MAT 322/523 MIDTERM II

WEDNESDAY APRIL 10, 2024
2:30–3:20PM

Name: ______________________________ ID: ______________

Instructions.

(1) Fill in your name and Stony Brook ID number and circle your lecture number at the top of this cover sheet.

(2) This exam is closed-book and closed-notes; no electronic devices.

(3) You have 80 minutes to complete this exam.

(4) You must justify all your answers and show all your work (unless the problem says otherwise). Even a correct answer without any justification will result in no credit.

1. Consider the function

$$f\colon (0,\infty)^2 \longrightarrow (0,\infty)^2$$
$$(x,y) \longmapsto (x^2 y, \log y)$$

(a) (10 pts) Show that f is a diffeomorphism.

Solution. It is clearly smooth, since the two coordinate functions $f_1(x,y) = x^2 y$ and $f_2(x,y) = \log y$ are smooth. Assume $f(x',y') = f(x,y)$. Then we get

$$\begin{cases} (x')^2 y' = x^2 y \\ \log y' = \log y \end{cases}.$$

The second equation implies $y = y'$ which turns the first equation into $(x')^2 = x^2$ and hence $x' = x$ because $x, x' > 0$, and hence f is injective. To show surjectivity assume $x^2 y = a$ and $\log y = b$. This gives $b = e^y$ and therefore $x = \sqrt{\frac{a}{e^y}}$. Finally we compute the derivative

$$Df(x,y) = \begin{pmatrix} 2xy & x^2 \\ 0 & \frac{1}{y} \end{pmatrix},$$

and so $\det Df(x,y) = 2x \neq 0$ since $x > 0$. □

(b) (10 pts) Compute $V(Df)$.

Solution. The definition of the volume is

$$V(Df(x,y)) = \sqrt{\det(Df(x,y)^T Df(x,y))}.$$

From Problem 1(b) we get $Df(x,y) = \begin{pmatrix} 2xy & x^2 \\ 0 & \frac{1}{y} \end{pmatrix}$. We compute the matrix product:

$$\begin{aligned} Df(x,y)^T Df(x,y) &= \begin{pmatrix} 2xy & 0 \\ x^2 & \frac{1}{y} \end{pmatrix} \begin{pmatrix} 2xy & x^2 \\ 0 & \frac{1}{y} \end{pmatrix} \\ &= \begin{pmatrix} 4x^2y^2 & 2x^3 y \\ 2x^3 y & x^4 + \frac{1}{y^2} \end{pmatrix}, \end{aligned}$$

so

$$\begin{aligned} V(Df(x,y)) = \sqrt{\det(Df(x,y)^T Df(x,y))} &= \sqrt{4x^2y^2\left(x^4 + \frac{1}{y}^2\right) - 4x^6y^2} \\ &= \sqrt{4x^2} = 2x. \end{aligned}$$

□

2. (20 pts) Consider the smooth 2-manifold in $\mathbb{R}^3$ given by

$$M := \{(x, y, z) \in \mathbb{R}^3 \mid z = F(x, y),\ (x, y) \in A\} \subset \mathbb{R}^3,$$

where $A \subset \mathbb{R}^2$ is some bounded open set and where $F\colon \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Show that

$$\operatorname{vol} M = \int_A \sqrt{1 + \left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2}.$$

[*Hint:* The manifold M admits a parametrization $\alpha\colon A \to \mathbb{R}^3$ defined by $\alpha(x, y) = (x, y, F(x, y))$.]

Solution. Since the manifold M admits a parametrization α, we have

$$\operatorname{vol} M = \int_M 1 dV = \int_A V(D\alpha(x, y)).$$

We compute the Jacobian $D\alpha(x, y)$ and the volume $V(D\alpha)$:

$$D\alpha(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{pmatrix}.$$

$$V(D\alpha(x, y)) = \sqrt{\det(D\alpha(x, y)^T D\alpha(x, y))} = \sqