i_\infty, 1 \times i_\infty, 1)$ is a homotopy equivalence. By [**46**, 2.3], but with Remark VI.2.7(iv) substituted for [**46**, 2.1] in the proof given there, the top arrow $B(\alpha_\infty i_\infty \pi, 1, 1)$ is a group completion. It therefore suffices to prove that $B(1, 1 \times i_\infty, 1)$ induces an isomorphism on homology. By [**46**, A.4], it suffices to show that each

$$
B_q(1, 1 \times i_\infty, 1) \colon B_q(Q, C' \times C_\infty, X) \longrightarrow B_q(Q, C' \times K_\infty, X)
$$

induces an isomorphism on homology. $B_q(Q, C' \times C_\infty, X)$ and $B_q(Q, C' \times K_\infty, X)$ are obtained by application of the functor Q to $(C' \times C_\infty)^q(X)$ and to $(C' \times K_\infty)^{[q]}(X)$, where the latter is the appropriate domain space, namely $(C' \times K_\infty)^{(q)}(X)$ in cases (1) and (2) and $(C' \times K_\infty)^{(q+1)}(Y)$ in case (3). By [**26**, I§4], it suffices to show that

$$
(C' \times C_\infty)^q(X) \longrightarrow (C' \times K_\infty)^{[q]}(X)
$$

induces an isomorphism on homology. Typical points of both sides have coordinates in various of the spaces of the relevant operads and in X or Y. We may filter by the number of coordinates in X or Y. The successive quotients may be thought of as generalized equivariant half-smash products [**45**, 2.5], and the map induced on such spaces by any morphism of (partial) E_∞ operads is a homology isomorphism. Indeed, the shuffle map shows that the homology of such spaces depends

only on the chains of X or Y and the chains of the operad coordinate. The latter chains are acyclic and free over the appropriate configuration of symmetric groups, hence standard techniques of homological algebra apply to yield the conclusion. $\square$

The following pair of results show that $\Omega^\infty TX$ gives the "right" spectrum in cases (2) and (3).

Proposition VII.3.2. *Let $E \in \mathscr{S}$. Then the maps*

$$\varepsilon(\phi_V)\colon (TE_0)(V) = B(\Sigma^V, C_V, E_0) \longrightarrow EV,$$

where $\phi_V\colon \Sigma^V E_0 \cong \Sigma^V \Omega^V EV \to EV$ is the evaluation map, define a natural map $\omega\colon TE_0 \to \nu E$ of prespectra. The unique $\widetilde{\omega}\colon \Omega^\infty TE_0 \to E$ of spectra such that $(\nu\widetilde{\omega})\iota = \omega$ induces an isomorphism on π_i for all $i \geq 0$.

PROOF. $\epsilon(\phi_V)$ is defined as in [**45**, 9.2 and p. 126], and the first statement is an easy verification from (VII.1), (VII.2), and (VII.3) and the definition, Definition II.1.1, of prespectra. Since $\omega\colon E_0 = (TE_0)\{0\} \to E_0$ is the identity map and E_0 is grouplike, (iv) of the theorem implies that $\widetilde{\omega}\colon \Gamma E_0 \to E_0$ is a weak homotopy equivalence. The second statement follows. $\square$

The proposition implies that ω becomes an isomorphism in the stable homotopy category $H\mathscr{S}$ if E is connective. In other words, E (and thus the cohomology theory it determines) can be recovered from the underlying $\mathscr{C}$-space E_0.

Proposition VII.3.3. *For $Y \in \mathscr{T}$, the composite map of spectra*

$$\Omega^\infty TCY \xrightarrow{\Omega^\infty T(\alpha_\infty \pi)} \Omega^\infty TQY \xrightarrow{\widetilde{\omega}} Q_\infty Y = \Omega^\infty \Sigma^\infty Y$$

is a strong deformation retraction.

PROOF. Recall that $(\Sigma^\infty Y)(V) = \Sigma^V Y$. The maps

$$\varepsilon(\beta_V)\colon B(\Sigma^V, C_V, C_V Y) \longrightarrow \Sigma^V Y$$

are strong deformation retractions by [**45**, 9.9 and 11.10]. With the evident maps $\sigma = B(1, \sigma, \sigma)$ and $B(\Sigma^f, C_f, C_f)$ as in (VII.2) and (VII.3), the $B(\Sigma^V, C_V, C_V Y)$ are the spaces of a prespectrum $T'CY$, and the maps $\varepsilon(\beta_V)$ define a strong deformation retraction of prespectra. There is an obvious inclusion of $T'CY$ in TCY and, since passage to spectra is a limit process over $V \subset \mathbb{R}^\infty$, this inclusion becomes an isomorphism upon application of the functor Ω^∞. The resulting deformation retraction $\Omega^\infty TCY \to Q_\infty Y$ is the specified composite since β_V is the composite

$$\Sigma^V C_V Y \xrightarrow{\Sigma^V \pi} \Sigma^V K_V Y \xrightarrow{\Sigma^V \alpha_V} \Sigma^V \Omega^V \Sigma^V Y \xrightarrow{\phi_V} \Sigma^V Y.$$

$\square$

The proposition gives an equivalence of infinite loop spaces between ΓCY and QY and is our preferred version of the Barratt–Quillen theorem.

[**45**, §14 and 15] and [**46**, §2 and 3] contain further discussion and various additional results about the coordinatized spectrum $B_\infty X$ specified by $B_i X = (\Omega^\infty TX)(\mathbb{R}^i)$. Of course, the little cubes operad used in those papers could be replaced by the little convex bodies operad introduced here without any change in the results or their proofs. In particular, we have the following consistency statement, which was used in the discussion of Bott periodicity in Chapter I. We give some details since the result needed there was more precise than the result proven in [**45, 46**].

Proposition VII.3.4. *Let X be a $\mathscr{C}'$-space. Then there is a map $\zeta\colon \Omega^\infty T\Omega^d X \to \Omega^d \Omega^\infty TX$ in $H\mathscr{S}$ such that the following diagram commutes in $H\mathscr{T}$.*

$$
\begin{array}{ccc}
 & \Omega^d X & \\
{\scriptstyle \iota}\swarrow & & \searrow{\scriptstyle \Omega^d \iota} \\
\Gamma \Omega^d X & \xrightarrow{\quad \zeta_0 \quad} & \Omega^d \Gamma X
\end{array}
$$

PROOF. [**46**, 3.1] gives maps of $\mathscr{C}$-spaces

$$X \xleftarrow{\varepsilon} Y_d X \xrightarrow{\delta} B_d \Omega^d X$$

such that δ and $\Omega^d E$ are weak equivalences. Think of $B_d \Omega^d X$ as the zero$^{\text{th}}$ space of the coordinatized spectrum $\Omega^{-d} B_\infty \Omega^d X = \{B_{d+i} \Omega^d X\}$. Then application of the functor $\Omega^d B_\infty$ and use of Proposition VII.3.2 gives maps of coordinatized spectra

$$(*) \qquad B_\infty \Omega^d X \cong \Omega^d \Omega^{-d} B_\infty \Omega^d X \xleftarrow{\Omega^d \widetilde{\omega}} \Omega^d B_\infty B_d \Omega^d X \xleftarrow{\Omega^d B_\infty Y_d X}$$

$$\xrightarrow{\Omega^d B_\infty \varepsilon} \Omega^d B_\infty X,$$

the first two of which are equivalences. On the zero$^{\text{th}}$ space level, we have the following commutative diagram of weak equivalences:

$$
\begin{array}{ccccccc}
\Omega^d X & \xrightarrow{\iota} & \Omega^d B_d \Omega^d X & \xleftarrow{\Omega^d \delta} & \Omega^d Y_d X & \xrightarrow{\Omega^d \varepsilon} & \Omega^d X \\
{\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \Omega^d \iota} & & \downarrow{\scriptstyle \Omega^d \iota} & & \downarrow{\scriptstyle \Omega^d \iota} \\
\Gamma \Omega^d X & \xleftarrow{\Omega^d \widetilde{\omega}_0} & \Omega^d \Gamma B_d \Omega^d X & \xleftarrow{\Omega^d \Gamma \delta} & \Omega^d \Gamma Y_d X & \xrightarrow{\Omega^d \Gamma \widetilde{\varepsilon}} & \Omega^d \Gamma X
\end{array}
$$

Inspection of the explicit construction of the intermediate space $Y_d X$ in [**45**, p. 148–151 (especially the bottom diagram on p. 150)] and use of [**45**, 14.9] demonstrated that the composite $(\Omega^d \varepsilon)(\Omega^d \delta)^{-1} \iota$ is equal in $H\mathscr{T}$ to the identity map of $\Omega^d X$. Now application of the functor ψ (which commutes with Ω) to the maps of $(*)$ and use of the equivalence $\psi \phi \simeq 1$ of Theorem II.1.8 gives the required map ζ of coordinate-free spectra. $\qquad\square$

As explained in [**46**, 3.7 (p. 75)], the previous result implies the following further consistency statement.

Proposition VII.3.5. *Let G be a monoid in $\mathscr{C}'[\mathscr{T}]$. Then BG and the delooping of $B_1 G$ are equivalent as infinite loop spaces.*

4. The two operad recognition principle

Assume given a locally contractible operad pair $(\mathscr{C}', \mathscr{G}')$, for example $(n, \mathscr{G}')$ where $\mathscr{G}'$ is locally contractible or any E_∞ operad pair, and define $(\mathscr{C}, \mathscr{G})$ to be the product (partial) operad pair $(\mathscr{C}' \times \mathscr{K}_\infty, \mathscr{G}' \times \mathscr{L})$. Let $(\pi, \rho) \colon (\mathscr{C}, \mathscr{G}) \to (\mathscr{K}_\infty, \mathscr{L})$ be the projection and regard elements of $\mathscr{G}(j)$ as linear isometries via ρ. Recall the definition, Definition IV.1.1, of a $\mathscr{G}$-prespectrum.

THEOREM VII.4.1. *Let (X, θ, ξ) be a $(\mathscr{C}, \mathscr{G})$-space. Then $TX = T(X, \theta)$ admits a natural structure of $\mathscr{G}$-prespectrum, hence $\Omega^\infty TX$ admits a natural structure of $\mathscr{G}$-spectrum.*

PROOF. By Example IV.2.3, the second clause will follow from the first. Let V_i, $1 \le i \le j$, be a finite dimensional sub inner product of $\mathbb{R}^\infty$ and let $g \in \mathscr{G}(j)$. We must specify appropriate maps

$$(*) \qquad \xi_j(g) \colon (TX)(V_1) \wedge \cdots \wedge (TX)(V_j) \longrightarrow (TX)(W), \text{ where } W = g(v_1 \oplus \cdots \oplus V_j).$$

These maps will be induced from composites

$$B(\Sigma^{V_1}, C_{V_1}, X) \times \cdots \times B(\Sigma^{V_j}, C_{V_j}, X)$$
$$\downarrow$$
$$|B_*(\Sigma^{V_1}, C_{V_1}, X) \times \cdots \times B_*(\Sigma^{V_j}, C_{V_j}, X)|$$
$$\downarrow$$
$$|B_*(\Sigma^{V_1}, C_{V_1}, X) \wedge \cdots \wedge B_*(\Sigma^{V_j}, C_{V_j}, X)|$$
$$\downarrow {\scriptstyle |\chi_j(g)_*|}$$
$$B(\Sigma^W, C_W, X)$$

Here the first arrow is the natural homeomorphism of [**45**, 11.5], the second arrow is derived by passage to smash products in each simplicial degree, and the third arrow is the realization of a map $\chi_j(g)_*$ of simplicial spaces still to be constructed. It is apparent from Lemma VII.2.3 that the maps

$$\lambda \colon \mathscr{L}(j) \times \mathscr{K}_\infty(i_1) \times \cdots \times \mathscr{K}_\infty(i_j) \longrightarrow \mathscr{K}_\infty(i_1 \cdots i_j)$$

are obtained by passage to adjoints and limits from maps

$$\lambda(\ell) \colon \mathscr{K}_{V_1}(i_1) \times \cdots \times \mathscr{K}_{V_j}(i_j) \longrightarrow \mathscr{K}_W(i_1 \cdots i_j), \quad W = \ell(V_1 \oplus \cdots \oplus V_j),$$

for $\ell \in \mathscr{L}(j)$. Therefore, in view of Definition VI.1.8 and the product structure on $(\mathscr{C}, \mathscr{G})$, there are unique dotted arrows $\xi_j(g)$ such that the following diagrams commute (the solid arrows $\xi_j(g)$ being given by Definition VI.1.8 and Proposition VI.1.9):

$$
\begin{array}{ccc}
C_{V_1}X \times \cdots \times C_{V_j}X & \overset{\subset}{\longrightarrow} & CX \times \cdots \times CX \\
{\scriptstyle \xi_j(g)}\downarrow & & \downarrow{\scriptstyle \xi_j(g)} \\
C_W X & \underset{\subset}{\longrightarrow} & CX
\end{array}
$$

This statement holds for any $\mathscr{G}_0$-space X and in particular for CX, CCX, etc. In view of the role played in Definition VI.1.8 by the original map $\xi_j(g) \colon X^j \to X$, we see that iterative application of the statement above yields unique dotted arrows $\xi_j(g)_q$, $q \geq 0$, such that the following diagrams commute:

$$
\begin{array}{ccc}
C^q_{V_1}X \times \cdots \times C^q_{V_j}X & \overset{\subset}{\longrightarrow} & C^q X \times \cdots \times C^q X \\
{\scriptstyle \xi_j(g)}\downarrow & & \downarrow{\scriptstyle \xi_j(g)} \\
C^q_W X & \underset{\subset}{\longrightarrow} & C^q X
\end{array}
$$

The $\xi_j(g)_q$ all pass to smash products. If we collect the smash product factors $tV_1, \ldots, tV_j$ together and apply tg to then, then we obtain from the $\xi_j(g)_q$ the further maps

$$\chi_j(g)_q \colon \Sigma^{V_1} C^q_{V_1} X \wedge \cdots \wedge \Sigma^{V_j} C^q_{V_j} X \longrightarrow \Sigma^W C^q_W X.$$

$B_q(\Sigma^V, C_V, X)$ is a subspace of $\Sigma^V C^q_V X$, and these maps restrict to the maps on q-simplices of the required simplicial map $\chi_j(g)_*$. The fact and degeneracy operations [**45**, 9.6] are respected because the maps θ, μ, and η are maps of $\mathscr{G}_0$-spaces and, for the zero$^{\text{th}}$ face (obtained from the $\beta_V \colon \Sigma^V C_V \to \Sigma^V$) because $\alpha_\infty \pi \colon CX \to QX$ is a map of $\mathscr{G}_0$-spaces (by Theorem VII.2.4) and the action of $\mathscr{G}$ on QX is induced by passage to limits from maps

$$\xi_j(g) \colon \Omega^{V_1} \Sigma^{V_1} X \wedge \cdots \wedge \Omega^{V_j} \Sigma^{V_j} X \longrightarrow \Omega^W \Sigma^W X.$$

The point is that all requisite compatibility pulls back to the level of finite dimensional inner product spaces from the compatibility statements (for K_∞ and C) codified in Definitions VI.1.6 to VI.1.8 and Proposition VI.1.9 and in Section VII§2. It is easy verified that the maps $\xi_j(g)$ of $(*)$ satisfy

the algebraic identities specified in Definition IV.1.1(a)–(c). Indeed, these identities are inherited from obvious identities for linear isometries and the identities given by the assertion that each $C^q X$ is a $\mathscr{G}_n$-space. Condition Definition IV.1.1(d) holds by the continuity of all of the functors employed in the definition of the $\xi_j(g)$. An easy diagram chase shows that the compatibility condition Definition IV.1.1(e) between the $\xi_j(g)$ and the maps σ of formula (VII.2) is satisfied. Finally, condition Definition IV.1.1(f) obviously holds if we define the map $\xi(f)\colon X \to X$ required for formula (VII.3) to be $\xi_1(g)$ for any $g \in \mathscr{G}(1)$ such that $g|_V = f$ $\square$

Thus E_∞ ring spaces determine E_∞ ring spectra. We know from Theorem VII.3.1 how the derived additive structure (that is, the spectrum structure) is related to the given additive E_∞ space structure. We next relate the derived multiplicative structure to be given multiplicative structure.

THEOREM VII.4.2. *For a $(\mathscr{C},\mathscr{G})$-space (X,θ,ξ), all of the maps $\varepsilon(\theta)$, $B(\alpha_\infty,\pi,1,1)$, γ^∞, $\tau(\eta)$, and ι specified in Theorem VII.3.1 are maps of $\mathscr{G}_0$-spaces, hence the first three of these maps are maps of $(\mathscr{C},\mathscr{G})$-spaces.*

PROOF. In view of [**45**, 9.6 and 9.9], $\varepsilon(\theta)$, $B(\alpha_\infty\pi,1,1)$, and $\tau(\eta)$ are geometric realizations of maps of simplicial $\mathscr{G}_0$-spaces and are therefore maps of $\mathscr{G}_0$-spaces by [**45**, 12.2]. The map γ^∞ is the limit over $V \subset \mathbb{R}^\infty$ of the maps

$$\gamma^V \colon |B_*(\Omega^V \Sigma^V, C_V; X)| \longrightarrow \Omega^V |B_*(\Sigma^V, C_V, X)|$$

(see [**45**, 9.7, 12.3, and 14.4]) and is easily verified to be a map of $\mathscr{G}_0$-spaces by explicit calculation (compare [**45**, 12.4]). It would be pointless to give the details since we know independently, by Lemma IV.1.6, that ι is a map of $\mathscr{G}_0$-spaces. $\square$

As promised, we have thus "group completed" the additive structure of an E_∞ ring space while carrying along the multiplicative structure. Again, the obvious special cases behave correctly.

Proposition VII.4.3. *Let E be a $\mathscr{G}$-spectrum. Then $\omega\colon TE_0 \to E$ is a morphism of $\mathscr{G}$-prespectra, hence $\widetilde{\omega}\colon \Omega^\infty TE_0 \to E$ is a morphism of $\mathscr{G}$-spectra.*

PROOF. By Definition IV.1.1(e), we have commutative diagrams

$$
\begin{array}{ccccc}
\Sigma^{v_1 \oplus \cdots \oplus v_j}(E_0 \wedge \cdots \wedge E_0) & \cong & \Sigma^{V_1} E_0 \wedge \cdots \wedge \Sigma^{V_j} E_0 & \xrightarrow{\sigma \wedge \cdots \wedge \sigma} & EV_1 \wedge \cdots \wedge EV_j \\
{\scriptstyle \Sigma^{\rho g}\xi_j(g)}\Big\downarrow & & & & \Big\downarrow{\scriptstyle \xi_j(g)} \\
\Sigma^W E_0 & & \xrightarrow{\hspace{4cm}\sigma\hspace{4cm}} & & EW
\end{array}
$$

for $g \in \mathscr{G}(j)$, $V_i \subset \mathbb{R}^\infty$, and $W = g(V_1 \oplus \cdots \oplus V_j)$. In view of the role played by the $\xi_j(g)$ on E_0 in the definition of the $\xi_j(g)$ on TE_0, it follows readily that the diagrams

$$
\begin{array}{ccc}
(TE_0)(V_1) \wedge \cdots \wedge (TE_0)(V_j) & \xrightarrow{\varepsilon(\phi_{V_1}) \wedge \cdots \wedge \varepsilon(\phi_{V_j})} & EV_1 \wedge \cdots \wedge EV_j \\
{\scriptstyle \xi_j(g)}\Big\downarrow & & \Big\downarrow{\scriptstyle \xi_j(g)} \\
(TE_0)(W) & \xrightarrow{\hspace{1.5cm}\varepsilon(\phi_W)\hspace{1.5cm}} & EW
\end{array}
$$

are commutative. This proves the first part, and the second part follows by Lemma IV.1.6. $\square$

Propositions VII.3.3 and VII.4.3 imply the following result.

Corollary VII.4.4. *For a $\mathscr{G}_0$-space Y, the composite deformation retraction*

$$\Omega^\infty TCY \xrightarrow{\Omega^\infty T(\alpha_\infty \pi)} \Omega^\infty TQY \xrightarrow{\widetilde{\omega}} Q_\infty Y$$

is a morphism of $\mathscr{G}$-spectra.

Indeed, even more is true. The inverse inclusion of $Q_\infty Y$ in $\Omega^\infty TCY$ and each h_t of the deformation obtained from [**45**, 9.9 and 11.10] by application of Ω^∞ are also morphisms of $\mathcal{G}$-spectra. This corollary can usefully be combined with the following consequence of [**46**, A.2(i)].

Corollary VII.4.5. *For a $\mathcal{G}_0$-space Y, the projection $\psi \colon CY \to C'Y$ is a map of $(\mathcal{C}, \mathcal{G})$-spaces and, if $\mathcal{C}'$ is an E_∞ operad, the induced map $\Omega^\infty T\psi \colon \Omega^\infty TCY \to \Omega^\infty TC'Y$ of $\mathcal{G}$-spectra is a weak homotopy equivalence.*

Consider, for example, the case $(\mathcal{C}', \mathcal{G}') = (\mathcal{Q}, \mathcal{Q})$. The corollaries and Proposition VII.3.3 imply that, for any (multiplicative) $\mathcal{Q}_0$-space Y, QY is weakly homotopy equivalent *as a $(\mathcal{Q} \times \mathcal{K}_\infty, \mathcal{Q} \times \mathcal{L})$-space* to ΓDY. When $Y = S^0$, $DY = \coprod K(\Sigma_j, 1)$ as a space. We have thus obtained a group completion of $\coprod K(\Sigma_j, 1)$ which is equivalent as an E_∞ ring space to QS^0. This is a greatly strengthened version of the Barratt–Quillen theorem. Note that we have made no use of the monoid structures on DS^0 and our discussion applies equally well to $C'S^0 = \coprod K(\Sigma_j, 1)$ for any E_∞ operad pair $(\mathcal{C}', \mathcal{G}')$. The force of the particular example $(\mathcal{Q}, \mathcal{Q})$ is the connection it establishes, via Example VI.5.1, between the category of finite sets and the sphere spectrum and thus, via Example VI.5.2, between algebraic K-groups and the stable homotopy groups of spheres (both with all internal structure in sight). Similarly, Remark VI.5.9 and the corollaries above imply the following result.

Corollary VII.4.6. *The E_∞ ring spectrum $\Omega^\infty TB\mathcal{O}\mathbb{Z}$ determined by the bipermutative category $\mathcal{O}\mathbb{Z}$ is equivalent to $Q_\infty(\mathbb{R}\mathrm{P}^{\infty+})$ and, under the equivalence $\Omega^\infty TB\mathcal{E} \simeq Q_\infty S^0$, the morphism of E_∞ ring spectra induced by the functor $e \colon \mathcal{E} \to \mathcal{O}\mathbb{Z}$ coincides with the natural split injection $Q_\infty S^0 \to Q_\infty(\mathbb{R}\mathrm{P}^{\infty+})$.*

5. The multiplicative E_∞ structure and localizations

Consider a $(\mathcal{C}, \mathcal{G})$-space (X, θ, ξ) where, as in the previous section, $(\mathcal{C}, \mathcal{G})$ is a product operad pair $(\mathcal{C}' \times \mathcal{K}_\infty, \mathcal{G}' \times \mathcal{L})$ with $\mathcal{C}'$ and $\mathcal{G}'$ locally contractible. We have a firm grasp on the $\mathcal{G}$-spectrum $\Omega^\infty T(X, \theta)$ and its relationship to X. Clearly the spectrum $\Omega^\infty T(X, \xi)$ is weakly contractible since its zero$^{\text{th}}$ space is a group completion of X in which the element $0 \in \pi_0 X$ becomes invertible. Thus we must delete components of X in order to obtain interesting spectra from its multiplicative E_∞ structure.

We make a simplifying assumption. As a commutative semi-ring, $\pi_0 X$ admits a unit $e \colon \mathbb{Z}^+ \to \pi_0 X$; indeed this morphism of semi-rings is obtained by application of π_0 to the unit $e \colon CS^0 \to X$. We assume henceforward that $e \colon \mathbb{Z}^+ \to \pi_0 X$ is an inclusion (as is the case in practice). Let M be a (multiplicative) submonoid of $\mathbb{Z}^+$ such that 0 is not in M and M contains at least one element other than 1. Let $\mathbb{Z}_M$ denote the localization of the integers at M (obtained by inverting the primes which divide elements of M). Define X_M to be the union of those components X_m of X such that $m \in M \subset \pi_0 X$ and note that X_M is a sub $\mathcal{G}$-space of X. We shall prove that the unit component of $\Gamma(X_M, \xi)$ is equivalent as an infinite loop space to the localization at M of the unit component of $\Gamma(X, \theta)$.

For an E_∞ space (Y, χ) and an element i of the group completion of $\pi_0 Y$, let $\Gamma_i(Y, \chi)$ denote the ith component of the zero$^{\text{th}}$ space $\Gamma(Y, \chi)$ of $\Omega^\infty T(Y, \chi)$. Let $\iota_\oplus \colon X \to \Gamma(X, \theta)$ and $\iota_\otimes \colon X_M \to \Gamma(X_M, \xi)$ denote the group completions obtained by specialization of Theorem VII.3.1. We shall make one further simplifying assumption (although it could perhaps be avoided at the price of some extra work).

Definition VII.5.1. X is said to be *convergent at M* if for each prime p which does not divide any element of M there exists an eventually increasing sequence $n_i(p)$ such that $(\iota_\oplus)_* \colon H_j(X_i; \mathbb{Z}_p) \to H_j(\Gamma_i(X, \theta); \mathbb{Z}_p)$ is an isomorphism for all $j \leq n_i(p)$. Here we allow $p = 0$, when $\mathbb{Z}_p$ is to be interpreted as the rational numbers.

This condition seems always to be satisfied in practice.

Example VII.5.2. X is convergent at M in the following cases.

(i) X is grouplike under θ, so that $\pi_0 X$ is a ring; here $\iota_\oplus \colon X_i \to \Gamma_i(X, \theta)$ is a weak homotopy equivalence.

(ii) $X = CY$ for some $\mathscr{G}_0$-space Y; here the result holds by inspection of the calculation of $H_*(CY; \mathbb{Z}_p)$ in [**26**, I§5].

(iii) For p not dividing any element of M, the additive translations $X_i \to X_{i+1}$ induce isomorphisms $H_j(X_i; \mathbb{Z}_p) \to H_j(X_{i+1}; \mathbb{Z}_p)$ for $j \leq n_i(p)$, where $\{n_i(p)\}$ is eventually increasing; here the result holds since, by [**46**, 3.9], $(\iota_\oplus)_*$ induces an isomorphism

$$\varinjlim H_*(X_i; \mathbb{Z}_p) \longrightarrow \varinjlim H_*(\Gamma_i(X, \theta); \mathbb{Z}_p) \cong H_*(\Gamma_0(X, \theta); \mathbb{Z}_p).$$

The last example applies to $X = B\mathscr{a}$ for the interesting bipermutative categories $\mathscr{a}$ displayed in Section VI§5.

We shall be considering spaces obtained by application of the one operad recognition principle of Theorem VII.3.1 to $\mathscr{G}$-spaces, hence all spaces in sight will be $\mathscr{G} \times \mathscr{K}_\infty$-spaces (where given $\mathscr{G}$-spaces are regarded as $\mathscr{G} \times \mathscr{K}_\infty$-spaces by pullback along the projection).

We shall allow ourselves to invert weak homotopy equivalences by working in the category $H\mathscr{T}$ (see Section II§2).

In the case $X = QS^0$, the idea of the following result is due to Sullivan. Tornehave [**76**, 5.8] proved this case and also proved a somewhat weaker result in the case $X = B\mathscr{GL}k_r$ [**77**, 3.1].

THEOREM VII.5.3. *Consider the following commutative diagram, in which all spaces are $\mathscr{G} \times \mathscr{K}_\infty$-spaces, all maps are composites of $\mathscr{G} \times \mathscr{K}_\infty$-maps and homotopy inverses of $\mathscr{G} \times \mathscr{K}_\infty$-maps, and the maps i are inclusions of components:*

(i) *If $\pi_0 X$ is a ring, then $(\Gamma i)\iota_\otimes \colon X_1 \to \Gamma_1(X_M, \xi)$ is a localization of X_1 at M.*

(ii) *If X is convergent, then $\Gamma\iota_\oplus \colon \Gamma(X_M, \xi) \to \Gamma(\Gamma_M(X, \theta), \xi)$ is a weak homotopy equivalence.*

Therefore, if X is convergent, the composite of $\mathscr{G} \times \mathscr{K}_\infty$-maps and inverses of $\mathscr{G} \times \mathscr{K}_\infty$-maps

$$\phi = (\Gamma\iota_\oplus)^{-1}(\Gamma i)\iota_\otimes \colon \Gamma_1(X, \theta) \longrightarrow \Gamma_1(X_M, \xi)$$

is a localization of $\Gamma_1(X, \theta)$ at M.

PROOF. The last statement will follow from (ii) and from (i) applied to $\Gamma(X, \theta)$. Write the set of elements of M in order as $1, m_1, m_2, \ldots$ and define $n_i = m_1 \cdots m_i \in M$. Fix $c_n \in \mathscr{C}(n)$ and write n for $\theta_n(c_n)(1^n) \in X_n$ for any positive integer n. Consider the sequence of spaces and maps

$$(*) \qquad\qquad X_1 \xrightarrow{\tau(m_1)} X_{n_1} \xrightarrow{\tau(m_2)} X_{n_2} \longrightarrow \cdots \longrightarrow X_{n_{i-1}} \xrightarrow{\tau(m_i)} X_{n_i} \longrightarrow \cdots$$

where $\tau(m_i)$ means multiplicative right translation by m_i; thus, for $m \in M$, $\tau(m)(x) = \xi_2(g)(x,m)$ for any fixed $g \in \mathcal{G}(2)$. By the definition of a group completion,

$$H_*(\Gamma_1(X_M, \xi); \Bbbk) = \varinjlim H_*(X_n; \Bbbk)$$

for any commutative ring $\Bbbk$, where the limit is taken over

$$\tau(m)_*: H_*(X_n, \Bbbk) \longrightarrow H_*(X_{mn}; \Bbbk), \quad m, n \in M.$$

(Compare [**46**, 1.2].) By cofinality, we see that $H_*(\Gamma_1(X_M, \xi); \Bbbk)$ is isomorphic to $H_*(\overline{X}_M; \Bbbk)$, where $\overline{X}_M$ denotes the mapping telescope of the sequence $(*)$. Moreover, by [**46**, 3.9], this isomorphism can be realized naturally by a map $\bar{\iota}_\otimes: \overline{X}_M \to \Gamma_1(X_M, \xi)$ in $H\mathcal{T}$ such that the following diagram commutes in $H\mathcal{T}$ (where j is the natural inclusion):

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\quad j \quad} & \overline{X}_M \\
{\scriptstyle \iota_\otimes}\downarrow & & \downarrow{\scriptstyle \bar{\iota}_\otimes} \\
\Gamma_1(X_1, \xi) & \xrightarrow{\Gamma i} & \Gamma_1(X_M, \xi)
\end{array}
$$

(Actually, the cited result is stated under cellular restrictions and with M free on one generator, but its proof transcribes trivially to the present context.) We prove first that $\widetilde{H}_*(\overline{X}_M; \mathbb{Z}_p) = 0$ if p divides some element of M. This will imply that multiplication by p is an isomorphism on $\widetilde{H}_*(\overline{X}_M; \mathbb{Z})$, hence that $H_*(\overline{X}_M; \mathbb{Z})$ is a $\mathbb{Z}_M$-module and $\Gamma_1(X_M, \xi)$ is an M-local space. $\overline{X}_M$ itself will be M-local if it is simple (or at least nilpotent). Denote the products on $H_*(X; \mathbb{Z}_p)$ coming from θ and ξ by $x * y$ and xy write $[n]$ for the homology class corresponding to the point $n \in X_n$. Let $x \in H_q(X; \mathbb{Z}_p)$, $q > 0$. We claim that $x \cdot [p^t] = 0$ for t sufficiently large. Indeed, by [**26**, II.1.5],

$$x \cdot [p] = x([1] * \cdots * [1]) = \sum x^{(1)} * \cdots * x^{(p)} = y^{[p]},$$

where $x \to \sum x^{(1)} \otimes \cdots \otimes x^{(p)}$ gives the iterated coproduct, y is the sum of the symmetric terms (all p $\otimes$-factors the same), and $y^{[p]}$ denotes the pth power of y under the $*$-product. Since $\deg y = q/p$, our claim follows by iteration. In the sequence $(*)$, p divides infinitely many of the m_i and our claim therefore implies that $\widetilde{H}_*(\overline{X}_M; \mathbb{Z}_p) = 0$. To prove (i), choose points $-n \in X_{-n}$, define $\rho(n)$ to be the additive right translation $\rho(n)(x) = \theta_2(c_2)(x, n)$, and observe that the definition, Definition VI.1.8 and Proposition VI.1.9, of a $(\mathcal{C}, \mathcal{G})$-space implies that the following ladder is homotopy commutative:

$$
\begin{array}{ccccccc}
X_1 & \xrightarrow{\tau(m_1)} & X_{n_1} & \xrightarrow{\tau(m_2)} & X_{n_2} & \longrightarrow & \cdots \\
{\scriptstyle \rho(-1)}\downarrow & & {\scriptstyle \rho(-n_1)}\downarrow & & & & \\
X_0 & \xrightarrow{\tau(m_1)} & X_0 & \xrightarrow{\tau(m_2)} & X_0 & \longrightarrow & \cdots
\end{array}
$$

The cited definitions also imply that $\tau(m)$ is homotopic to the mth power operation $x \to \theta_m(c_m)(x^m)$. Thus the bottom arrows $\tau(m_i)$ induce multiplication by m_i on homotopy groups, hence the mapping telescope of the bottom sequence is a localization of X_0 at M. Since the vertical arrows are homotopy equivalences and $\overline{X}_M$ is simple (as a limit of simple spaces), $j: X_1 \to \overline{X}_M$ and $\bar{\iota}_\otimes j = (\Gamma i)\iota_\otimes: X_1 \to \Gamma_1(X_M, \xi)$ are also localizations at M. To prove (ii), note that the first parts of the proof apply to $\Gamma(X, \theta)$ as well as to X and consider the commutative ladder

$$
\begin{array}{ccccccc}
X_1 & \xrightarrow{\tau(m_1)} & X_{n_1} & \xrightarrow{\tau(m_2)} & X_{n_2} & \longrightarrow & \cdots \\
{\scriptstyle \iota_\oplus}\downarrow & & {\scriptstyle \iota_\oplus}\downarrow & & {\scriptstyle i_\oplus}\downarrow & & \\
\Gamma_1(X, \theta) & \xrightarrow{\tau(m_1)} & \Gamma_{n_1}(X, \theta) & \xrightarrow{\tau(m_2)} & \Gamma_{n_2}(X, \theta) & \longrightarrow & \cdots
\end{array}
$$

Since X is convergent at M, the induced map $\overline{X}_M \to \overline{\Gamma}_M(X, \theta)$ of mapping telescopes induces isomorphisms on homology with coefficients in $\mathbb{Z}_p$ if p does not divide any element of M. Therefore

the same statement holds for

$$\Gamma \iota_\oplus \colon \Gamma_1(X_M, \xi) \longrightarrow \Gamma_1(\Gamma_M(X, \theta), \xi).$$

Since these spaces are M-local, this proves (ii) on the 1-component and therefore on all components.
$\square$

By application of the theorem in the situations of Corollaries VII.4.4 and VII.4.5, with $\mathscr{C}'$ an E_∞ operad, we obtain the following result. Recall from [45, 8.14] that $\pi_0 CY$ is the free commutative monoid generated by the based set $\pi_0 Y$ and that $\pi_0 QY$ is the group completion of $\pi_0 CY$. Let $C_M Y$, $C'_M Y$, and $Q_M Y$ denote the unions of the components corresponding to M in CY, $C'Y$ and QY.

Corollary VII.5.4. *Let Y be a $\mathscr{G}_0$-space and consider the following commutative diagram in $H\mathscr{T}$, in which all spaces are $\mathscr{G} \times \mathscr{K}_\infty$-spaces and all maps are composites of $\mathscr{G} \times \mathscr{K}_\infty$-maps and inverses of $crG \times \mathscr{K}_\infty$-maps*

$$
\begin{array}{ccccccc}
\Gamma_1(C'Y, \theta) & \xleftarrow{\Gamma\psi} & \Gamma_1(CY, \theta) & \xrightarrow{\Gamma\alpha_\infty\pi} & \Gamma_1(QY, \theta) & \xrightarrow{\;\tilde{\omega}\;} & Q_1 Y \\
{\scriptstyle\phi}\downarrow & & {\scriptstyle\phi}\downarrow & & {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle(\Gamma i)\iota_\otimes} \\
\Gamma_1(C'_M Y, \xi) & \xleftarrow{\Gamma\psi} & \Gamma_1(C_M Y, \xi) & \xrightarrow{\Gamma\alpha_\infty\pi} & \Gamma_1(Q_M Y, \xi) & =\!=\!= & \Gamma_1(Q_M Y, \xi)
\end{array}
$$

  (i) *All horizontal arrows are weak homotopy equivalences.*
  (ii) *All vertical arrows are localizations at M.*

Consider the case $(\mathscr{C}', \mathscr{G}') = (\mathcal{Q}, \mathcal{Q})$ and $Y = S^0$. Here $D_M S^0 = \coprod_{m \in M} K(\Sigma_m, 1)$ and $Q_1 S^0 = SF$. Thus, as an infinite loop space, the 1-component $\Gamma_1(D_M S^0, \xi)$ of our group completion of $\coprod_{m \in M} K(\Sigma_m, 1)$ is equivalent to the localization of SF at M. This statement is a version of the main theorem of Tornehave's paper [76]. The force of the particular example $(\mathcal{Q}, \mathcal{Q})$ is the connection it establishes, via Example VI.5.1, between the category of finite sets under $\otimes$ and the theory of stable spherical fibrations.

Algebraic and topological K-theory[*]

We here apply the machinery of the previous two chapters to obtain E_∞ ring spectra which represent various cohomology theories of interest. The emphasis will be on the construction and analysis of approximations derived from discrete categories for spaces and spectra relevant to the J-theory diagram studied in Chapter V.

In Section VIII§1, after showing that the ordinary cohomology theories with coefficients in commutative rings are represented by E_∞ ring spectra, we define higher K-groups of a permutative or bipermutative category $\mathscr{a}$ as the homotopy groups of its associated spectrum or E_∞ ring spectrum $\Omega^\infty TB\mathscr{a}$; when $\mathscr{a} = \mathscr{GL}A$ for a discrete ring A, our definition yields Quillen's higher K-groups of A [**59**, **61**]. When A is commutative, our construction rather trivial gives the ring structure on K_*A. We have already calculated $KO_*\mathbb{Z}$ in Corollary VII.4.6, and, in Remark VIII.3.6, we shall relate this to Quillen's results about $K_*\mathbb{Z}$ [**60**]. Beyond these observations, we have no new applications to algebraic K-theory. The calculational power of infinite loop space theory lies primarily in connection with fine structure, such as homology operations (and the arguments in Section VIII§4 will demonstrate how powerful this structure can be). It is not geared towards analysis of homotopy types (other than deloopings of known ones). In view of the present primitive state of calculations in algebraic K-theory, it is too early to tell how useful the rich extra structure which we shall obtain on the representing spectra for the relevant cohomology theories will turn out to be. We end Section VIII§1 with a discussion of the relationship between representation theory and the internal structure on the zero$^{\text{th}}$ spaces of spectra derived from bipermutative categories.

In Section VIII§2, we prove that the real and complex (connective) topological K-theories are represented by the E_∞ ring spectra $kO = \Omega^\infty TB\mathscr{O}$ and $kU = \Omega^\infty TB\mathscr{U}$. We then use Brauer lifting to transport Bott periodicity from kO and kU to $\Omega^\infty TB\mathscr{O}\overline{k}_q$ (q odd) and $\Omega^\infty TB\mathscr{GL}\overline{k}_q$, all completed away from q. These results imply that Brauer lifting on the completed zero$^{\text{th}}$ space level is an infinite loop map, a result first proven by the second author [**75**], in the complex case, by different methods. We also use recent results of Adams and Priddy [**8**] and of Madsen, Snaith, and the second author [**42**], together with a representation theoretical calculation, to prove that Brauer lifting gives an infinite loop map on the multiplicative as well as on the additive infinite loop space level.

One point of these results is that they allow us to study infinite loop properties of the Adams operations, and of maps derived from them, by use of the Frobenius automorphism in Section VIII§3. In Theorem VIII.3.2, we obtain discrete models j_p^δ for the spectra j_p introduced in Section V§5. These models result by completion of E_∞ ring spectra at p and, at p, the classifying space $B(SF; j_p^\delta)$ for j_p^δ-oriented spherical fibrations is $B\operatorname{Coker}J$ endowed with an infinite loop space structure. In Theorem VIII.3.4, we use Brauer lifting to demonstrate that a large portion of the J-theory diagram, centering around $B\operatorname{Coker}J$, is a commutative diagram of infinite loop spaces and maps.

In Section VIII§4, we construct an exponential infinite loop map $\Gamma_0 B\mathscr{GL}k_r \to \Gamma_1 B\mathscr{GL}k_r$ (away from r) and prove that, with $r = r(p)$, it becomes an equivalence when localized at an odd prime p. The domain and range here are discrete models for J_p and $J_{\otimes p}$, and the constructed map factors through the unit map $e\colon SF \to \Gamma_1 B\mathscr{GL}k_r$ of j_p^δ. It follows that BSF splits as $BJ_p \times B\operatorname{Coker}J$ as

---

infinite loop space at p. These results were first proven, quite differently, by the second author [**77**]. The present proofs do not use Brauer lifting and illustrate the richness of structure of E_∞ ring spaces. All of the constructions we use work equally well at the prime 2, but the key calculation fails; here the orientation sequence

$$SF \xrightarrow{\;e\;} J^\delta_{\otimes 2} \xrightarrow{\;\tau\;} B(SF; j^\delta_2) \xrightarrow{\;q\;} BSF$$

(where $J^\delta_{\otimes 2}$ is the 1-component of the zero$^{\text{th}}$ space of j^δ_2) may be regarded as a codification of how the infinite loop space sF is built up from $B\,\text{Coker}\,J$ and a discrete model for $J_{\otimes 2}$.

We agree to replace any space not of the homotopy type of a CW-complex by a weakly equivalent CW-complex, without change of notation (so as to allow the construction of inverse maps to weak equivalences without further verbiage).

1. Examples; algebraic K-theory

Let A be a commutative topological semi-ring or, equivalently, an (n, n)-space (see Lemma VI.2.4). By Theorems VII.2.4 and VII.4.1, $\Omega^\infty TA$ is an E_∞ ring spectrum (in fact, since $n \times \mathscr{L}$, an $\mathscr{L}$-spectrum) and its zero$^{\text{th}}$ space ΓA is an E_∞ ring space. By Theorem VII.4.2, $\iota\colon A \to \Gamma A$ is a group completion of the additive structure of A which is compatible with its multiplicative structure. Of course, ΓA is not a ring. The original precise structure on A has been weakened to structure precise only up to all higher coherence homotopies.

Now let A be discrete. Then ΓA is homologically discrete, in the sense that $H_i \Gamma A = 0$ for $i > 0$, and $\iota_*\colon A = \pi_0 A \to \pi_0 \Gamma A$ is the completion of A to a ring. If A is already a ring, then $\iota\colon A \to \Gamma A$ is a homotopy equivalence and the E_∞ ring spectrum $\Omega^\infty TA$ is thus an Eilenberg–Mac Lane spectrum $HA = \mathscr{K}(A, 0)$. Therefore any ordinary cohomology theory with coefficients in a commutative ring is represented by an E_∞ ring spectrum.

Less trivial examples arise from categories associated to commutative rings. We proceed from the general to the particular.

If $(\mathscr{a}, \oplus, 0, c)$ is a permutative category, then $B\mathscr{a}$ is a $\mathscr{Q}$-space, by Proposition VI.4.2 and Theorem VII.3.1 gives an infinite loop space $\Gamma B\mathscr{a}$ and a map $\iota\colon B\mathscr{a} \to \Gamma B\mathscr{a}$ which is a group completion. We define the algebraic K-groups of $\mathscr{a}$ by

$$K_i \mathscr{a} = \pi_i(\Gamma B\mathscr{a}, 0) \quad \text{for } i \geq 0.$$

If $(\mathscr{a}, \oplus, 0, c, \otimes, 1, \widetilde{c})$ is a bipermutative category, then $B\mathscr{a}$ is a $(\mathscr{Q}, \mathscr{Q})$-space, by Proposition VI.4.4, and Theorems VII.2.4 and VII.4.2 give that $\Gamma B\mathscr{a}$ is a $(\mathscr{Q} \times \mathscr{K}_\infty, \mathscr{Q} \times \mathscr{L})$-space and that ι is compatible with the multiplicative as well as the additive structure. Moreover, Lemma VI.2.5 gives that $K_* \mathscr{a}$ is a commutative (in the graded sense) and associative ring with unit. Additive right translation by one defines a homotopy equivalence $\rho(1)$ from the zero component $\Gamma_0 B\mathscr{a}$ to the 1-component $\Gamma_1 B\mathscr{a}$. Since $(\Gamma_0 B\mathscr{a}, \theta)$ and $(\Gamma_1 Bsra, \xi)$ are E_∞ spaces, we therefore have two 0-connected spectra, one coming from $\oplus$ and the other from $\otimes$, both of which have the higher K-groups of $\mathscr{a}$ as homotopy groups. These spectra will generally be very different, but Theorem VIII.4.1 below will show that in certain interesting cases they do not become equivalent when localized at an appropriate prime.

Now let the permutative category $\mathscr{a}$ be of the form specified in Remark VI.5.8, so that $\mathscr{a}$ can be thought of as a disjoint union of topological groups $G(n)$ for $n \geq 0$. Then $B\mathscr{a} = \coprod BG(n)$. Define BG to be the limit of the translations $\rho(1)\colon BG(n) \to BG(n+1)$. As explained in [**46**, 3.9], there is a well-defined natural homotopy class

$$\overline{\lambda}\colon BG \longrightarrow \Gamma_0 B\mathscr{a}$$

such that the restriction of $\overline{\iota}$ to $BG(n)$ is homotopic to the composite of $\iota\colon BG(n) \to \Gamma_n B\mathscr{a}$ and the translation $\Gamma_n B\mathscr{a} \to \Gamma_0 B\mathscr{a}$; the fact that $\iota\colon B\mathscr{a} \to \Gamma B\mathscr{a}$ is a group completion implies that $\overline{\iota}$ induces an isomorphism on homology (without any coefficients). Therefore ΓBG is homologically equivalent to $BG \times \mathbb{Z}$.

To relate the constructions above to Quillen's algebraic K-theory, we must review some of his results and definitions [**59, 61, 62**]. Recall that a group is said to be perfect if it is equal to its commutator subgroup. Let X be a connected CW-complex and let N be a perfect normal subgroup of $\pi_1 X$. Then there is a map $f: X \to X^+$ unique up to homotopy, such that the kernel of $\pi_1 f$ is N and f induces an isomorphism on homology with any coefficients (wee Wagoner [**79**]). If N is the commutator subgroup of $\pi_1 X$, then X^+ is a simple space. If Y is a connected space such that $\pi_1 Y$ contains no non-trivial perfect subgroup, then $f^*: [X^+, Y] \to [X, Y]$ is an isomorphism.

In this connection, we record the following useful triviality.

Lemma VIII.1.1. *Let $f: X \to X'$ be a map of connected spaces which induces an isomorphism on integral homology. Then*

$$f^*: [X', \Omega Z] \longrightarrow [X, \Omega Z]$$

is an isomorphism for any space Z.

PROOF. Let A be a discrete ring (associative with unit). The commutator subgroup EA of GLA is perfect, and Quillen defined

$$K_i A = \pi_i(BGLA^+) \quad \text{for } i \geq 1.$$

Consider the permutative category $\mathscr{GL}A$ of Example VI.5.2. By the universal property of f (or the lemma), $\bar{\iota}: BGLA \to \Gamma_0 B\mathscr{GL}A$ induces a map $\widetilde{\iota}: BGLA^+ \to \Gamma_0 B\mathscr{GL}A$ such that $\widetilde{\iota} \circ f$ is homotopic to $\bar{\iota}$. Since $\bar{\iota}$ and f are homology isomorphisms, so is $\widetilde{\iota}$. Since $BGLA^+$ and $\Gamma_0 B\mathscr{GL}A$ are simple, $\widetilde{\iota}$ is therefore a homotopy equivalence. Thus

$$K_i A = K_i \mathscr{GL}A \quad \text{for } i \geq 1.$$

$\square$

Now let A be commutative. Then $\mathscr{GL}A$ is bipermutative and $K_* \mathscr{GL}A$ is thus a ring. here $K_0 \mathscr{GL}A = \mathbb{Z}$. If instead of $\mathscr{GL}A$ we use a bipermutative category $\mathscr{P}A$ of finitely generated projective modules (as exists by Proposition VI.3.5), then we would obtain a commutative graded ring $K_* \mathscr{P}A$ such that $K_i \mathscr{P}A = K_i A$ for all $i \geq 0$ (by [**46**, p. 85]).

Alternative constructions of spectra having the $K_i A$ as homotopy groups can be obtained by use of the black boxes of Boardman and Vogt, Segal, Anderson, the second author, and Barratt and Eccles [**20, 68, 10, 76, 16**]. It seems likely that all of these constructions yield spectra equivalent to ours, but a proof would be tedious and unrewarding. Gersten and Wagoner [**30, 79**] have constructed a spectrum having the $K_i A$ as homotopy groups by means of ring theoretic arguments within algebraic K-theory. The relationship between their spectrum and ours will be determined in Chapter IX.

The discussion of Sections VIII§2 and VIII§3 below suggests that the algebraic K-theory of discrete commutative rings can be thought of as analogous to complex K-theory and that the appropriate analog of real K-theory can be defined by

$$KO_i A = K_i \mathscr{O}A \quad \text{for } i \geq 1,$$

where $\mathscr{O}A$ is the bipermutative category of Example VI.5.3. The ring homomorphism

$$K_* \mathscr{O}A \longrightarrow K_* \mathscr{GL}A$$

indued by the inclusion of $\mathscr{O}A$ in $\mathscr{GL}A$ can be thought of as analogous to complexification. This idea is presumably not new: it can be viewed as the starting point for Karoubi's treatment of Hermitian K-theory [**34**]. (However, by Corollary VII.4.6, $KO_* \mathbb{Z}$ is not very interesting.)

The following immediately consequence of Remark VI.5.8 and Theorems VII.3.1 and VII.4.2 plays a key role in many of the topological applications. It reduces to group theory the analysis

of the action maps θ and ξ on the E_∞ ring space $\Gamma B\mathscr{a}$ derived from a bipermutative category $\mathscr{a} = \coprod G(n)$. Let

$$c_p \colon \Sigma_p \wr G(n) \longrightarrow G(pn) \quad \text{and} \quad \widetilde{c}_p \colon \Sigma_p \wr G(n) \longrightarrow G(p^n)$$

be the homomorphisms of groups specified by

$$c_p(\sigma, g_1, \ldots, g_p) = (g_{\sigma^{-1}(1)} \oplus \cdots \oplus g_{\sigma^{-1}(p)})\sigma(n, \ldots, n)$$

and

$$c_p(\sigma, g_1, \ldots, g_p) = (g_{\sigma^{-1}(1)} \times \cdots \times g_{\sigma^{-1}(p)})\sigma \langle n, \ldots, n \rangle .$$

(See Notations VI.1.1 and VI.1.4 for notations.) Let ψ denote both projections $\mathcal{Q} \times \mathscr{K}_\infty \to \mathcal{Q}$ and $\mathcal{Q} \times \mathscr{L} \to \mathcal{Q}$.

Proposition VIII.1.2. *For any permutative category $\mathscr{a}$ of the form $\coprod G(n)$, the following diagram is homotopy commutative:*

$$
\begin{array}{ccccc}
[\mathcal{Q}(p) \times \mathscr{K}_\infty(p)] \times_{\Sigma_p} BG(n)^p & \xrightarrow{\psi \times 1} & \mathcal{Q}(p) \times_{\Sigma_p} BG(n)^p \cong B(\Sigma_p \wr G(n)) & \xrightarrow{Bc_p} & BG(pn) \\
{\scriptstyle 1 \times \iota^p} \downarrow & & & & \downarrow {\scriptstyle \iota} \\
[\mathcal{Q}(p) \times \mathscr{K}_\infty(p)] \times_{\Sigma_p} (\Gamma_n B\mathscr{a})^p & & \xrightarrow{\quad\quad\quad\quad \theta_p \quad\quad\quad\quad} & & \Gamma_{pn} B\mathscr{a}
\end{array}
$$

If $\mathscr{a}$ is bipermutative, then the following diagram is commutative:

$$
\begin{array}{ccccc}
[\mathcal{Q}(p) \times \mathscr{L}(p)] \times_{\Sigma_p} BG(n)^p & \xrightarrow{\psi \times 1} & \mathcal{Q}(p) \times_{\Sigma_p} BG(n)^p \cong B(\Sigma_p \wr G(n)) & \xrightarrow{B\widetilde{c}_p} & BG(n^p) \\
{\scriptstyle 1 \times \iota^p} \downarrow & & & & \downarrow {\scriptstyle \iota} \\
[\mathcal{Q}(p) \times \mathscr{L}(p)] \times_{\Sigma_p} (\Gamma_n B\mathscr{a})^p & & \xrightarrow{\quad\quad\quad\quad \xi_p \quad\quad\quad\quad} & & \Gamma_{n^p} B\mathscr{a}
\end{array}
$$

Let $\mathscr{a} = \mathscr{GL}A$ for a commutative topological ring A and let G be a compact topological group. For a representation $\rho \colon G \to GL(n, A)$ and a subgroup π of Σ_p, define the additive and multiplicative wreath product representations $\pi \wr p$ and $\pi \wr_\otimes p$ to be the composites

$$\pi \wr G \xrightarrow{1 \wr \rho} \pi \wr GL(n, A) \xrightarrow{c_p} GL(pn, A)$$

and

$$\pi \wr G \xrightarrow{1 \wr \rho} \pi \wr GL(n, A) \xrightarrow{\widetilde{c}_p} GL(n^p, A)$$

The proposition reduces analysis of θ_p and ξ_p on $\Gamma B\mathscr{GL}A$ to analysis of these wreath products. Pragmatically, however, there is an essential difference. The operation $\pi \wr \rho$ is additive in ρ, hence passes to representation rings, and is trivially seen to satisfy the character formula

$$\chi(\pi \wr \rho)(\theta; g_1, \ldots, g_p) = \sum_{\sigma(i) = i} \chi(\rho)(g_i).$$

In contrast, $\pi \wr_\otimes \rho$ is multiplicative but not additive in ρ,[2] and there is no general formula for the calculation of $\chi(\pi \wr_\otimes \rho)$ in terms of $\chi(\rho)$.

Of course, E_∞ maps are structure-preserving before passage to homotopy, whereas representation theoretic techniques apply only after passage to homotopy. This suggests use of the following (not quite standard) definition.

---

[2] With $k = p$ and $j_r = 2$, the diagram of Lemma VI.4.3 implies the formula $\pi \wr_\otimes (\rho + \sigma) = \pi \wr_\otimes \rho + \pi \wr_\otimes \sigma + \sum_{i=1}^{p-1} \frac{1}{p}(i, p - i)\pi \wr \rho^i \sigma^{p-i}$.

Definition VIII.1.3. Let p be a prime, let π be the cyclic group of order p embedded as usual in Σ_p, and let W be any contractible space on which π acts freely (for example $\mathscr{C}(p)$ for any E_∞ operad $\mathscr{C}$). An H_∞^p-space (X,θ) is a (homotopy associative) H-space X together with a map $\theta \colon W \times_\pi X^p \to X$ such that for each $w \in W$ the restriction of θ to $X^p \cong w \times X^p$ is homotopic to the p-fold iterate of the product on X. An H_∞^p-map $f \colon (X,\theta) \to (X',\theta')$ is an H-map $f \colon X \to X'$ such that the following diagram is homotopy commutative:

$$
\begin{array}{ccc}
W \times_\pi X^p & \xrightarrow{1 \times f^p} & W \times_\pi (X')^P \\
\theta \downarrow & & \downarrow \theta' \\
X & \xrightarrow{\quad f \quad} & X'
\end{array}
$$

Clearly an E_∞ map, an in particular an infinite loop map, is an H_∞^p-map. Mod p homology operations are defined in terms of θ_* [**26**, I§1] and are thus preserved by H_∞^p-maps. If X and X' are infinite loop spaces derived from permutative categories (of the usual form) and if appropriate $\varprojlim^1$ terms vanish, then Proposition VIII.1.2 reduces the determination of whether or not an H-map $f \colon X \to X'$ is an H_∞^p-map to representation theory. The following remarks give the details of this reduction.

Remark VIII.1.4. Let Y be an infinite loop space with induced H_∞^p-structure $\xi \colon W \times_\pi Y^p \to Y$. (We use the letter ξ since we choose to think of Y as multiplicative, that being appropriate to our applications of these remarks in the next section.) Fix points m and m^{-1} in the components Y_m and $Y_{m^{-1}}$ of Y. Since the product $\#$ and inverse map χ on Y are infinite loop maps and thus H_∞^p-maps, the following diagrams are homotopy commutative; they show how ξ on $W \times_\pi Y_1^p$ relates to ξ on $W \times_\pi Y_m^p$:

$$
\begin{array}{ccccc}
W \times_\pi Y_1^p & \xrightarrow{1 \times \tau(m)^p} & W \times_\pi Y_m^p & \xrightarrow{(1,\omega_{m^{-1}})} & (W \times_\pi Y_m^p) \times (W \times_\pi \{m^{-1}\}^p) \\
\xi \downarrow & & & & \downarrow \xi \times \xi \\
Y_1 & \xleftarrow{\hspace{4cm} \# \hspace{4cm}} & & & Y_{m^p} \times Y_{m^{-p}}
\end{array}
$$

where $\tau(m)(y) = y \# m$ and $\omega_m(w, y_1, \ldots, y_p) = (w, m, \ldots, m)$, and

$$
\begin{array}{ccc}
W \times_\pi Y_{m^{-1}}^p & \xrightarrow{1 \times \chi^p} & W \times_\pi Y_m^p \\
\xi \downarrow & & \downarrow \xi \\
Y_{m^{-p}} & \xleftarrow{\quad \chi \quad} & Y_{m^p}
\end{array}
$$

These diagrams combine to give the homotopy commutative diagram

$$
\begin{array}{ccccc}
W \times_\pi Y_1^p & \xrightarrow{1 \times \tau(m)^p} & W \times_\pi Y_m^p & \xrightarrow{(1,\omega)} & (W \times_\pi Y_m^p) \times (W \times_\pi \{m\}^p) \\
\xi \downarrow & & & & \downarrow \xi \times \xi \\
Y_1 & \xleftarrow{\quad \# \quad} Y_{m^p} \times Y_{m^{-p}} & \xleftarrow{\quad 1 \times \chi \quad} & & Y_{m^p} \times Y_{m^p}
\end{array}
$$

Let Z be another infinite loop space and suppose that $\pi_0 Y = \pi_0 Z$. Let $f \colon Y_1 \to Z_1$ be an H-map. Since $Y \simeq Y_1 \times \pi_0 Y$ and $Z \simeq Z_1 \times \pi_0 Z$ as H-spaces [**26**, I.4.6], f is the 1-component of an H-map $f \colon Y \to Z$ with $\pi_0 f = 1$. Assume that $Y = \Gamma(X,\xi)$, where (X,ξ) is an E_∞ space with $\pi_0 X$ a multiplicative submonoid of $\mathbb{Z}^+$, and let $\iota \colon X \to Y$ be the natural E_∞ map. Write m for the consistently chosen basepoints in X_m, Y_m, and Z_m. All parts of the following diagram commute,

by the facts above, except from the front and lower back right rectangles:

By [**46**, 3.9], if $\overline{X}$ is the mapping telescope of (a cofinal subsequence of) the X_m, then there is a homology isomorphism $\overline{\iota} \colon \overline{X} \to Y_1$ whose restriction to X_m is homotopic to $X_m \xrightarrow{\iota} Y_m \xrightarrow{\tau(m^{-1})} Y_1$. $1 \times \overline{\iota}^p \colon W \times_\pi \overline{X}^p \to W \times_\pi Y_1^p$ and the natural map $\mathrm{Tel}(W \times_\pi X_m^p) \to W \times_\pi \overline{X}^p$ are homology isomorphisms, hence, by Lemma VIII.1.1 and the diagram, we conclude that if $\varinjlim^1[W \times_\pi X_m^p, \Omega Z_1] = 0$ and if $f\iota \colon X \to Z$ is an H_∞^p-map, then $f \colon Y_1 \to Z_1$ is an H_∞^p-map. When the X_m are classifying space of groups and Z_m is $BO \times \{m\}$ or $BU \times \{m\}$, Proposition VIII.1.2 reduces to the verification that $f\iota$ is an H_∞^p-map to representation theory.

The following remarks recast the notions of H_∞^p-space and H_∞^p-map.

Remark VIII.1.5. Let (X, θ) be an H_∞^p-space. Let $\nu \colon E \to B$ be a principal π-covering classified by $\mu \colon B \to W/\pi$, let $\widetilde{\mu} \colon E \to W$ cover μ, and define $\overline{\nu} \colon B \to W \times_\pi E^p$ by $\overline{\nu}(b) = (\widetilde{\mu}e, \alpha e, \ldots, \alpha^p e)$, where $\nu e = b$ and α generates π. For $g \in [E^+, X]$, define the transfer $\tau(g) \in [B^+, X]$ to be the composite

$$B \xrightarrow{\overline{\nu}} W \times_\pi E^p \xrightarrow{1 \times g^p} W \times_\pi X^p \xrightarrow{\theta} X.$$

When ν is $W \times X^p \to W \times_\pi X^p$ and $g \colon W \times X^p \to X$ is the projection on the last coordinate, $(1 \times g^p)\overline{\nu} = 1$. Therefore an H-map $X \to Y$ between H_∞^p-spaces is an H_∞^p-map if and only if it commutes with transfer. For examples such as $\rho^r \colon BSO \to BSO_\otimes$ [**42**], it is useful to observe that a simple diagram chase from the definition, Definition VI.1.10, of an E_∞ ring space implies that if (X, θ, ξ) is an E_∞ ring space, then

$$\tau_\otimes(f + g) = \tau_\otimes(f) + \tau_\otimes(g) + \sum_{i=1}^{p-1} \frac{1}{p}(i, p-i)\tau(f^i g^{p-i}),$$

where τ and $\tau_\otimes$ denote the transfers associated to θ and ξ and $+$ and $\cdot$ denote the products induced by θ and ξ on the functor $[?, X]$.

The criterion (below Theorem V.7.2) of Madsen, Snaith, and the second author [**42**] for determining when an H-map $f \colon \widehat{BU}_p \to \widehat{BU}_p$ or $f \colon \widehat{BSO}_p \to \widehat{BSO}_p$ is an infinite loop map translates to the assertion that f is an infinite loop map if and only if it is an H_∞^p-map. The Adams–Priddy theorem Theorem V.4.2, together with Theorems II.2.13 and II.2.14, yields the following very useful consequence.

THEOREM VIII.1.6. *Let X and Y be infinite loop spaces of the homotopy type of BSO or of BSU localized or completed at a set of primes T. Then an H-map $X \to Y$ is an infinite loop map if and only if its completion at p is an H_∞^p-map for all primes $p \in T$.*

2. Bott periodicity and Brauer lifting

Write $\mathscr{G}$ for either of the bipermutative categories $\mathscr{O}$ or $\mathscr{U}$ specified in Example VI.5.4 and write G for either O or U. Define $kG = \Omega^\infty TB\mathscr{G}$. The homology isomorphism $\bar{\iota}\colon BG \to \Gamma_0 B\mathscr{G}$ is a homotopy equivalence, hence the zero$^{\text{th}}$ space $\Gamma B\mathscr{G}$ of kG is equivalent to $BG \times \mathbb{Z}$. Since $\iota\colon \coprod BG(n) \to \Gamma BG$ is a map of H-spaces for both $\oplus$ and $\otimes$, it follows that ΓBG represents the ring-valued functor KGX on finite dimensional CW-complexes X. The external tensor product $KGX \otimes KGY \to KG(X \times Y)$ is defined on maps $f\colon X^+ \to \Gamma B\mathscr{G}$ and $g\colon Y^+ \to \Gamma B\mathscr{G}$ (where the plus notation again denotes addition of a disjoint basepoint) as the composite

$$(X \times Y)^+ = X^+ \wedge Y^+ \xrightarrow{f \wedge g} \Gamma B\mathscr{G} \wedge \Gamma B\mathscr{G} \xrightarrow{\otimes} \Gamma B\mathscr{G}.$$

Let $d = 8$ or $d = 2$ for $G = 0$ or $G = u$. One formulation of Bott periodicity asserts that

$$\otimes\colon KGX \otimes KGS^d \longrightarrow KG(X \times S^d)$$

is an isomorphism or, equivalently, that tensoring with a generator $[b] \in \pi_d BG = \pi_d \Gamma_0 B\mathscr{G}$ defines an isomorphism

$$\widetilde{K}GX \longrightarrow \widetilde{K}G(\Sigma^d X).$$

Bott [**21**] deduced the latter isomorphism by verifying that the adjoint $BG \to \Omega_0^d BG$ of the composite

$$BG \wedge S^d \xrightarrow{1 \wedge b} BG \wedge BG \xrightarrow{\otimes} BG$$

is homotopic to the iterated Bott map discussed in Section I§1. Under the equivalence $\bar{\iota}\colon BG \to \Gamma_0 BG$, this adjoint corresponds to the restriction to 0-components of zero$^{\text{th}}$ spaces of the adjoint $\beta\colon kG \to \Omega^d kG$ of the composite map in $H\mathscr{S}$

$$kG \wedge S^d \xrightarrow{1 \wedge b} kG \wedge \Gamma_0 B\mathscr{G} \xrightarrow{\phi} kG$$

given by Lemma II.2.9. In view of Definition II.3.10, we conclude that $\beta_0\colon \Gamma B\mathscr{G} \to \Omega^d \Gamma B\mathscr{G}$ agrees under the equivalence $\Gamma B\mathscr{G} \simeq BG \times \mathbb{Z}$ with the Bott map $BG \times \mathbb{Z} \to \Omega^d(BG \times \mathbb{Z})$.

By Proposition II.3.2 and Corollaries II.3.9 and II.3.14, it follows immediately that kO and kU are isomorphic in $H\mathscr{S}$ to the connective ring spectra obtained from the periodic Bott spectra by killing their homotopy groups in negative degrees. We have thus proven the following result.

THEOREM VIII.2.1. *kO and kU represent real and complex connective K-theory (as ring-valued cohomology theories).*

In particular, the diagrams of Proposition VIII.1.2 now reduce the analysis of homology operations on BO and BU to representation theory. Application of the additive diagram to BO and BU was first justified by Boardman [unpublished] and has been exploited by Priddy [**54**] in mod 2 homology and by Snaith [**70**] in K-theory.

Remark VIII.2.2. As proven by Bott [**21**], real periodicity factors as the composite of the two natural isomorphisms

$$\widetilde{K}OX \otimes \widetilde{K}SpS^4 \longrightarrow \widetilde{K}Sp(\Sigma^4 X) \quad \text{and} \quad \widetilde{K}SpY \otimes \widetilde{K}SpS^4 \longrightarrow \widetilde{K}O(\Sigma^4 Y).$$

A full understanding of these transformations in our context would seem to require a theory of E_∞ module spectra over E_∞ ring spectra.

We now turn to Brauer lifting. Fix a prime q and let $k = \bar{k}_q$ be an algebraic closure of the field of q elements. Let k_r denote the field with $r = q^a$ elements contained in k, so that $k = \varprojlim k_r$. Fix and embedding $\mu\colon k^* \to \mathbb{C}^*$ of multiplicative groups. Recall from Green [**31**, Theorem 1] that

if $\rho\colon G \to GL(n, k)$ is a representation of a finite groups G and if $\rho(g)$ has roots $\xi_i(g)$, then the complex-valued function

$$\chi_\rho(g) = \sum_{i=1}^{n} \mu\xi_i(g)$$

is the character of a unique (virtual) representation $\lambda(\rho) \in RG$. Quillen [58, p. 79] proved that if q is odd and ρ takes values in $O(n, k)$, then χ_ρ is the character of a (necessarily unique) real representation $\lambda(\rho) \in ROG$. If $\pi\colon H \to GL(m, k)$ is a representation of another finite group H, then

$$\chi_{\pi\oplus\rho}(h, g) = \chi_\pi(h) + \chi_\rho(g) \quad\text{and}\quad \chi_{\pi\otimes\rho}(h, g) = \chi_\pi(h)\chi_\rho(g).$$

Therefore, if R_kG denotes the representation ring of G over k, the following diagrams are commutative:

(A)
$$\begin{array}{ccc}
R_kH \times R_kG & \xrightarrow{\ \oplus\ } & R_k(H \times G) \\
{\scriptstyle \lambda\times\lambda}\downarrow & & \downarrow{\scriptstyle \lambda} \\
RH \times RG & \xrightarrow{\ \oplus\ } & R(H \times G)
\end{array}
\quad\text{and}\quad
\begin{array}{ccc}
R_kH \times R_kG & \xrightarrow{\ \otimes\ } & R_k(H \times G) \\
{\scriptstyle \lambda\times\lambda}\downarrow & & \downarrow{\scriptstyle \lambda} \\
RH \times RG & \xrightarrow{\ \otimes\ } & R(H \times G)
\end{array}$$

By Adams' formula [1, 4.1(vi)], the following diagram also commutes, where, for $r = q^a$, ϕ^r denotes the iterated Frobenius automorphism (Example VI.5.5):

(B)
$$\begin{array}{ccc}
R_kG & \xrightarrow{\ \phi^r\ } & R_kG \\
{\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \lambda} \\
RG & \xrightarrow{\ \psi^r\ } & RG
\end{array}$$

Thus $\psi^r\lambda(\rho) = \lambda(\rho)$ if $\rho\colon G \to GL(n, k)$ factors through $GL(n, k_r)$. The analogs of the diagrams above also commute in the real case and relate the orthogonal representation ring RO_kG to ROG.

Of course, passage to the classifying maps and then to Grothendieck groups gives ring homomorphisms $RG \to KU(BG)$ and $ROG \to KO(BG)$, and these become isomorphisms when the left sides are completed with respect to the IG-adic topology [14]. Moreover, by [14, 4.2 and 7.1 (and p. 13,17)], $KU^{-1}(BG) = 0$ and $KO^{-1}(BG)$ is a finite dimensional vector space over $\mathbb{Z}_2$. Let $\lambda(n, r)\colon BGL(n; k_r) \to BU$ represent the element of $KU(BGL(n; k_r))$ obtained by application of λ to the difference of the inclusion of $GL(n, k_r)$ in $GL(n, k)$ and the trivial representation of degree n. Since λ is natural in G and additive, the maps $\lambda(n, r)$ are compatible (up to homotopy) as n and r increase. Since the relevant $\varinjlim^1$ terms vanish, there result unique homotopy classes

(∗) $\lambda\colon BGL\overline{k}_q \longrightarrow BU$ and, if q is odd, $\lambda\colon BO\overline{k}_q \longrightarrow BO$

compatible with the $\lambda(n, r)$. The main step in Quillen's proof of the Adams conjecture was the following result [58, 1.6].

THEOREM VIII.2.3. *The maps λ of (∗) induce isomorphisms on cohomology with coefficients in $\mathbb{Z}_p$ for each prime $p \neq q$.*

As we have explained in Section VIII§1, the group completion property of the recognition principle Theorem VII.3.1 gives homology isomorphisms

$$\bar{\iota}\colon BGLk \longrightarrow \Gamma_0 B\mathscr{GL}k \quad\text{and}\quad \bar{\iota}\colon BOk \longrightarrow \Gamma_0 B\mathscr{O}k.$$

Invoking Lemma VIII.1.1, we define

(∗∗) $\widetilde{\lambda}\colon \Gamma_0 B\mathscr{GL}\overline{k}_q \longrightarrow BU$ and, if q is odd, $\widetilde{\lambda}\colon \Gamma_0 B\mathscr{O}\overline{k}_q \longrightarrow BO$

to be the unique homotopy classes such that $\widetilde{\lambda} \circ \bar{\iota} \simeq \lambda$. (Of course, we could also invoke the properties of the plus construction, but its use would add nothing to the discussion.)

We shall need the following observation.

Lemma VIII.2.4. *The following diagrams are homotopy commutative:*

$$
\begin{array}{ccc}
\Gamma_0 B\mathscr{GL}k \times \Gamma_0 B\mathscr{GL}k & \xrightarrow{\oplus} & \Gamma_0 B\mathscr{GL}k \\
\widetilde{\lambda}\times\widetilde{\lambda}\downarrow & & \downarrow\widetilde{\lambda} \\
BU \times BU & \xrightarrow{\oplus} & BU
\end{array}
\quad and \quad
\begin{array}{ccc}
\Gamma_0 B\mathscr{GL}k \wedge \Gamma_0 B\mathscr{GL}k & \xrightarrow{\otimes} & \Gamma_0 B\mathscr{GL}k \\
\widetilde{\lambda}\wedge\widetilde{\lambda}\downarrow & & \downarrow\widetilde{\lambda} \\
BU \times BU & \xrightarrow{\otimes} & BU
\end{array}
$$

and similarly with $\Gamma_0 B\mathscr{GL}k$ and BU replaced by $\Gamma_0 B\mathscr{O}k$ and BO.

PROOF. Since $\widetilde{KU}(X \wedge Y) \to \widetilde{KU}(X \times Y)$ is a monomorphism for any X and Y, it suffices to consider the second diagram with smash products replaced by Cartesian products. For both diagrams, it suffices to prove commutativity after composition with $\bar{\iota} \times \bar{\iota}$ and thus, since the relevant $\varprojlim^1$ terms vanish, after further composition with the inclusions of $BGL(m, k_r) \times BGL(n, k_s)$. Now the conclusion is immediate from the diagrams Equation (A). $\qquad\square$

Similarly, diagram Equation (B) implies the following result:

Lemma VIII.2.5. *The following diagrams are homotopy commutative:*

$$
\begin{array}{ccc}
\Gamma_0 B\mathscr{GL}k & \xrightarrow{\phi^r} & \Gamma_0 B\mathscr{GL}k \\
\widetilde{\lambda}\downarrow & & \downarrow\widetilde{\lambda} \\
BU & \xrightarrow{\psi^r} & BU
\end{array}
\quad and \quad
\begin{array}{ccc}
\Gamma_0 B\mathscr{O}k & \xrightarrow{\phi^r} & \Gamma_0 B\mathscr{O}k \\
\widetilde{\lambda}\downarrow & & \downarrow\widetilde{\lambda} \\
BO & \xrightarrow{\psi^r} & BO
\end{array}
$$

At this point, it will be convenient to introduce a generic (and abusive) simplification of notation, to be used throughout the rest of the book. We shall write Y^δ for specified "discrete models" for topologically significant spaces or spectra Y. In each case, Y^δ will be derived from the classifying spaces of discrete categories by means of suitable topological constructions. In particular, we have the following notations.

Definition VIII.2.6. Define BU^δ and BO^δ to be the completions away from q of the spaces $\Gamma_0 B\mathscr{GL}\bar{k}_q$ and (with q odd) $\Gamma_0 B\mathscr{O}\bar{k}_q$. Revert to the convention that $G = O$ or $G = U$ and define $\widehat{\lambda}\colon BG^\delta \to \widehat{BG}[1/q]$ to be the completion away from q of the map $\widetilde{\lambda}$ of $(**)$. Define kO^δ and kU^δ to be the completions away from q of the E_∞ ring spectra $\Omega^\infty TB\mathscr{O}\bar{k}_q$ and $\Omega^\infty TB\mathscr{GL}\bar{k}_q$. Then BG^δ is the 0-component of the zero$^{\text{th}}$ space of kG^δ.

The following result is an immediate consequence of Theorem VIII.2.3 and the homological characterization of completions (see Section II§2).

Corollary VIII.2.7. $\widehat{\lambda}\colon BG^\delta \to \widehat{BG}[1/q]$ *is a homotopy equivalence.*

This justifies our thinking of BG^δ as a model for BG. Of course, the use of completions rather than localizations is essential here since $\pi_{2i}BGL\bar{k}_q^+ = 0$ and $\pi_{2i+1}BGL\bar{k}_q^+ = \mathbb{Z}[q^{-1}]/\mathbb{Z}$ [**59**, p. 585]. We can now verify that kG^δ represents the completion away from q of real or complex connective K-theory and that the equivalence $\widehat{\lambda}$ is an infinite loop map.

THEOREM VIII.2.8. *There is a unique isomorphism $\Lambda\colon kG^\delta \widehat{kG}[1/q]$ of ring spectra in $H\mathscr{S}$ such that the 0-component of the zeroth map of Λ is equivalent to $\widehat{\lambda}\colon BG^\delta \to \widehat{BG}[1/q]$, $G = O$ or $G = U$.*

PROOF. $\pi_0(kG_0^\delta)$ and $\pi_0(\widehat{kG}[1/q]_0)$ are both canonically isomorphic to the ring $\widehat{\mathbb{Z}}[1/q] = \prod_{p \neq q} \widehat{\mathbb{Z}}_{(p)}$, and there is a unique (continuous) isomorphism of rings from one to the other. Denote this isomorphism by $\pi_0\widehat{\lambda}$. By Definition II.3.10 and a trivial diagram chase from Lemma VIII.2.4, $\pi_0\widehat{\lambda}$ and $\widehat{\lambda}\colon BG^\delta \to \widehat{BG}[1/q]$ together determine an equivalence $\widehat{\lambda}\colon kG_0^\delta \to \widehat{kG}[1/q]_0$ of ring spaces. Write b^δ for the composite

$$
S^d \xrightarrow{b} BG \xrightarrow{\gamma} \widehat{BG}[1/q] \xrightarrow{\widehat{\lambda}^{-1}} BG^\delta.
$$

Thus $\widehat{\lambda}_*[b^\delta] = [b]$. Let $\beta^\delta \colon kG^\delta \to \Omega^d kG^\delta$ be adjoint to the composite

$$kG^\delta \wedge S^d \xrightarrow{1 \wedge b^\delta} kG^\delta \wedge BG^\delta \xrightarrow{\phi} kG^\delta.$$

Then $(kG^\delta, \beta^\delta)$, $(\widehat{kG}[1/q], \beta)$, and the map $\widehat{\lambda}$ of zero$^{\text{th}}$ spaces satisfy the hypotheses of Corollary II.3.14 (and Corollary II.3.9). The conclusion follows from those results and Proposition II.3.2. □

The following addendum is the reason that this result is of topological interest; it shows that the Frobenius automorphisms ϕ^r and Adams operations ψ^r (both completed away from q) agree under Λ.

THEOREM VIII.2.9. *Te following diagram commutes in $H\mathscr{S}$, $G = O$ or $G = U$:*

$$
\begin{array}{ccc}
kG^\delta & \xrightarrow{\phi^r} & kG^\delta \\
{\scriptstyle \Lambda}\downarrow & & \downarrow{\scriptstyle \Lambda} \\
\widehat{kG}[1/q] & \xrightarrow{\psi^r} & \widehat{kG}[1/q]
\end{array}
$$

PROOF. ϕ^r is induced by passage from bipermutative categories to E_∞ ring spaces to E_∞ ring spectra to completions away from q. Lemma VIII.2.5 implies that the two composites induce the same map in $h\mathscr{V}$ on zero$^{\text{th}}$ spaces and that this map is one of ring spaces. The conclusion follows by Remark II.3.15 and the uniqueness clause of Corollary II.3.14. □

Let $BG_\otimes^\delta$ denote the 1-component of the zero$^{\text{th}}$ space of kG^δ. Clearly $O(1, \overline{k}_q) = \mathbb{Z}_2$. $GL(1, \overline{k}_q) = \overline{k}_q^*$, and the completion away from q of the infinite loop map $B\mu \colon B\overline{k}_q^* \to B\mathbb{C}^*$ is an equivalence by a simple homological calculation. Let $BSG_\otimes^\delta$ denote the simply connected cover of $BG_\otimes^\delta$. The same proof as that of Lemma V.3.1 yields the following observation.

Lemma VIII.2.10. *$BO_\otimes^\delta$ and $BU_\otimes^\delta$ are equivalent as infinite loop spaces to $BO(1) \times BSO_\otimes^\delta$ and to $\widehat{BU}(1)[1/q] \times BSU_\otimes^\delta$.*

The 1-component $BG_\otimes^\delta \to BG_\otimes$ of zero$^{\text{th}}$ map of $\Lambda \colon kG^\delta \widehat{kG}[1/q]$ is clearly compatible with the splitting given by the lemma and Lemma V.3.1. It is therefore an infinite loop map by the following theorem.

THEOREM VIII.2.11. *$\Lambda \colon BSG_\otimes^\delta \to \widehat{BSG}[1/q]$ is an infinite loop map, $G = O$ or $G = U$.*

PROOF. By Theorem VIII.1.6, it suffices to prove that the completion of Λ at each prime $p \neq q$ is an H_∞^p-map, and this will hold if it does so on the localized level with SG replaced by G. For clarity of notation, we treat the case $G = U$. The only additional point needed in the real case is that the relevant representation theoretical constructs, in particular the decomposition homomorphism, restrict appropriate, and the requisite information is contained in the appendix of Quillen's paper [**58**]. Let M be the monoid of positive integers prime to p, let $X = B_M \mathscr{GL} \overline{k}_q$, let $Y = \Gamma(X, \xi)$, let $\iota \colon X \to Y$ be the natural E_∞ map, and let $Z = \Gamma(B_M \mathscr{GL}\mathbb{C}, \xi)$. Of course, $\mathscr{GL}\mathbb{C}$ and $\mathscr{U}$ can be used interchangeably here, $\mathscr{GL}\mathbb{C}$ being given its usual topology. By Theorem VII.5.3 (and the diagram in its statement), we have infinite loop maps

$$\phi \colon \Gamma_M(B\mathscr{GL}\overline{k}_q, \theta) \longrightarrow Y \quad \text{and} \quad \phi \colon \Gamma_M(B\mathscr{GL}\mathbb{C}, \theta) \longrightarrow Z$$

which restrict on components to localizations at p. We agree to write $BU \times \{m\} = \Gamma_m(B\mathscr{GL}\mathbb{C}, \theta)$ for $m \in M$; thus Z_m is the localization of $BU \times \{m\}$ at p. There is a unique map $f \colon Y_1 \to Z_1$ such that $f \circ \phi \simeq \phi(\widetilde{\lambda} * 1)$, where $\widetilde{\lambda} * 1$ is the translation to 1-components of the map $\widetilde{\lambda}$ of ($**$); f is an H-map since $\widetilde{\lambda} * 1$ is so by the second diagram of (A). Since the map $\Lambda \colon BU_\otimes^\delta \to \widehat{BU}_\otimes[1/q]$ is the completion of $\widetilde{\lambda} * 1$ away from q, it clearly suffices to prove that $f \colon Y_1 \to Z_1$ is an H_∞^p-map. By Remark VIII.1.4, f extends to an H-map $Y \to Z$ and it suffices to prove that the composite

$f \circ \iota \colon X \to Z$ is an H^p_∞-map. The Brauer lifts of the inclusions $GL(m, k_r) \to GL(m, \overline{k}_q)$ fit together to give a unique map $\beta \colon BGL(m, \overline{k}_q) \to BU \times \{m\}$, and we also write β for its composite with the classifying map of any representation $H \to GL(m, \overline{k}_q)$. It follows easily from the definition of $\widetilde{\lambda}$ and the proof of Theorem VII.5.3 that $f \circ \iota \simeq \phi \circ \beta \colon X_m \to Z_m$. Thus we must prove that the following diagram is homotopy commutative:

$$
\begin{array}{ccccc}
W \times_\pi BGL(m, \overline{k}_q)^p & \simeq & B(\pi \wr GL(m, \overline{k}_q)) & \xrightarrow{B\widetilde{c}_p} & BGL(m^p, \overline{k}_q) \\
{\scriptstyle 1 \times \beta^p} \downarrow & & & & \downarrow {\scriptstyle \beta} \\
W \times_\pi (BU \times \{m\})^p & & & & BU \times \{m^p\} \\
{\scriptstyle 1 \times \phi^p} \downarrow & & & & \downarrow {\scriptstyle \phi} \\
W \times_\pi (Z_m)^p & & \xrightarrow{\hspace{3cm} \xi \hspace{3cm}} & & Z_{m^p}
\end{array}
$$

By an obvious limit argument, it suffices to prove this with $\overline{k}_q$ replaced by k_r (for each $r = q^a$) in the top row. By Lemma VIII.1.1 and a transfer argument, the resulting diagram will homotopy commute if it does so with domain restricted to $W \times_\pi (BH)^p$ for a p-Sylow subgroup H of $GL(m, k_r)$ (since the index of $\pi \wr H$ in $\pi \wr GL(m, k_r)$ is prime to p and since Z_{m^p} is a p-local space). Let ℓ be a finite field between k_r and $\overline{k}_q$. By a trivial diagram chase, it suffices to prove that the diagram above homotopy commutes after replacement of $\overline{k}_q$ by ℓ in the top row and restriction of the domain to $W \times_\pi (BH)^p$. Let p^e be the maximal order of an element of $\pi \wr H$. Construct a field $K \subset \mathbb{C}$ which contains all $(p^e)^{\text{th}}$ roots of unity and has a discrete valuation whose valuation ring A has quotient field ℓ, $k_r \subset \ell \subset \overline{k}_q$. Consider the following diagram, where ρ is the inclusion of H in $GL(m, \ell)$, $r \colon \mathscr{GL}A \to \mathscr{GL}\ell$ is induced by the quotient map $A \to \ell$, and $i \colon \mathscr{GL}A \to \mathscr{GL}\mathbb{C}$ is induced by the inclusion $A \to \mathbb{C}$:

$$
\begin{array}{ccccc}
B(\pi \wr H) & \xrightarrow{\hspace{1cm} B(1\wr\rho) \hspace{1cm}} & B(\pi \wr GL(m, \ell)) & \xrightarrow{\hspace{1cm} B\widetilde{c}_p \hspace{1cm}} & BGL(m^p, \ell) \\
& \searrow {\scriptstyle B(1\wr\widetilde{\rho})} & & \nearrow & \\
{\scriptstyle B(1\wr i\widetilde{\rho})} \downarrow & B(\pi \wr GL(m, A)) & \xrightarrow[{\scriptstyle B(1\wr r)} \quad {\scriptstyle B\widetilde{c}_p}]{} & BGL(m^p, A) & \quad {\scriptstyle Br} \\
& \swarrow {\scriptstyle B(1\wr i)} & & \swarrow {\scriptstyle Bi} & \downarrow {\scriptstyle \beta} \\
B(\pi \wr GL(m, \mathbb{C})) & \xrightarrow{\hspace{0.8cm} B\widetilde{c}_p \hspace{0.8cm}} & BGL(m^p, \mathbb{C}) & \xrightarrow{\hspace{0.8cm} \iota \hspace{0.8cm}} & BU \times \{m^p\}
\end{array}
$$

The two interior parallelograms obviously commute. Let $\widehat{K}$ be the completion of K. Then $R_K H = R_{\widehat{K}} H$ [**69**, 14.6], and the decomposition homomorphism $d \colon R_{\widehat{K}} H \to R_\ell H$ is an isomorphism [**69**, 15.2 and 15.5]. Thus $d \colon R_K H \to R_\ell H$ is also an isomorphism, and, since d maps $R^+_K H$ onto $R^+_\ell H$ any representation $\rho \colon H \to GL(m, \ell)$ lifts to a representation $\widetilde{\rho} \colon H \to GL(m, A)$ [**69**, especially the remarks on pp. 136,139,141]. This lifting to an honest rather than virtual representation is crucial. By Proposition VIII.1.2 (applied to $\mathscr{GL}C$ under $\otimes$), the fact that ϕ is an infinite loop map, and a trivial diagram chase, it suffices to prove that the composite $BH \xrightarrow{B(i\widetilde{\rho})} BGL(m, \mathbb{C}) \xrightarrow{\iota} BU \times \{m\}$ is homotopic to β and the the outer rectangle of the diagram is homotopy commutative. Actually, because the homomorphism $\mu \colon k^*_q \to \mathbb{C}^*$ in our definition of Brauer lifting, these assertions will in general be off by certain Adams operations. To rectify this, we need only choose μ consistently with the requisite decomposition homomorphism (in a manner independent of p). We do this as follows. Let $\mathscr{Z}_0 = (q)$ in $B_0 = Z$ and let $A_0 = \mathbb{Z}_{(q)}$. Inductively, given $\mathscr{Z}_{j-1}$, B_{j_1}, and A_{j-1}, let A_j be the localization of the ring of cyclotomic integers $B_j = \mathbb{Z}[\exp(2\pi i/(q^j - 1)(q^{j-1} - 1) \cdots (q - 1))]$ at a chosen prime ideal $\mathscr{Z}_j$ which contains $\mathscr{Z}_{j-1} \subset B_{j-1}$. Let $K_j \subset \mathbb{C}$ be the field of fractions of A_j, let ℓ_j be the quotient of A_j by its maximal ideal, let $i \colon A_j \to \mathbb{C}$ be the inclusion, and let $r \colon A_j \to \ell_j$ be the quotient map. Obviously, char $\ell_j = q$ and $\varinjlim \ell_j \overline{k}_q$. Moreover, A_j contains a group ν_j of $(q^j - 1) \cdots (q - 1)^{\text{st}}$ roots of unity which r maps isomorphically onto the corresponding subgroup

$\overline{\nu}_j$ of ℓ_j^*, these isomorphisms being compatible as j varies. We specify $\mu\colon \overline{k}_q^* \to \mathbb{C}^*$ by letting its restriction to $\overline{\nu}_j$ be $i \circ r^{-1}$. In the construction of our last diagram, we agree to choose $A = A_j$ and $\ell = \ell_j$ for j sufficiently large. It is then obvious that the character of $i\widetilde{\rho}\colon H \to GL(m, \mathbb{C})$ is χ_ρ, so that $\iota \circ B(i\widetilde{\rho}) \simeq \beta$. Similarly, if G is a finite p-group with no elements of order greater than p^e and if $\sigma\colon G \to GL(m^p, A)$ is a representation, then the character of $i\sigma\colon G \to GL(m^p, \mathbb{C})$ is $\chi_{r\sigma}$ and $\iota \circ B(i\sigma)\colon BG \to BU \times \{m^p\}$ is therefore homotopic to β. With $\sigma = \widetilde{c}_p \circ (1 \wr \widetilde{\rho})\colon H \to GL(m^p, A)$, it follows that the outer triangle of our diagram is indeed homotopy commutative. $\qquad\square$

3. Finite fields, Frobenius and $B \operatorname{Coker} J$

Throughout this section, all spaces and spectra are to be completed at a fixed point p and $r = q^a$ (q odd) is to be $r(p)$. Thus $r = 3$ if $p = 2$ and r reduces mod p^2 to a generator of the group of units of $\mathbb{Z}_{p^2}$ if $p > 2$. We retain the notations of the previous section and continue wit the discussion of discrete models for various of the spaces and maps in the J-theory diagram of Section V§3.

We have an equivalence of orientation sequences

$$
\begin{array}{ccccccc}
SF & \xrightarrow{\ e\ } & BO^\delta_\otimes & \xrightarrow{\ \tau\ } & B(SF; kO^\delta) & \xrightarrow{\ q\ } & BSF \\
\| & & \downarrow{\scriptstyle \Lambda} & & \downarrow{\scriptstyle B\Lambda} & & \| \\
SF & \xrightarrow{\ e\ } & BO_\otimes & \xrightarrow{\ \tau\ } & B(SF; kO) & \xrightarrow{\ q\ } & BSF
\end{array}
$$

(compare Lemma V.2.4) and an equivalence of fibration sequences

$$
\begin{array}{ccccccc}
Spin^\delta_\otimes & \longrightarrow & BC^\delta_p & \longrightarrow & B(SF; kO^\delta) & \xrightarrow{\ c(\phi^r)\ } & BSpin^\delta_\otimes \\
{\scriptstyle \Omega\Lambda}\downarrow & & \downarrow & & \downarrow{\scriptstyle B\Lambda} & & \downarrow{\scriptstyle \Lambda} \\
Spin_\otimes & \longrightarrow & BC_p & \longrightarrow & B(SF; kO) & \xrightarrow{\ c(\psi^r)\ } & BSpin_\otimes
\end{array}
$$

Here $BSpin^\delta_\otimes$ is the 2-connected cover of $BSO^\delta_\otimes$ and $Spin^\delta_\otimes$ is its loop space. $c(\phi^r)$ is the universal cannibalistic class (defined above Proposition V.2.2) determined by $\phi^r\colon kO^\delta \to kO^\delta$ and is an infinite loop map because ϕ^r is the completion of a map of E_∞ ring spectra. The fibre BC^δ_p of $c(\phi^r)$ is thus an infinite loop space, and we think of it as BC_p endowed with an infinite loop space structure. We shall prove in Theorem VIII.3.4 that both diagrams above are commutative diagrams of infinite loop spaces and maps.

In order to obtain a better understanding of the infinite loop space BC^δ_p, we construct discrete models for the spectra j_p and jO_2 of Definition V.5.16. Recall the functor $\Omega^\infty T$ from E_∞ ring spaces to E_∞ ring spectra of Theorem VII.4.1.

Definition VIII.3.1. Define $j_2^\delta = \Omega^\infty T B n k_3$ and $jO_2^\delta = \Omega^\infty T B \mathcal{O} k_3$. For $p > 2$, define $j_p^\delta = \Omega^\infty T B \mathcal{GL} k_r$. The bipermutative categories $n k_3$, $\mathcal{O} k_3$, and $\mathcal{GL} k_r$ are specified in Examples VI.5.2, VI.5.3 and VI.5.7 (and the specified E_∞ ring spectra are understood to be completed at p). Let J_p^δ and $J_{\otimes p}^\delta$ denote the 0-component and 1-component of the zero$^{\text{th}}$ space of j_p^δ (which is equivalent to $J_p^\delta \times \widehat{\mathbb{Z}}_{(p)}$), and let JO_2^δ and $JO_{\otimes 2}^\delta$ denote the 0-component and 1-component of $(jO_2^\delta)_0$.

The following theorem is based on ideas and results of Quillen [57, 59] and Fiedorowicz and Priddy [28].

THEOREM VIII.3.2. *There are equivalences* $\nu\colon j_p^\delta \to j_p$ *and* $\bar{\nu}\colon jO_2^\delta \to jO_2$ *such that the following diagrams commute in* $H\mathscr{S}$:

$$
\begin{array}{ccc}
j_p^\delta & \xrightarrow{\ \kappa^\delta\ } & kO^\delta \\
\nu\downarrow & & \downarrow\Lambda \\
j_p & \xrightarrow{\ \kappa\ } & kO
\end{array}
\qquad and \qquad
\begin{array}{ccccc}
j_2^\delta & \longrightarrow & jO_2^\delta & \xrightarrow{\ \kappa^\delta\ } & kO^\delta \\
\nu\downarrow & & \bar{\nu}\downarrow & & \downarrow\Lambda \\
j_2 & \longrightarrow & jO_2 & \xrightarrow{\ \kappa\ } & kO
\end{array}
$$

where the κ^δ *are induced by inclusions of bipermutative categories (when* $p > 2$, *in the sense that* $j_p^\delta \xrightarrow{\kappa^\delta} kO^\delta \to kU^\delta$ *is induced by* $\mathscr{GL}k_r \to \mathscr{GL}\bar{k}_q$).

PROOF. It will be convenient to treat the cases $p > 2$ and $p = 2$ separately. We adopt the obvious discrete models analog of the notations in Notation V.5.14.

(i) $p > 2$: In view of Theorem VIII.2.9 (and [**48**, I(2.12)]), we have the following comparisons of fibration sequences in $H\mathscr{S}$, where $F\psi^r$ and $F\phi^r$ denote the relevant fibres:

Since $p > 2$, $bo \simeq bspin$. The arrows labelled c^δ are induced by passage to completed spectra from the inclusion $\mathscr{O}\bar{k}_q \to \mathscr{GL}\bar{k}_q$ of bipermutative categories. By [**1**, 5.1] and Theorem VIII.2.8, the dotted arrows are all equivalences in $H\mathscr{S}$. On the level of bipermutative categories, $\phi^r\colon \mathscr{GL}\bar{k}_q \to \mathscr{GL}\bar{k}_q$ restricts to the identity on $\mathscr{GL}k_r$. By passage to completed spectra, we conclude that the composite of $\phi^r - 1$ with the map $j_p^\delta \to kU^\delta$ induced by the inclusion of $\mathscr{GL}k_r$ in $\mathscr{GL}\bar{k}_q$ is trivial. There results a lift $\mu\colon j_p^\delta \to F\phi^r$, and μ obviously induces an isomorphism on π_0. Since $KU^{-1}BG = 0$ for a finite group G [**14**, 4.2] and since the zero$^{\text{th}}$ space functor commutes with fibres [**48**, VIII], the 0-component of the zero$^{\text{th}}$ map of μ is determined by the homotopy commutative diagram

$$
\begin{array}{ccc}
 & & J_p^\delta \\
 & {}^{\mu_0}\nearrow & \downarrow \\
\text{fibre}(\phi^r - 1) & \longrightarrow \ BU^\delta & \xrightarrow{\ \phi^r - 1\ } BU^\delta
\end{array}
$$

Quillen [**59**, p. 576] proved that μ_0 induces an isomorphism on mod p homology and is therefore a homotopy equivalence (since J_p^δ and the fibre of $\phi^r - 1$ are p-complete simple spaces). It follows that μ induces isomorphisms on π_i for all i and is thus an equivalence. The desired equivalence $\nu\colon j_p^\delta \to j_p$ is $c^{-1} \circ \Lambda \circ \mu$.

(ii) $p = 2$: Let $FO\phi^3$ and $F\phi^3$ denote the fibres of $\phi^3 - 1\colon kO^\delta \to bso^\delta$ and of $\phi^3 - 1\colon kO^\delta \to bspin^\delta$. By Lemma V.5.15 and Theorems VIII.2.8 and VIII.2.9, comparisons of fibrations yield a commutative diagram

$$
\begin{array}{ccccc}
F\phi^3 & \longrightarrow & FO\phi^3 & \longrightarrow & kO^\delta \\
\Lambda\downarrow & & \downarrow\Lambda & & \downarrow\Lambda \\
j_2 & \longrightarrow & jO_2 & \longrightarrow & kO
\end{array}
$$

in which the maps Λ are equivalences. The composite of $\phi^3 - 1\colon kO^\delta \to bso^\delta$ with $\kappa^\delta\colon jO_2^\delta \to kO^\delta$ is trivial since $\phi^3\colon \mathcal{O}\overline{k}_3 \to \mathcal{O}\overline{k}_3$ restricts to the identity on $\mathcal{O}k_3$ and since $[jO_2^\delta, bso^\delta] \cong [jO_2^\delta, bo^\delta]$ by the proof of Lemma V.5.15 and the fact that $H^2 jO_2^\delta = H^0 kO^\delta$ (where H denotes mod 2 cohomology). There results a lift $\overline{\mu}\colon jO_2^\delta \to FO\phi^3$, and $\overline{\mu}$ obviously induces an isomorphism on π_0. Restriction to the 0-component of zero$^{\text{th}}$ spaces gives a homotopy commutative diagram

$$
\begin{array}{ccc}
 & JO_2^\delta & \\
\overset{\overline{\mu}_0}{\swarrow} & \downarrow & \\
\text{fibre}(\phi^3 - 1) \longrightarrow & BO^\delta \xrightarrow{\ \phi^3 - 1\ } & BSO^\delta
\end{array}
$$

Here $\overline{\mu}_0$ is not determined by the diagram, but Fiedorowicz and Priddy [28] have proven that any H-map $\overline{\mu}_0$ which makes the triangle homotopy commute induces an isomorphism on mod 2 homology and is therefore a homotopy equivalence. (Friedlander [29], following up on Quillen's ideas about étale cohomology [57], earlier obtained a particular equivalence $\overline{\mu}_0$, not necessarily an infinite loop map.) Thus $\overline{\mu}$ is an equivalence in $H\mathcal{S}$. Next, consider the following diagram in $H\mathcal{S}$:

The right triangle commutes and induces θ in such a manner that $F\phi^3$ is canonically equivalent to the fibre of θ and the solid arrow diagram is a braid of fibrations (by [48, I(2.13)]). θ restricts non-trivially to Ωbso^δ, and we define $\xi = \theta\overline{\mu}$. We need a slight calculation to construct μ. Recall that $H^* bso \cong \Sigma^2(A/A\operatorname{Sq}^3)$ (e.g., by [8]). Since $H^2 kO^\delta = 0$, $\phi^3 - 1\colon kO^\delta \to bso^\delta$ induces the trivial map on H^* and we have an exact sequence

$$
0 \longrightarrow A/A\operatorname{Sq}^1 + A\operatorname{Sq}^2 \longrightarrow H^2 FO\psi^3 \longrightarrow \Sigma(A/A\operatorname{Sq}^3) \longrightarrow 0 \ .
$$

Thus $H^1 FO\phi^3 = \mathbb{Z}_2$ and its unique non-zero class θ restricts to the generator of $H^*\Omega bso^\delta$. By inspection of the fibration $F\pi \to jO_2^\delta \xrightarrow{\ \pi\ } \mathcal{K}(\widehat{\mathbb{Z}}_{(2)}, 0)$ with 0-connected fibre, we see that this is consistent with the known fact (e.g. [28]) that $H^1 JO_2^\delta = H^1 BOk_3$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ with non-zero classes corresponding to the determinant, the spinor norm, and their product; we denote the last of these classes by ξ_0. In view of Examples VI.5.3 and VI.5.7, J_2^δ is equivalent to the fibre of $\xi_0\colon JO_2^\delta \to K(\mathbb{Z}_2, 1)$ (see [28]). Thus the cofibre of $j_2^\delta \to jO_2^\delta$ is $\mathcal{K}(\mathbb{Z}_2, 1)$ by the long exact homotopy sequence. Clearly, the cofibre map $jO_2^\delta \to \mathcal{K}(\mathbb{Z}_2, 1)$ must be the non-trivial map ξ, hence j_2^δ must be equivalent to the fibre of ξ. We conclude

(by [**48**, I(2.12)]) that there exists $\mu \colon j_2^\delta \to F\phi^3$ which makes the diagram above commute in $H\mathscr{S}$, and μ is an equivalence by the five lemma. The desired equivalences $\nu \colon j_2^\delta \to j_2$ and $\overline{\nu} \colon jO_2^\delta \to jO_2$ are $\Lambda \circ \mu$ and $\Lambda \circ \overline{\mu}$.

$\square$

Since $\phi^r \kappa^\delta = \kappa^\delta \colon j_p^\delta \to kO^\delta$, it follows from the definition of $c(\phi^r) \colon B(SF; kO^\delta) \to BSpin_\otimes^\delta$ (in Section V§2) that the restriction of $c(\phi^r)$ to $B(SF; j_p^\partial)$ is the trivial infinite loop map. There results a lift $\zeta^\delta \colon B(SF; j_p^\delta) \to BC_p^\delta$, and the proof of Theorem V.5.17 yields the following corollary.

Corollary VIII.3.3. $\zeta^\delta \colon B(SF; j_p^\delta) \to BC_p^\delta$ *is an equivalence of infinite loop spaces.*

In Section V§5, j_p was regarded as a ring spectrum by pullback along ν^{-1}. On 1-components of zero$^{\text{th}}$ spaces, ν and $\overline{\nu}$ restrict to composite equivalences

$$J_\otimes^\delta \xrightarrow{\mu_0} F\phi_\otimes^r \xrightarrow{\Lambda} J_{\otimes p} \quad \text{and} \quad JO_{\otimes 2}^\delta \xrightarrow{\overline{\mu}_0} FO\phi_\otimes^\delta \xrightarrow{\Lambda} JO_{\otimes 2}$$
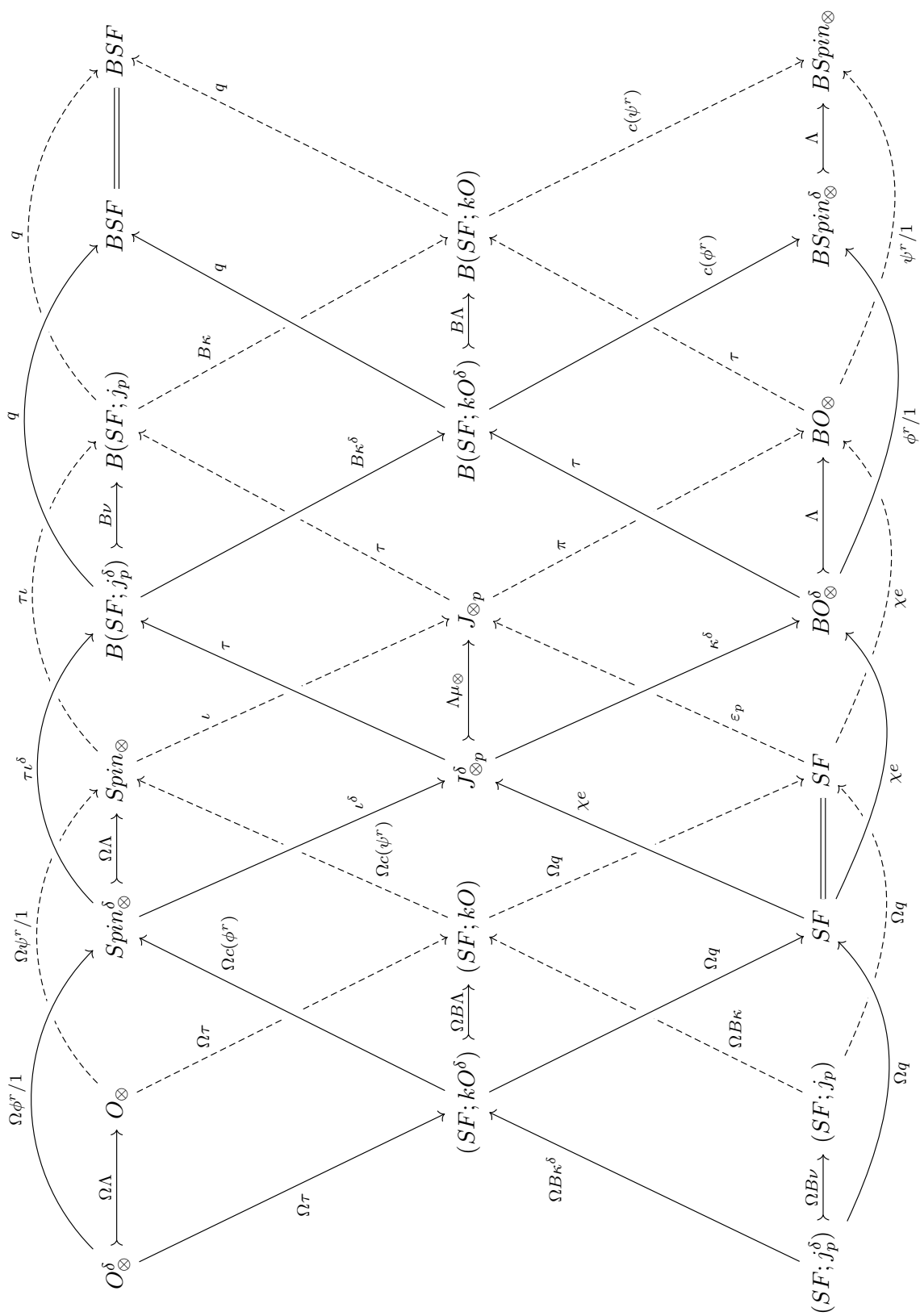
where $F\phi_\otimes^\delta$ and $FO\phi_\otimes^\delta$ denote the fibres of $\phi^r/1 \colon BO_\otimes^\delta \to BSpin_\otimes^\delta$ and of $\phi^3/1 \colon BO_\otimes^\delta \to BSO_\otimes^\delta$. We shall see in the following theorem that the maps Λ may be regarded as infinite loop maps in view of Theorem VIII.2.11. When $p > 2$, μ_0 is easily seen to be an H-map; when $p = 2$, not even this much is clear in view of the non-uniqueness of $\overline{\mu}_0$. However, the proof of the following theorem will yield a possibly different (when $p = 2$) map $\mu_\otimes \colon J_{\otimes p}^\delta \to F\phi_\otimes^r$ which is an equivalence of infinite loop spaces, and an analogous argument gives an equivalence $\overline{\mu}_\otimes \colon JO_{\otimes 2}^\delta \to FO\varphi_\otimes^3$ of infinite loop spaces. The composite $\Lambda\mu_\otimes$ plays a central role in the "multiplicative Brauer lift diagram" displayed on the following page. The dotted arrow portion of the diagram is an elaboration of part of the J-theory diagram of Section V§3 (completed at p, with $r = r(p)$). The following result asserts that discrete models yield an approximation to this part of the J-theory diagram by a commutative diagram of infinite loop spaces and maps and that this approximation is in fact consistent with all preassigned geometric infinite loop space structures in sight. In other words, our ad hoc discrete models notation behaves as if it were a functor naturally equivalent to the identity.

THEOREM VIII.3.4. *The solid arrow ($\longrightarrow$) and the dotted arrow ($--\rightarrow$) portions of the multiplicative Brauer lift diagram are braid of fibrations, the horizontal ($\rightarrowtail$) arrows are all equivalences, and the entire diagram is a commutative diagram of infinite loop spaces and infinite loop maps.*
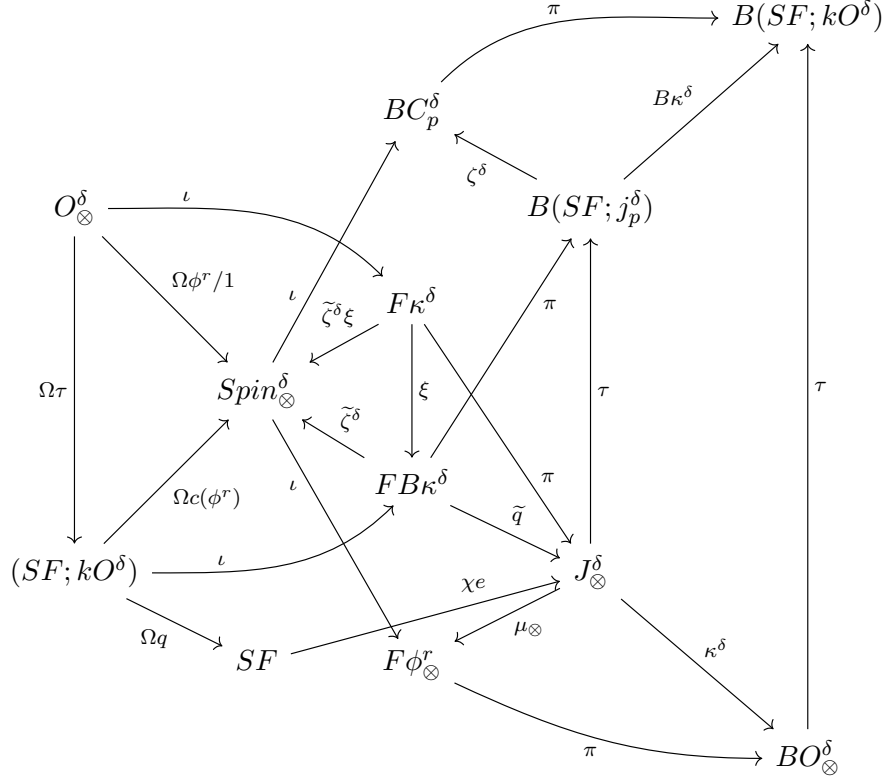
PROOF. First focus attention on the solid arrow portion of the diagram. It features two orientation sequences (for j_p^δ and kO^δ) and the obvious comparison between them. We must construct an infinite loop map

$$\iota^\delta \colon Spin_\otimes^\delta \longrightarrow J_{\otimes p}^\delta$$

such that ι^δ is equivalent to the fibre $\pi \colon F\kappa^\delta \to J_{\otimes p}^\delta$ of κ^δ and $\tau\iota^\delta \colon Spin_\otimes^\delta \to B(SF; j_p^\delta)$ is equivalent to the fibre $\pi \colon FB\kappa^\delta \to B(SF; j_p^\delta)$ of $B\kappa^\delta$, these equivalences being compatible with infinite loop equivalences $\zeta^\delta \colon B(SF; j_p^\delta) \to BC_p^\delta$ and $\mu_\otimes \colon J_{\otimes p}^\delta \to F\phi_\otimes^r$. Thus consider the following diagram (in which, as in Section V§3, the letters π and ι are used generically for natural maps of fibration

sequences):

$$
\begin{array}{c}
\text{(commutative braid diagram with nodes } B(SF;kO^\delta),\ BC_p^\delta,\ O_\otimes^\delta,\ B(SF;j_p^\delta),\ F\kappa^\delta,\ Spin_\otimes^\delta,\ FB\kappa^\delta,\ J_\otimes^\delta,\ (SF;kO^\delta),\ SF,\ F\phi_\otimes^r,\ BO_\otimes^\delta \\
\text{and maps } \pi,\ B\kappa^\delta,\ \iota,\ \zeta^\delta,\ \Omega\phi^r/1,\ \widetilde{\zeta^\delta}\xi,\ \widetilde{\zeta^\delta},\ \xi,\ \Omega\tau,\ \Omega c(\phi^r),\ \widetilde{q},\ \chi e,\ \mu_\otimes,\ \Omega q,\ \tau,\ \kappa^\delta)
\end{array}
$$

$\tau\colon J_{\otimes p}^\delta \to B(SF;j_p^\delta)$ is equivalent to the fibre of $q\colon B(SF;j_p^\delta)$, and $q\colon B(SF;kO^\delta)\to BSF$ induces $\widetilde{q}\colon FB\kappa^\delta \to J_{\otimes p}^\delta$ (by base change). By [**48**, I.(2.13)], (which is a precise form of Verdier's axiom for fibrations), there is a canonical equivalence $\xi\colon F\kappa^\delta \to FB\kappa^\delta$ such that

$$\xi\circ\iota \simeq i\circ\Omega\tau \quad\text{and}\quad \widetilde{q}\circ\xi \simeq \pi.$$

Clearly, ζ^δ induces an equivalence $\widetilde{\zeta^\delta}\colon FB\kappa^\delta \to Spin_\otimes^\delta$ such that

$$\zeta^\delta\circ\pi \simeq \iota\circ\widetilde{\zeta^\delta} \quad\text{and}\quad \widetilde{\zeta^\delta}\circ\delta \simeq \Omega c(\phi^r).$$

Define $\iota^\delta = \widetilde{q}(\widetilde{\zeta^\delta})^{-1}\colon Spin_\otimes^\delta \to J_{\otimes p}^\delta$. It remains to construct $\mu_\otimes\colon J_{\otimes p}^\delta \to F\phi_\otimes^r$, and we note that

$$(\widetilde{\zeta^\delta}\circ\xi)\circ\iota \simeq \widetilde{\zeta^\delta}\circ\iota\circ\Omega\tau \simeq \Omega c(\phi^r)\circ\Omega\tau \simeq \Omega\phi^r/1\colon O_\otimes^\delta \longrightarrow Spin_\otimes^\delta.$$

The constructions so far all result by passage to zero$^\mathrm{th}$ spaces from the analogous constructions on spectra, hence we may regard the diagram as one of connective spectra $H\mathscr{S}$. By abuse, we retain the notations of the diagram on the spectrum level. Here cofibrations and fibrations agree up to sign, by [**48**, XI], hence standard arguments with cofibrations show that $\widetilde{\zeta^\delta}\circ\xi$ induces $\mu_{\otimes p}^\delta \to F\phi_\otimes^r$ such that

$$\pi\circ\mu_\otimes = \kappa^\delta \quad\text{and}\quad \mu_\otimes\circ\pi = \iota\circ\widetilde{\zeta^\delta}\xi \quad\text{in } H\mathscr{S},$$

and $\mu_\otimes$ is an equivalence by the five lemma. Now passage back to zero$^\mathrm{th}$ spaces and comparison of the diagram above to the multiplicative Brauer lift diagram complete of infinite loop spaces and maps and a braid of fibrations. On the space level, we have already constructed all of the horizontal equivalences, and we could of course assign infinite loop space structures to their ranges by requiring them to be infinite loop maps. The problem that remains is to check the consistency of the resulting new infinite loop space structures with the geometrically constructed infinite loop space

structures already existing on $BO_\otimes$, $BSpin_\otimes$, $B(SF; kO)$, and their loop spaces. By Lemmas V.3.1 and VIII.2.10 and Theorem VIII.2.11, we have that

$$\Lambda: BO_\otimes^\delta \longrightarrow BO_\otimes \quad \text{and} \quad \Lambda: BSpin_\otimes^\delta \longrightarrow BSpin_\otimes$$

are both infinite loop maps. We may therefore specify $\psi^r/1$ as an infinite loop map by $\psi^r/1 = \Lambda \circ \phi^r/1 \circ \Lambda^{-1}$ (compare Lemma V.7.6). The equivalence $\Lambda: F\phi^r_\otimes \to J_{\otimes p}$ obtained by comparison of fibrations is then an infinite loop map if $J_{\otimes p}$ is given an infinite loop space structure as the fibre of $\psi^r/1$. Next, specify $\varepsilon_p = \Lambda\mu_\otimes \circ \chi e: SF \to J_{\otimes p}$ as an infinite loop map. On the space level, parts of the multiplicative Brauer lift diagram already known to commute then imply that

$$\varepsilon_p \circ \Omega q \simeq \iota \circ \Omega c(\psi^r): (SF; kO) \longrightarrow J_{\otimes p} \quad \text{and} \quad \pi \circ \varepsilon_p \simeq \chi e: SF \longrightarrow BO_\otimes.$$

These were the defining conditions for the map labelled ε_p in Section V§3. We have that $\pi\varepsilon_p = \chi e = \Lambda\chi e: SF \to BO_\otimes$ as infinite loop maps in view of Theorem V.7.9. Delooping once, we conclude (by [48, I(2.12)]) that there is an infinite loop map $B'\Lambda: B(SF; kO^\delta) \to B(SF; kO)$ such that

$$q \circ B'\Lambda = q: B(SF; kO^\delta) \longrightarrow BSF \quad \text{and} \quad \tau \circ \Lambda = B'\Lambda \circ \tau: BO_\otimes^\delta \longrightarrow B(SF; kO)$$

as infinite loop maps. We must verify that $B'\Lambda$ is homotopic to $B\Lambda$, and of course $q \circ B\Lambda \simeq q$ and $\tau \circ \Lambda \simeq B\Lambda \circ \tau$ on the space level. Thus $B\Lambda/B'\Lambda$ factors as $\tau\omega$ for some map $\omega: B(SF; kO^\delta) \to BO_\otimes$, and $\tau\omega\tau \simeq *$. Since $B(SF; kO^\delta)$ has the homotopy type of $BSpin \times BC_p$, by Theorems V.4.7 and V.4.8, and since $[BC_p, BO_\otimes] = 0$, by Theorem V.7.8, ω may be regarded as a map $BSpin \to BO_\otimes$. It clearly induces the trivial homomorphism on rational (indeed, on integral) cohomology, and Lemmas V.2.8 and V.2.10 imply that it is null homotopic. Thus $B\Lambda \simeq B'\Lambda$. We may now specify $c(\psi^r)$ as an infinite loop map by $c(\psi^r) = \Lambda \circ c(\phi^r) \circ (B\Lambda)^{-1}$. Similarly, we specify $B(SF; j_p)$ as an infinite loop space by requiring $B\nu$ to be an infinite loop map and we specify $B\kappa = B\Lambda \circ B\kappa^\delta \circ (B\nu)^{-1}: B(SF; j_p) \to B(SF; kO)$ and $\tau = B\nu \circ \tau \circ (\Lambda\mu_\otimes)^{-1}: J_{\otimes p} \to B(SF; j_p)$ as infinite loop maps. The remaining verifications are trivial.   □

We single out the following part of the theorem for emphasis (compare Remark V.5.13).

Corollary VIII.3.5. *The composite* $SF \xrightarrow{\chi e} J_{\otimes p}^\delta \xrightarrow{\mu_\otimes} F\phi^r_\otimes \xrightarrow{\Lambda} J_{\otimes p}$ *may be taken as the map* $\varepsilon_p: SF \to J_{\otimes p}$ *of the J-theory diagram.*

The force of this assertion lies mainly at the prime 2. At odd primes it is almost trivial, since there we have

$$[SF, SO_\otimes] \cong [Q_0 S^0, SO_\otimes] \cong [B\Sigma_\infty, SO_\otimes] = 0,$$

by Proposition VII.3.4 and [14], so that ε_p is uniquely determined by the fact that its composite with $\pi: J_{\otimes p} \to BO_\otimes$ is homotopic to $\chi e: SF \to BO_\otimes$.

We digress to give the following application of the corollary, which summarizes Quillen's results [60] about $K_*\mathbb{Z}$.

Remark VIII.3.6. For any commutative topological ring A, we have a commutative diagram of bipermutative categories

$$\begin{array}{ccc} & \mathscr{O}\mathbb{Z} & \longrightarrow & \mathscr{O}A \\ \mathscr{E} \nearrow & \downarrow & & \downarrow \\ \searrow & \mathscr{GL}\mathbb{Z} & \longrightarrow & \mathscr{GL}A \end{array}$$

and a derived commutative diagram of K-groups in positive degrees

$$\begin{array}{ccc} & KO_*\mathbb{Z} \longrightarrow KO_*A \\ K_*\mathscr{E} \nearrow & \downarrow \qquad\qquad \downarrow \\ \searrow & K_*\mathbb{Z} \longrightarrow K_*A \end{array} \quad \text{or} \quad \begin{array}{ccc} & \pi_*\Gamma_1 B\mathscr{O}\mathbb{Z} \longrightarrow \pi_*\Gamma_1 B\mathscr{O}A \\ \pi_*SF \nearrow & \downarrow \qquad\qquad\qquad \downarrow \\ \searrow & \pi_*\Gamma_1 B\mathscr{GL}\mathbb{Z} \longrightarrow \pi_*\Gamma_1 B\mathscr{GL}A \end{array}$$

where the second diagram results from the first by translation from 0-components to 1-components of zero$^{\text{th}}$ spaces of spectra and where $\Gamma_1 B\mathscr{E}$ is identified with SF via Proposition VII.3.4 and Corollaries VII.4.4 and VII.4.5. By Corollary VII.4.6, $K_*\mathscr{E} = \pi_*^s$ maps monomorphically onto a direct summand of $KO_*\mathbb{Z}$, the complementary summand being isomorphic to $\pi_*^s(\mathbb{R}P^\infty)$. When $A = \mathbb{R}$, $KO_*A = K_*A = \pi_*BO_\otimes$ and Remark V.5.6 shows that the element $\mu_i \in \pi_i SF$, $i \equiv 1$ or $2 \bmod 8$, defines a direct summand $\mathbb{Z}_2$ in $K_i\mathbb{Z}$. Let $_pJ_i$ denote the p-torsion in the image of $j_*\colon \pi_i Spin \to \pi_i SF$. When $p > 2$ and $A = k_{r(p)}$, the p-torsion subgroup of K_*A is isomorphic to $_pJ_*$ and, by Corollaries V.4.6 and VIII.3.5, $_pJ_*$ is a direct summand of $K_*\mathbb{Z}$. Finally, consider $_2J_i$. The image of $_2J_i = \mathbb{Z}_2$ in $K_i\mathbb{Z}$ is unknown, $i \equiv 0$ or $1 \bmod 8$ (and $i \geq 8$). Let $A = k_3$. The 2-torsion subgroup of $KO_{4i-1}k_3 = K_{4i-1}nk_3$ is $_2J_{4i-1}$. Write JU_2 and JU_2^δ for the fibre of $\psi^3 - 1\colon BU \to BU$ and for $\Gamma_0 B\mathscr{G}\mathscr{L}k_3$ (completed at 2). By the proof of Theorem VIII.3.2, there is an equivalence $\bar{\nu}_0\colon JU_2^\delta \to JU_2$ under which the natural map $JO_2^\delta \to JU_2^\delta$ corresponds to the map $c\colon JO_2 \to JU_2$ induced by complexification. By [1, 5.2], $\pi_{4i-1}JO_2$ and $\pi_{4i-1}JU_2$ are the same group and c_{4i-1} is the identity if i is even and multiplication by 2 if i is odd. Therefore, by Corollaries V.4.6 and VIII.3.5, $_2J_{8i-1}$ is a direct summand of $K_*\mathbb{Z}$ and the image in $K_*\mathbb{Z}$ of the element of order 2 in $_2J_{8i-5}$ maps to zero in K_*k_3. Quillen [60] proved that $_2J_{4i-1}$ maps monomorphically to $K_{4i-1}\mathbb{Z}$ by noting that Adams' e-invariant can be identified with the map $\pi_{4i-1}SF \to \pi_{4i-1}X$ induced by the unique lift $\zeta\colon SF \to X$ of $\chi e\colon SF \to BO_\otimes$ to the fibre X of the Pontryagin character $BO_\otimes \to \prod_{i\geq1} K(\mathbb{Q}, 4i)$ and observing that ζ is necessarily factors through $\Gamma_1 B\mathscr{G}\mathscr{L}\mathbb{Z}$ because the Chern classes of representations of discrete groups are torsion classes. Karoubi [34] found that $_2J_3$ is not a direct summand of $K_3\mathbb{Z}$, and Lee and Szczarba [37] proved the deep result that $K_3\mathbb{Z}$ is exactly $\mathbb{Z}_{48}$. $\Gamma_1 B\mathscr{G}\mathscr{L}\mathbb{Z}$

4. The splitting of SF at odd primes

Again, all spaces and spectra are to be completed at a fixed prime $p \neq q$ and $r = q^a$ is to be $r(p)$. Actually, almost all spaces in sight will have finite homotopy groups, hence localization will agree with completion.

Theorem VIII.3.4 focuses attention on the orientation sequence

$$SF \xrightarrow{\ e\ } J_{\otimes p}^\delta \xrightarrow{\ \tau\ } B(SF; j_p^\delta) \xrightarrow{\ q\ } BSF.$$

The map τ is null homotopic by the splitting of SF in Corollary V.4.6 and Theorems V.4.7 and V.4.8 (and Corollary VIII.3.5). When $p = 2$, [70, 9.11] or [26, II.12.2] show that there is no splitting $SF \simeq C_2 \times J_2$ as H-spaces, and presumably the first delooping of τ already fails to be null homotopic. When $p > 2$, we shall prove an exponential law for $\mathscr{G}\mathscr{L}k_r$ and shall use it to split SF and $B(SF; kO^\delta)$ as infinite loop spaces; it will follow that τ is trivial as an infinite loop map.

Let M denote the monoid $\{r^n \mid n \geq 0\}$. Subscripts M will denote unions of components indexed on M. Since $Q_\infty S^0$ is the free spectrum generated by S^0 (by Definition II.1.6) and also the free $\mathcal{Q} \times \mathscr{L}$-spectrum generated by S^0 (by Lemmas VI.2.4 and VI.2.5), there is an exponential unit map of spectra $e_r\colon Q_\infty S^0 \to \Omega^\infty T(Q_M S^0, \xi)$ specified on S^0 by $0 \to 1$ and $1 \to x$ for any chosen point $x \in Q_r S^0$ and also a unit map of $\mathcal{Q} \times \mathscr{L}$-spectra $e\colon Q_\infty S^0 \to \Omega^\infty T(B\mathscr{G}\mathscr{L}k_r, \theta)$. By Examples VI.5.2 and VI.5.6, we have a unit functor $e\colon \mathscr{E} \to \mathscr{G}\mathscr{L}k_r$ and a forgetful functor $f\colon \mathscr{G}\mathscr{L}k_r \to \mathscr{E}$. Let $e_r = fe\colon \mathscr{E} \to \mathscr{E}$ (which is an exponential map of permutative categories) and let $g = ef\colon \mathscr{G}\mathscr{L}k_r \to \mathscr{G}\mathscr{L}k_r$. Recall from Example VI.5.1 that $B\mathscr{E} = DS^0$. With these notations, freeness and Corollaries VII.4.4, VII.4.5 and VII.5.4 and Theorem VII.5.3 yield the following homotopy commutative diagram in which all maps indicated by $\simeq$ are homotopy equivalences and $\alpha_p^\delta\colon J_p^\delta \to SF$ is defined to be the

composite from the lower left to the upper right corner:

$$
\begin{array}{ccc}
Q_0S^0 & \xrightarrow{\quad\alpha_p^\delta e\quad} & SF
\end{array}
$$

Here $(\Gamma i)\iota_\otimes$ and the maps ϕ are equivalences since they are localizations at M and we are further localizing or completing all spaces at p. The maps $e\colon Q_0S^0 \to J_p^\delta$ and $e\colon SF \to J_{\otimes p}^\delta$ are the restrictions of the 0 and 1 components of the zero$^{\text{th}}$ map of the unit of j_p^δ. Of course, the equalities which involve J_p^δ and $J_{\otimes p}^\delta$ require p to be odd, but we can construct a precisely analogous diagram

$$
\begin{array}{ccc}
Q_0S^0 & \xrightarrow{\alpha_2^\delta e} & SF \\
{\scriptstyle e}\downarrow \quad {}^{\alpha_2^\delta}\nearrow & & \downarrow{\scriptstyle e} \\
J_2^\delta & \xrightarrow[e\alpha_2^\delta]{} & J_{\otimes 2}^\delta
\end{array}
$$

by use of $n k_3$ rather than $\mathscr{GL}k_r$. An analogous diagram can also be constructed by use of $\mathscr{O}k_r$. Henceforward, we assume that p is odd.

By the results of Sections VII§4 and VII§5 cited above, all maps in our diagram are composites of maps of $\mathscr{Q}\times\mathscr{LK}_\infty$-spaces and homotopy inverses of maps of $\mathscr{Q}\times\mathscr{LK}_\infty$-spaces, and the diagram induces a similar commutative diagram in $H\mathscr{S}$. Clearly all three operads are required: it is $\mathscr{K}_\infty$ which acts naturally on Q_0S^0, $\mathscr{L}$ on SF, and $\mathscr{Q}$ on $B\mathscr{GL}k_r$ (in two ways). Because of the different geometric sources of the actions, the statement that all of our maps preserve them is highly nontrivial.

THEOREM VIII.4.1. *The composite $J_p^\delta \xrightarrow{\delta_p^\delta} SF \xrightarrow{e} J_{\otimes p}^\delta$ is a homotopy equivalence.*

PROOF. By our diagram, the specified composite may also be described as

$$
J_p^\delta = \Gamma_0(B\mathscr{GL}k_r,\theta) \xrightarrow{\Gamma g} \Gamma_1(B_M\mathscr{GL}k_r,\xi) \xleftarrow{\phi^{-1}} \Gamma_1(B\mathscr{GL}k_r,\theta) = J_{\otimes p}^\delta \ .
$$

It clearly suffices to prove that $\phi^{-1}\circ\Gamma g$ induces an isomorphism on mod p homology H_*. By Quillen [59, §8], there exist elements y_i of degree $2i(p-1)$ and z_i of degree $2i(p-1)-1$, $i \geq 1$, such that

$$
H_*J_p^\delta \cong H_*BGL(\infty,k_r) = P\{y_i\} \otimes E\{z_i\} \ .
$$

Actually, we shall only need that $H_*J_p^\delta$ is additively no larger than stated. This is the easy part of Quillen's work and depends only on the form of the p-Sylow subgroups of the $GL(n;k_r)$ [59, p. 573–574]. The rest of the computation of $H_*J_p^\delta$ will fall out of the argument to follow (and is thus independent of Brauer lifting). The remainder of the proof depends solely on general properties of the Pontryagin products $*$ and $\#$ and the homology operations Q^s and $\widetilde{Q}^s$ determined on E_∞ ring spaces by θ and ξ respectively [26, II§1,2], together with particular properties of H_*QS^0 [26, I§4,

II§5]. Write $\#$ by juxtaposition on elements, and write $[n]$ for the homology class of a component n. By [**26**, II.2.8], modulo linear combinations of $*$-products between positive degree elements,

(a) $\widetilde{Q}^s[r] \equiv \frac{1}{p}(r^p - r)Q^s[1] * [r^p - p]$.

The coefficient is non-zero because r reduces mod p^2 to a generator of the group of units of $\mathbb{Z}_{p^2}$. By [**26**, II.2.8], for $x \in H_*Q_mS^0$ or $x \in H_*\Gamma_m(B\mathscr{GL}k_r, \theta)$,

(b) $mx \equiv (x * [1 - m])[m]$.

Let $k = r^{-p}\frac{1}{p}(r^p - r)$; then (a) and (b) imply

(c) $\widetilde{Q}^s[r] \equiv k(Q^s[1] * [1 - p])[r^p]$.

Since $e_r \colon QS^0 \to \Gamma(Q_MS^0, \xi)$ and Γg are exponential, they send

$$Q^s[1] * [-p] \quad \text{to} \quad \widetilde{Q}^s[r][r^{-p}].$$

In view of (c), it follows that $(\Gamma i \circ \iota_\otimes)^{-1}e_r$ and $\phi^{-1}\Gamma g$ send

$$Q^s[1] * [-p] \quad \text{to} \quad kQ^s[1] * [1 - p],$$

modulo elements decomposable under the translate $\underline{*}$ of the $*$-product from the zero component to the one component. By the multiplication table for $\#$ on H_*SF [**26**, II.5.6], it follows immediately that the composite

$$P\{Q^s[1] * [-p]\} \otimes E\{\beta Q^s[1] * [-p]\} \subset H_*Q_0S^0 \xrightarrow{(\Gamma i \circ \iota_\otimes)_*^{-1}(e_r)_*} H_*SF$$

is a monomorphism (this being the step which would fail if $p = 2$). Since $(\Gamma i \circ \iota_\otimes)^{-1}e_r$ factors through $e \colon Q_0S^0 \to J_p^\delta$, we conclude by a count of dimensions that

$$H_*J_p^\delta = P\{Q^q[1] * [-p]\} \otimes E\{\beta Q^s[1] * [-p]\}$$

as a Hopf algebra (under $*$) over the Steenrod algebra A. Moreover, by translation $x \to x * [1]$, we now have a basis in which we know $H_*J_{\otimes p}^\delta$ as a coalgebra over A (because J_p^δ and $J_{\otimes p}^\delta$ are the 0 and 1 components of $\Gamma B\mathscr{GL}k_r$). We already know that

$$\phi_*^{-1}(\Gamma g)_*(Q^1[1] * [-p]) = kQ^1[1] * [1 - p] \neq 0,$$

and it follows by standard techniques that $\phi_*^{-1}(\Gamma g)_*$ is an isomorphism. In detail, $\phi_*^{-1}(\Gamma g)_*$ is a morphism of connected Hopf algebras of the same finite dimension in each degree and will be an isomorphism if it is a monomorphism on primitive elements. Let p_s, of degree $2s(p - 1)$, be the sth even degree basic primitive element of $H_*J_p^\delta$, namely the sth Newton polynomial in the $Q^s[1] * [-p]$. Since, by [**26**, I.1.1], $P_*^rQ^s[1] = (-1)^r(r, s(p - 1) - pr)Q^{s-r}[1]$, a standard calculation gives

$$P_*^r p_s = (-1)^r(r, s(p - 1) - pr - 1)p_{s-r}.$$

Therefore some $P_*^r p_s \neq 0$, $r > 0$, unless $s = p^k$ for some $k \geq 0$ when

$$P_*^1 p_{s+1} = p_s \quad \text{and} \quad P_*^p p_{s+1} = -p_{s+1-p} \text{ if } k \geq 2$$

and

$$P_*^{p-1} p_{s+p-1} = p_s \quad \text{and} \quad P_*^p p_{s+p-1} = 2p_{s-1} \text{ if } k = 1.$$

Thus, by induction on s, $\phi_*^{-1}(\Gamma g)_*(p_s) \neq 0$ for all $s \geq 1$. Let b_s, of degree $2s(p-1)-1$, be the sth odd degree basic primitive element of $H_*J_p^\delta$, so that $b_s \equiv \beta Q^s[1] * [-p]$ modulo elements decomposable under $*$ (and $\beta p_s = (-1)^{s+1}sb_s$). Since, again by [**26**, I.1.1], $P_*^r\beta Q^s[1] = (-1)^r(r, s(p - 1) - pr - 1)\beta Q^{s-r}[1]$, another calculation gives

$$P_*^r b_s = (-1)^r(r, s(p - 1) - pr - 1)b_{s-r}.$$

The coefficient here is the same as that in the even degree case, hence the same special cases show that $\phi_*^{-1}(\Gamma g)_*(b_s) \neq 0$ for all s since $\phi_*^{-1}(\Gamma g)_*(b_1) \neq 0$. The proof is complete. $\square$

In the following corollaries, we write $*$ or $\#$ for the product on infinite loop spaces according to whether we choose to think of them as additive or multiplicative. Recall from Corollary VIII.3.3 that $(SF; j_p^\delta) = \Omega B(SF; j_p^\delta)$ is equivalent as an infinite loop space to $C_p^\delta = \Omega B C_p^\delta$.

Corollary VIII.4.2. *The composites*

$$J_p^\delta \times (SF; j_p^\delta) \xrightarrow{\alpha_p^\delta \times \Omega q} SF \times SF \xrightarrow{\#} SF$$

and

$$BJ_p^\delta \times B(SF; j_p^\delta) \xrightarrow{B\alpha_p^\delta \times q} BSF \times BSF \xrightarrow{\#} BSF$$

are equivalences of infinite loop spaces.

PROOF. Ωq is equivalent to the fibre of $e\colon SF \to J_{\otimes p}^\delta$, hence the theorem implies that the first composite, and thus also the second, induces isomorphism on homotopy groups. $\qquad\square$

Choose an infinite loop map $w\colon BSF \to B(SF; j_p^\delta)$ such that $wq = 1$ as infinite loop maps.

Corollary VIII.4.3. *The map*

$$B(SF; kO) \xrightarrow{(c(\psi^r), wq)} BO_\otimes \times B(SF; j_p^\delta)$$

is an equivalence of infinite loop spaces.

PROOF. Since $q\colon B(SF; j_p^\delta) \to BSF$ factors through $q\colon B(SF; kO^\delta) \to BSF$, this follows from Theorem V.4.4 (last line), Theorem V.4.8(i), and the J-theory diagram of Section V§3 together with the multiplicative Brauer lift diagram of Theorem VIII.3.4. $\qquad\square$

The original diagram of this section suggests that $(SF; j_p^\delta) \simeq C_p^\delta$ is the multiplicative analog of the additive infinite loop space $C_{\oplus p}^\delta$ defined as the fibre of $e\colon Q_0 S^0 \to J_p^\delta$. Of course, we know that J_p^δ and $J_{\otimes p}^\delta$ are equivalent infinite loop spaces. In contrast, although $C_{\oplus p}^\delta$ and C_p^δ are evidently homotopy equivalent, there is no equivalence of infinite loop spaces between because their homology operations differ [**26**, I§4, II§6].

Corollary VIII.4.4. *The composite*

$$J_p^\delta \times C_{\oplus p}^\delta \xrightarrow{\tau^{-1}\alpha_p^\delta \times \pi} Q_0 S^0 \times Q_0 S^0 \xrightarrow{*} Q_0 S^0$$

is a homotopy equivalence (but $\tau^{-1}\alpha_p^\delta$ is not an infinite loop map), where $\tau\colon Q_0 S^0 \to SF$ is the translation $x \to x * 1$.

In fact, by [**26**, I§4] and our proof of Theorem VIII.4.1, the image of $(\tau^{-1}\alpha_p^\delta)_*$ generates $H_* Q_0 S^0$ as an algebra over the Dyer–Lashof algebra (under $*$ and the Q^s), and this statement even remains true at the prime 2. By [**26**, II§1], we also have the following technical consequence of our proof which has been used in the homological study of BF and $BTop$ in [**26**, II]. Conceptually, we have here used crude information on homology operations to obtain the geometric splitting of SF, and we here used the geometric splitting to obtain more subtle information.

Corollary VIII.4.5. $(\alpha_p^\delta)_*\colon H_* J_p^\delta \to H_* SF$ *takes the elements $Q^s[1] * [-p]$ and $\beta Q^s[1] * [-p]$ to generators of the subalgebra*

$$P\left\{Q^s[1] * [1 - p]\right\} \otimes E\left\{\beta Q^s[1] * [1 - p]\right\}$$

of $H_* SF$ considered as an algebra under the $\underline{*}$ product.

The point is that no higher operations $Q^I[1]$, $\ell(I) > 1$, contribute to the image of $(\alpha_p^\delta)_*$ on the specified generators. Since $(\alpha_p^\delta)_*$ is multiplicative with respect to $\#$, rather than $\underline{*}$, on $H_* SF$, such operations can contribute to the image of $(\alpha_p^\delta)_*$ on decomposable elements.

Remark VIII.4.6. The second author's original proof of Theorem VIII.4.1 gave different informa-
tion. Since p is odd, we may think of J_p and $J_{\otimes p}$ as the fibres of $\psi^r - 1$ and $\psi^r/1$ on BU and $BU_\otimes$.
Since $[J_p, U] = KU^{-1}BGL(\infty, k_r) = 0$ (by Theorem VIII.3.2 and [**14**]), the composite equivalence
$\varepsilon_p \alpha_p$ of Corollary V.4.6 is characterized by homotopy commutativity of the diagram

$$
\begin{array}{ccc}
J_p & \xrightarrow{\ \pi\ } & BU \\
{\scriptstyle \varepsilon_p \alpha_p} \downarrow & & \downarrow {\scriptstyle \rho^r} \\
J_{\otimes p} & \xrightarrow{\ \pi\ } & BU_\otimes
\end{array}
$$

where ρ^r is the cannibalistic class determined by ψ^r and the standard orientation $BU \to B(U; kU)$.
A representation theoretical calculation ([**77**, 4.1]) shows that the diagram

$$
\begin{array}{ccccc}
J_p^\delta & \xrightarrow{\ \nu\ } & J_p & \xrightarrow{\ \pi\ } & BU \\
{\scriptstyle e_p \alpha_p^\delta} \downarrow & & & & \downarrow {\scriptstyle \rho^r} \\
J_{\otimes p}^\delta & \xrightarrow{\ \nu\ } & J_{\otimes p} & \xrightarrow{\ \pi\ } & BU_\otimes
\end{array}
$$

is homotopy commutative, where the maps ν are the 0 and 1 components of the zero$^{\text{th}}$ map of the
equivalence $\nu \colon j_p^\delta \to j_p$ of Theorem VIII.3.2. Thus $\epsilon_p \alpha_p \simeq \nu e \alpha_p^\delta \nu^{-1}$ and $e\alpha_p^\delta$ is an equivalence.

$\alpha_p^\delta \nu^{-1} \colon J_p \to SF$ is an infinite loop map while $\alpha_p \colon J_p \to SF$ makes the J-theory diagram
homotopy commutative and, in particular, is such that $j \colon Spin \to SF$ factors through it. In view
of Corollary VIII.3.5 and the previous remarks, it is natural to hope that these two maps are
homotopic or, at least, that α_p can be chosen as an infinite loop map. This would certainly hold if
Conjecture V.7.14 (the complex Adams conjecture on the infinite loop level) were satisfied.

Very recently, Friedlander [Stable Adam's conjecture. Preprint] and Seymour have announced
proofs of Conjecture V.7.14. Unfortunately, Segal's machinery [**68**] seems essential to Friedlander's
proof, hence it is not yet known that his infinite loop structure on SF agrees with ours (and ours
is essential to such basic facets of the theory as the orientation sequences). We have not seen the
details of Seymour's argument.

CHAPTER IX

Pairings in infinite loop space theory

Maps of spectra of the general form $D \wedge E \to F$ are central to stable homotopy theory. The purpose of this chapter is to develop a theory of pairings that allows one to recognize such maps in the guise of appropriate space level maps $X \wedge Y \to Z$, where X, Y, and Z are E_∞ spaces. Since $X \wedge Y$ will not itself be an E_∞ space, such a theory is certainly not implicit in the recognition principle already obtained in Chapter VII. It will be convenient to work with (weak) prespectra and their pairings in the sense of Whitehead [**80**] in throughout this chapter. The relationship between these notions and the stable category has been explained in Section II§3.

While a theory of pairings is an obvious desideratum of any complete treatment of infinite loop space theory and should have many other applications, the need for it emerged in attempts to compare our machine-built spectra $\Omega^\infty TB\mathscr{P}A$, where $\mathscr{P}A$ is the category of finitely generated projective modules over a ring A, to the Gersten–Wagoner spectra [**30**, **79**]. Let CA be the ring of infinite, but row and column finite, matrices with entries in A and let SA be the quotient of CA by the ideal generated by the finite matrices. Gersten and Wagoner showed that ΩKSA is equivalent to KA, where KA denotes $BGL(\infty, A)^+ \times K_0 A$, and thus produced an Ω-prespectrum $GWA = \{KS^i A \mid i \geq 0.\}$.

Since free modules are cofinal among projective modules, $BGL(\infty, A)$ may be regarded as $\varinjlim B \operatorname{Aut} P$, $P \in \mathscr{P}A$ (up to homotopy type; compare [**46**, p. 85]). By the universal property of the plus construction (above Lemma VIII.1.1), the tensor product functor $\mathscr{P}A \times \mathscr{P}B \to \mathscr{P}(A \otimes_{\mathbb{Z}} B)$ induces a map $\mu \colon KA \wedge KB \to K(A \otimes_{\mathbb{Z}} B)$. Suppose given a functor E from rings to Ω-prespectra, written $EA = \{E_i A \mid i \geq 0\}$, such that $E_0 A = KA$ and suppose that E admits an external tensor product, by which we understand a natural pairing $(EA, EB) \to E(A \otimes_{\mathbb{Z}} B)$ which extends the map μ of zero[th] spaces. With these data, Fiedorowicz [**27**] has proven that there is a natural map $f : EA \to GWA$ of Ω-prespectra such that $f_0 \colon E_0 A = KA \to KA$ is the identity. It follows (by Lemmas II.2.10 and II.2.11) that the associated connective spectra of EA and GWA are equivalent.

Now let EA denote $\Omega^\infty TB\mathscr{P}A$ regarded as an Ω-prespectrum. Certainly $E_0 A$ is KA (up to homotopy type). The results of this chapter will imply that E admits an external tensor product and thus that EA is the associated connective spectrum of GWA.

We develop suitably related notions of pairings of symmetric monoidal categories, of permutative categories, and of $\mathscr{Q}$-spaces in Section IX§1. We prove that pairings of $\mathscr{Q}$-spaces induce pairings of Ω-prespectra in Section IX§2. Schematically, our results can be summarized as follows:

Pairings of symmetric monoidal categories
$\Downarrow$blow up
pairings of permutative categories
$\Downarrow B$
pairings of $\mathscr{Q}$-spaces
$\Downarrow \Omega^\infty T$
pairings of Ω-prespectra

While there is an evident intuitive relationship between the present theory and the E_∞ ring theory of the earlier chapters, I have not attempted to combine the two lines of thought into a single more general theory.

1. Pairings of categories and of $\mathcal{Q}$-spaces

Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be symmetric monoidal categories with products $\oplus$ and units 0. A pairing $\otimes\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a functor $\otimes$ such that $A \otimes 0 = 0$ and $0 \otimes B = 0$ together with a coherent natural bidistributivity isomorphism

$$(*) \qquad d\colon (A \oplus A') \otimes (B \oplus B') \cong ((A \otimes B) \oplus (A \otimes B')) \oplus ((A' \otimes B) \oplus (A' \otimes B'))$$

for $A, A' \in \mathcal{O}\mathcal{A}$ and $B, B' \in \mathcal{O}\mathcal{B}$; the extra parentheses are needed since $\oplus$ on $\mathcal{C}$ is not assumed to be associative.

The category theorist will recognize that this is not really a definition. Precision would require elucidation of the meaning of coherence, via a specification of just which diagrams involving d and the associativity, unity, and commutativity isomorphisms a, b, and c are required to commute. The details would be analogous to those in La Plaza [**35**]. We prefer to be informal since the intuition should be clear. Of course, the example to keep in mind is $\otimes\colon \mathscr{P}A \times \mathscr{P}B \to \mathscr{P}(A \otimes_\mathbb{Z} B)$.

We would like to define a pairing of permutative categories by requiring the isomorphisms d to be identity maps. However, the expansion of the right sides of $(*)$ when the left sides are

$$((A \oplus A') \oplus A'') \otimes (B \oplus B') \quad \text{and} \quad (A \oplus (A' \oplus A'')) \otimes (B \oplus B')$$

demonstrates that, in the absence of strict commutativity, this requirement would be unreasonable. We are led to the following definition.

Definition IX.1.1. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be permutative categories, with products $\oplus$ and units 0, and assume given subsets $\mathscr{G}\mathcal{A}$ of $\mathcal{O}\mathcal{A}$ and $\mathscr{G}\mathcal{B}$ of $\mathcal{O}\mathcal{B}$ which generate $\mathcal{O}\mathcal{A}$ and $\mathcal{O}\mathcal{B}$ under $\oplus$. A pairing $\otimes\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a functor $\otimes$ such that $A \otimes 0 = 0$, $0 \otimes B = 0$, and for all sequences $\{A_1, \ldots, A_j\}$ of objects in $\mathscr{G}\mathcal{A}$ and all sequences $\{B_1, \ldots, B_k\}$ of objects in $\mathscr{G}\mathcal{B}$

$$(**) \qquad (A_1 \oplus \cdots \oplus A_j) \otimes (B_1 \oplus \cdots \oplus B_k) = (A_1 \otimes B_1) \otimes \cdots \otimes (A_1 \otimes B_k) \otimes \cdots \otimes$$
$$(A_j \otimes B_1) \otimes \cdots \otimes (A_j \otimes B_k)$$

and the following diagram commutes for all permutations $\sigma \in \Sigma_j$ and $\tau \in \Sigma_k$, where the unlabelled isomorphisms are given by the commutativity isomorphisms of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$:

$$
\begin{array}{ccc}
(A_1 \oplus \cdots \oplus A_j) \otimes (B_1 \oplus \cdots \oplus B_k) & =\!=\!=\!=\!=\!= & (A_1 \otimes B_1) \oplus \cdots \oplus (A_j \oplus A_k) \\
\| \wr & & \| \wr \\
(A_{\sigma(1)} \oplus \cdots \oplus A_{\sigma(j)}) \otimes (B_{\tau(1)} \oplus \cdot \oplus B_{\tau(k)}) & =\!=\!= & (A_{\sigma(1)} \otimes B_{\tau(1)}) \oplus \cdots \oplus (A_{\sigma(j)} \otimes A_{\tau(k)})
\end{array}
$$

By use of the commutativity isomorphism of $\mathcal{C}$, $(**)$ determines a natural bidistributivity isomorphism d as in $(*)$. The commutative diagrams above give coherence. Thus a pairing of permutative categories is also a pairing of symmetric monoidal categories. It is an instructive exercise to verify that $\otimes\colon \mathscr{G}\mathscr{L}A \times \mathscr{G}\mathscr{L}B \to \mathscr{G}\mathscr{L}(A \otimes_\mathbb{Z} B)$ is a pairing of permutative categories. In this case, the generating sets of objects have the single element 1, $(**)$ is trivial, and only the diagrammatic relationship between the commutativity isomorphisms need be checked.

Recall the functor Φ of Proposition VI.3.2 from symmetric monoidal categories to permutative categories.

Proposition IX.1.2. *A pairing* $\otimes\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ *of symmetric monoidal categories naturally determines a pairing* $\otimes\colon \Phi\mathcal{A} \times \Phi\mathcal{B} \to \Phi\mathcal{C}$ *of permutative categories such that the diagram*

$$
\begin{array}{ccc}
\Phi\mathcal{A} \times \Phi\mathcal{B} & \xrightarrow{\ \otimes\ } & \Phi\mathcal{C} \\
{\scriptstyle \pi\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
\mathcal{A} \times \mathcal{B} & \xrightarrow{\ \otimes\ } & \mathcal{C}
\end{array}
$$

commutes up to coherent natural isomorphism.

PROOF. The space of objects of $\Phi\mathcal{A}$ is the free monoid with unit 0 generated by $\mathcal{O}\mathcal{A}$, hence we take $\mathcal{O}\mathcal{A}$ as the generating set in $\mathcal{O}\Phi\mathcal{A}$ and similarly for $\mathcal{B}$. Recall that, with the product on $\Phi\mathcal{A}$ again written as $\oplus$, π is specified on objects by

$$\pi(A_1 \oplus \cdots \oplus A_j) = A_1 \oplus (A_2 \oplus (A_3 \oplus \cdots (A_{j-1} \oplus A_j)\cdots)).$$

As in [**46**, 4.2] or Proposition VI.3.5, the morphisms from A to A' in $\Phi\mathcal{A}$ are the morphisms from πA to $\pi A'$ in $\mathcal{A}$, with composition and the commutativity isomorphism c determined in an evident way from these data on $\mathcal{A}$. Define $\otimes\colon \Phi\mathcal{A} \times \Phi\mathcal{B} \to \Phi\mathcal{C}$ by $(**)$ on objects. On morphisms $f\colon A \to A'$ and $g\colon B \to B'$ in $\Phi\mathcal{A}$ and $\Phi\mathcal{B}$, the morphism $f \otimes g\colon A \otimes B \to A' \otimes B'$ in $\Phi\mathcal{C}$ is specified by the composite

$$\pi(A \otimes B) \cong \pi(A) \otimes \pi(B) \xrightarrow{f\otimes g} \pi(A') \otimes \pi(B') \cong \pi(A' \otimes B')$$

in $\mathcal{C}$, where the unlabelled isomorphisms are uniquely determined by the coherent natural isomorphisms a, c, and d of $\mathcal{C}$ and are the isomorphisms required for the diagram in the statement of the proposition. The commutativity of the diagram in Definition IX.1.1 follows from the coherence of the given pairing $\otimes\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$. Indeed, the omitted formal definition of coherence here can be specified simply by listing those diagrams which suffice for the present proof. $\square$

Now recall the categorical E_∞ operad $\mathcal{Q}$ of Section VI§4. The tensor product $\Sigma_j \times \Sigma_k \to \Sigma_{jk}$ (of Notation VI.1.4) induces a functor $\widetilde{\Sigma}_j \times \widetilde{\Sigma}_k \to \widetilde{\Sigma}_{jk}$ and thus, by application of the classifying space functor B, a map $\otimes\colon \mathcal{Q}(j) \times \mathcal{Q}(k) \to \mathcal{Q}(jk)$.

Definition IX.1.3. Let X, Y, and Z be $\mathcal{Q}$-spaces. A pairing $f\colon X \times Y \to Z$ is a map f which factors through $X \wedge Y$ and is such that the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{Q}(j) \times X^j \times \mathcal{Q}(k) \times Y^k & \xrightarrow{\ \theta_j \times \theta_k\ } & X \times Y \\
{\scriptstyle 1\times\tau\times 1}\downarrow & & \searrow{\scriptstyle f} \\
\mathcal{Q}(j) \times \mathcal{Q}(k) \times X^j \times Y^k & & Z \\
{\scriptstyle 1\times 1\times n}\downarrow & & \nearrow{\scriptstyle \theta_{jk}} \\
\mathcal{Q}(j) \times \mathcal{Q}(k) \times (X \wedge Y)^{jk} & \xrightarrow{\ \otimes \times f^{jk}\ } & \mathcal{Q}(jk) \times Z^{jk}
\end{array}
$$

where

$$\nu(x_1, \ldots, x_j, y_1, \ldots, y_k) = (x_1 \wedge y_1, \ldots, x_1 \wedge y_k, \ldots, x_j \wedge y_1, \ldots, x_j \wedge y_k).$$

I do not have a definition (or any prospective applications) for a notion of pairing of $\mathcal{C}$-spaces for a general E_∞ operad $\mathcal{C}$. One could, of course, simply appeal to Remark VI.2.7(iii), which shows that $\mathcal{C}$-spaces can be replaced by equivalent $\mathcal{Q}$-spaces.

Proposition IX.1.4. *If* $\otimes\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ *is a pairing of permutative categories, then* $f = B\otimes\colon B\mathcal{A} \times B\mathcal{B} \to B\mathcal{C}$ *is a pairing of* $\mathcal{Q}$-spaces.

PROOF. The basepoint of $B\mathcal{a}$ is given by the object 0 (regarded as a 0-simplex), and f factors through $B\mathcal{a} \wedge B\mathcal{B}$ by the nullity of zero. The diagram of Definition IX.1.1 implies the commutativity of the following coherence diagram:

$$\begin{array}{ccc}
\widetilde{\Sigma}_j \times \mathcal{a}^j \times \widetilde{\Sigma}_k \times \mathcal{B}^k & \xrightarrow{c_j \times c_k} & \mathcal{a} \times \mathcal{B} \\
{\scriptstyle 1 \times \tau \times 1}\downarrow & & \\
\widetilde{\Sigma}_j \times \widetilde{\Sigma}_k \times \mathcal{a}^j \times \mathcal{B}^k & & \searrow^{f} \\
{\scriptstyle 1 \times 1 \times n}\downarrow & & \mathcal{C} \\
\widetilde{\Sigma}_j \times \widetilde{\Sigma}_k \times (\mathcal{a} \times \mathcal{B})^{jk} & \xrightarrow{\otimes \times \otimes^{jk}} & \widetilde{\Sigma}_{jk} \times \mathcal{C}^{jk} \nearrow_{c_{jk}}
\end{array}$$

were the functor ν is defined just as was the map ν in the previous definition. The conclusion follows upon application of B. $\qquad\square$

2. The recognition principle for pairings

We here extend the one operad recognition principle of Section VII§3 to a recognition principle for pairings. Although the present theory is basically an elaboration of the additive theory, it will still be necessary, for technical reasons, to work with the little convex bodies (partial) operads $\mathscr{K}_n$ rather than with the little cubes $\mathscr{C}_n$. Define $\otimes \colon \mathscr{K}_m(j) \times \mathscr{K}_n(k) \to \mathscr{K}_{m+n}(jk)$ by

$$\langle c_1, \ldots, c_j \rangle \otimes \langle c_1', \ldots, c_k' \rangle = \langle c_1 \times c_1', \ldots, c_1 \times c_k', \ldots, c_j \times c_1', \ldots, c_j \times c_k' \rangle.$$

Let $\mathcal{Q}_n$ denote $\mathcal{Q} \times \mathscr{K}_n$ for $n \geq 1$. The maps $\otimes \colon \mathcal{Q}(j) \times \mathcal{Q}(k) \to \mathcal{Q}(jk)$ and the maps just defined together determine maps $\otimes \colon \mathcal{Q}_m(j) \times \mathcal{Q}_n(k) \to \mathcal{Q}_{m+n}(jk)$. Let D_n denote the (partial) monad associated to $\mathcal{Q}_n$. We begin by using the maps $\otimes$ to define a "pairing of monads" $D_m \wedge D_n \to D_{m+n}$.

Proposition IX.2.1. *For based spaces X and Y, the composite maps*

$$\mathcal{Q}_m(j) \times X^j \times \mathcal{Q}_n(k) \times Y^k \xrightarrow{1 \times \tau \times 1} \mathcal{Q}_m(j) \times \mathcal{Q}_n(k) \times X^j \times Y^k \xrightarrow{\otimes \times \nu}$$
$$\mathcal{Q}_{m+n}(jk) \times (X \wedge Y)^{jk},$$

where ν is as specified in Definition IX.1.3, induce maps

$$\lambda_{mn} \colon D_m X \wedge D_n Y \longrightarrow D_{m+n}(X \wedge Y)$$

such that the following diagrams commute:

$$\begin{array}{ccccc}
D_m^{(2)} X \wedge D_n^{(2)} Y & \xrightarrow{\lambda_{mn}} & D_{m+n}^{(1)}(D_m X \wedge D_n Y) & \xrightarrow{D_{m+n}(\lambda_{mn})} & D_{m+n}^{(2)}(X \wedge Y) \\
{\scriptstyle \mu \wedge \mu}\downarrow & & & & \downarrow{\scriptstyle \mu} \\
D_m X \wedge D_n Y & & \xrightarrow{\lambda_{mn}} & & Dm+n(X \wedge Y)
\end{array}$$

and

$$\begin{array}{ccc}
 & X \wedge Y & \\
{\scriptstyle \eta \wedge \eta}\swarrow & & \searrow{\scriptstyle \eta} \\
D_m X \wedge D_n Y & \xrightarrow{\lambda_{mn}} & D_{m+n}(X \wedge Y)
\end{array} \quad .$$

PROOF. $D_m X$ is constructed from $\coprod \mathcal{Q}_m(j) \times X^j$ by use of appropriate equivariance and basepoint identifications, and its product μ and unit h are induced from the structural maps γ and unit 1 for the operads $\mathcal{Q}$ and $\mathscr{K}_m$ (see Definition VI.1.2 and [**45**, 2.4]). The proof consists of a check, in principle of $\mathcal{Q}$ and the $\mathscr{K}_m$ separately, of the commutation relations between $\otimes$ and the defining data of the specified operads. The details are closely analogous to those already formulated in Definitions VI.1.6 to VI.1.8 and VI.1.10 and Proposition VI.1.9 (specialized to the case $k = 2$) and will therefore be omitted. The top row of the first diagram must be interpreted in the sense of partial

monads, the superscripts indicating restrictions of powers such that only composable pairs of little convex bodies are in sight (see Definition VII.1.1 and the discussion following Theorem VII.2.1). For $D^{(1)}_{m+n}(D_m X \wedge D_n Y)$, composability is to be interpreted in terms of $\otimes$ on the $\mathscr{K}$'s. Indeed, we may specify this space to be the inverse image of $D^{(2)}_{m+n}(X \wedge Y)$ under $D_{m+n}(\lambda_{mn})$ and then check that it contains the image of $D^{(2)}_m X \wedge D^{(2)}_n Y$ under the map

$$\lambda_{mn} \colon D_m D_m X \wedge D_n D_n Y \longrightarrow D_{m+n}(D_m X \wedge D_n Y).$$

$\square$

Let $\beta_n \colon \Sigma^n D_n \to D_n$ be adjoint of the composite of the projection $\pi \colon D_n \to K_n$ and the morphism of monads $\alpha_n \colon K_n \to \Omega^n \Sigma^n$ of Theorem VII.2.2. Recall (for twisting maps) that we are writing suspension coordinates on the right.

Proposition IX.2.2. *The following diagrams commute for all X and Y*

$$
\begin{array}{ccccc}
\Sigma^m D_m X \wedge \Sigma^d D_n Y & \xrightarrow{1 \wedge \tau \wedge 1} & \Sigma^{m+n}(D_m X \wedge D_n Y) & \xrightarrow{\Sigma^{m+n}\lambda_{mn}} & \Sigma^{m+n} D_{m+n}(X \wedge Y) \\
{\scriptstyle \beta_m \wedge \beta_n} \downarrow & & & & \downarrow {\scriptstyle \beta_{m+n}} \\
\Sigma^m X \wedge \Sigma^n Y & & \xrightarrow{\hspace{3cm} 1 \wedge \tau \wedge 1 \hspace{3cm}} & & \Sigma^{m+n}(X \wedge Y)
\end{array}
$$

Proof. We may define $\lambda_{mn} \colon K_m X \wedge K_n Y \to K_{m+n}(X \wedge Y)$ just as in the previous proposition, and then $\pi \lambda_{mn} = \lambda_{mn}(\pi \wedge \pi)$. Moreover, as we pointed out in [**45**, 8.3] in the case of little cubes, the following diagram is commutative:

$$
\begin{array}{ccc}
K_m X \wedge K_n Y & \xrightarrow{\lambda_{mn}} & K_{m+n}(X \wedge Y) \\
{\scriptstyle \alpha_m \wedge \alpha_n} \downarrow & & \downarrow {\scriptstyle \alpha_{m+n}} \\
\Omega^m \Sigma^m X \wedge \Omega^n \Sigma^n Y & \xrightarrow{\wedge} & \Omega^{m+n} \Sigma^{m+n}(X \wedge Y)
\end{array}
$$

The conclusion follows by passage to adjoints. $\square$

Recall that $\sigma \colon \mathscr{K}_n \to \mathscr{K}_{n+1}$ is the morphism of operads specified by $c \to c \times 1$. We need to know that, up to homotopy, σ is independent of the choice of privileged coordinate. The following analog of [**45**, 4.9] for little cubes will give the idea. Let $\sigma' \colon \mathscr{K}_n \to \mathscr{K}_{n+1}$ be specified by $c \to 1 \times c$ on little convex bodies.

Lemma IX.2.3. *The maps σ and σ' from $\mathscr{K}_n(j)$ to $\mathscr{K}_{n+1}(j)$ are Σ_j-equivariantly homotopic.*

Proof. Define orthogonal transformations $g, g' \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$g(s, x) = (x, s) \quad \text{and} \quad g'(s, x) = \begin{cases} (s, x) & \text{if } n \text{ is even} \\ (1 - s, x) & \text{if } n \text{ is odd} \end{cases}$$

for $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$. Since g an g' both have degree $(-1)^n$, there is a path $h \colon I \to O(n)$ from g to g'. For a little convex body $c \colon \mathbb{R}^n \to \mathbb{R}^n$, we have

$$g \sigma'(c) g^{-1} = \sigma(c) \quad \text{and} \quad g' \sigma'(c)(g')^{-1} = \sigma'(c).$$

The required homotopy is given on little convex bodies c by conjugation of $\sigma'(c)$ with the orthogonal transformations h_t. $\square$

It is not just the existence but the form of the homotopies that is essential for our purposes. For example, the following result is immediate from the previous proof.

Proposition IX.2.4. *The bottom part of the following diagram commutes and the top part commutes up to homotopy for all X and Y:*

$$
\begin{array}{ccc}
& D_{m+1}X \wedge D_nY & \\
\overset{\sigma \wedge 1}{\nearrow} & & \overset{\lambda_{m+1,n}}{\searrow} \\
D_mX \wedge D_nY \xrightarrow{\lambda_{mn}} D_{m+n}(X \wedge Y) \xrightarrow{\sigma} D_{m+n+1}(X \wedge Y) \\
\underset{1 \wedge \sigma}{\searrow} & & \underset{\lambda_{m,n+1}}{\nearrow} \\
& D_mX \wedge D_{n+1}Y &
\end{array}
$$

At this point, we recall Whitehead's definition [**80**] of a pairing.

Definition IX.2.5. A pairing $\phi\colon (T', T'') \to T$ of prespectra consists of maps $\phi_{mn}\colon T'm \wedge T''_n \to T_{m+n}$ for $m, n \geq 0$ such that, for homotopy, the bottom part of the following diagram commutes and the top part of the diagram commutes up to the sign $(-1)^n$:

$$
\begin{array}{ccc}
\Sigma T'_m \wedge T''_n & \xrightarrow{\sigma_m \wedge 1} & T'_{m+1} \wedge T''_n \\
{\scriptstyle 1 \wedge \tau}\downarrow & & \quad\searrow{\scriptstyle \phi_{m+1,n}} \\
\Sigma(T'_m \wedge T''_n) & \xrightarrow{\Sigma \phi_{mn}} \Sigma T_{m+n} \xrightarrow{\sigma_{m+n}} T_{m+n+1} \\
{\scriptstyle 1}\downarrow & & \quad\nearrow{\scriptstyle \phi_{m,n+1}} \\
T'_m \wedge \Sigma T''_n & \xrightarrow{1 \wedge \sigma_n} & T'_m \wedge T''_{n+1}
\end{array}
$$

For a $\mathscr{Q}$-space X, we have the prespectrum TX specified by

$$T_iX = B(\Sigma^i, D_i, X),$$

where $T_0X = X$, with structural maps

$$\sigma_i = B(1, \sigma, 1)\colon \Sigma T_iX \cong B(\Sigma^{i+1}, D_i, X) \longrightarrow B(\Sigma^{i+1}, D_{i+1}, X) = T_{i+1}X.$$

We have the following recognition principle for pairings.

THEOREM IX.2.6. *A pairing $f\colon X \wedge Y \to Z$ of $\mathscr{Q}$-spaces naturally induces a pairing $\phi\colon (TX, TY) \to TZ$ of prespectra such that $\phi_{0,0} = f$.*

PROOF. As in the first diagram of Proposition IX.2.2, the maps λ_{mn} can be iterated to yield

$$\lambda_{mn}^{(q)}\colon D_m^{(q)} \wedge D_n^{(q)}Y \longrightarrow D_{m+n}^{(q)}(X \wedge Y).$$

By composing with $D_{m+n}^{(q)}f$, smashing with spheres $S^m \wedge S^n = S^{m+n}$ (which are taken as one-point compactifications of Euclidean spaces here), and using a twist map τ, we obtain maps

$$\phi_{mnq}\colon B_q(\Sigma^m, D_m, X) \wedge B_q(\Sigma^n, D_n, Y) \longrightarrow B_q(\Sigma^{m+n}, D_{m+n}, Z).$$

By the definition of a pairing of $\mathscr{Q}$-spaces, we have the commutative diagrams

$$
\begin{array}{ccc}
D_mX \wedge D_nY & \xrightarrow{\lambda_{mn}} D_{m+n}(X \wedge Y) \xrightarrow{D_{m+n}f} D_{m+n}Z \\
{\scriptstyle \theta \wedge \theta}\downarrow & & \downarrow{\scriptstyle \theta} \\
X \wedge Y & \xrightarrow{\hspace{8cm}} & Z
\end{array}
$$

where the θ are composites of projections and the given actions of D on X, Y, and Z. In view of the definition of the face and degeneracy operators [**45**, 9.6] and the commutative diagrams of Propositions IX.2.1 and IX.2.2, it follows that ϕ_{mn*} is a map of simplicial spaces. For any simplicial based spaces U and V, the natural homeomorphism $|U| \times |V| \cong |U \times V|$ induces a map $|U| \wedge |V| \to |U \wedge V|$, and we therefore obtain a map $\phi_{mn}\colon T_mX \wedge T_nY \to T_{m+n}Z$ on passage to

geometric realization. Certainly $\phi_{0,0} = f$. On the level of q-simplices, the diagram of the previous definition can be written as follows:

$$D_m^{(q)}X \wedge S^m \wedge S^1 \wedge D_n^{(q)}Y \wedge S^n \xrightarrow{\sigma_{mq}\wedge 1} D_{m+1}^{(q)}X \wedge S^{m+1} \wedge D_n^{(q)}Y \wedge S^n$$

$$1\wedge\tau \downarrow \qquad\qquad\qquad\qquad\qquad \searrow \phi_{m+1,n,q}$$

$$D_m^{(q)}X \wedge S^m \wedge D_n^{(q)}Y \wedge S^n \wedge S^1 \xrightarrow{\phi_{mnq}\wedge 1} D_{m+n}^{(q)}Z \wedge S^{m+n} \wedge S^1 \xrightarrow{\sigma_{m+n,q}} D_{m+n+1}^{(q)}Z \wedge S^{m+n+1}$$

$$1 \uparrow \qquad\qquad\qquad\qquad\qquad \nearrow \phi_{m,n+1,q}$$

$$D_m^{(q)}X \wedge S^m \wedge D_n^{(q)}Y \wedge S^n \wedge S^1 \xrightarrow{1\wedge\sigma_{nq}} D_m^{(q)}X \wedge S^m \wedge D_{n+1}^{(q)}Y \wedge S^{n+1}$$

The bottom part commutes by Proposition IX.2.4. Provided that we first twist S^n past S, application of the homotopy of Proposition IX.2.4 to $D_m^{(q)}X \wedge D_n^{(q)}Y$ and of the orthogonal transformations which give that homotopy to S^{m+1+n} yields a homotopy for the top part of the diagram. These homotopies as q varies are compatible with the face and degeneracy operators (for each parameter $t \in I$) and so determine the required homotopy on passage to geometric realizations. It is for this compatibility with face operators that use of little convex bodies rather than little cubes is essential.   □

While the passage via Ω^∞ from pairings of prespectra to maps in the stable category has already been discussed in Theorem II.3.3 and Proposition II.3.4, we should perhaps say a bit about the more elementary passage from pairings of prespectra to pairings of Ω-prespectra. Provided that we are willing to neglect phantom maps, the functor Ω^∞ can be redefined homotopically by

$$(\Omega^\infty T)_i = \text{Tel}\,\Omega^j T_{i+j},$$

with $\iota\colon T_i \to (\Omega^\infty T)_i$ being given by the 0th term of the limit system. Given a pairing $\phi\colon (T', T'') \to T$, the maps

$$\overline{\phi}_{ij}\colon \Omega^m T'_{i+m} \wedge \Omega^n T''_{j+n} \longrightarrow \Omega^{m+n} T_{i+j+m+n}$$

specified by $\overline{\phi}_{ij}(f \wedge g)(x \wedge y) = \phi_{i+m,j+n}(f(x) \wedge g(y))$ for $x \in S^m$ and $y \in S^n$ are such that the following diagrams are homotopy commutative:

$$\Omega^{m+1}T'_{i+m+1} \wedge \Omega^n T''_{j+n}$$

$$\Omega^m\widetilde{\sigma}_{i+m}\wedge 1 \nearrow \qquad\qquad \searrow \overline{\phi}_{ij}$$

$$\Omega^m T'_{i+m} \wedge \Omega^n T''_{j+n} \xrightarrow{\overline{\phi}_{ij}} \Omega^{m+n}T_{i+j+m+n} \xrightarrow{\Omega^{m+n}\widetilde{\sigma}_{i+j+m+n}} \Omega^{m+n+1}T_{i+j+m+n+1}$$

$$1\wedge\Omega^n\widetilde{\sigma}_{j+n} \searrow \qquad\qquad \nearrow \overline{\phi}_{ij}$$

$$\Omega^m T'_{i+m} \wedge \Omega^{n+1} T''_{j+n+1}$$

there being a permutation of loop coordinates in the upper part which cancels the sign inserted in the definition of a pairing. Still neglecting phantom maps, there result maps

$$\widetilde{\phi}_{ij}\colon (\Omega^\infty T')_i \wedge (\Omega^\infty T'')_j \simeq \text{Tel}(\Omega^m T'_{i+m} \wedge \Omega^n T''_{j+n}) \longrightarrow (\Omega^\infty T)_{i+j}$$

which give a pairing $\widetilde{\phi}\colon (\Omega^\infty T', \Omega^\infty T'') \to \Omega^\infty T$. The following diagrams are clearly homotopy commutative:

$$\begin{array}{ccc}
T'_i \wedge T''_j & \xrightarrow{\phi_{ij}} & T \\
\iota\wedge\iota \downarrow & & \downarrow \iota \\
(\Omega^\infty T')_i \wedge (\Omega^\infty T'')_j & \xrightarrow{\widetilde{\phi}_{ij}} & (\Omega^\infty T)_{i+j}
\end{array}$$

In the context of Definition IX.2.5, the group completion property of the recognition principle implies that the map ϕ_{00} is characterized by the case $i = j = 0$ of this diagram (compare [**46**, 3.9], Definition VII.1.1, and the paragraphs above the latter result).

One could obtain a genuine pairing $\widetilde{\phi}$, without neglect of phantom maps, by an elaboration of the discussion just given in terms of the mapping cylinder techniques of [**43**, Theorem 4]. However, the extra precision would be insignificant in view of Proposition II.3.4.

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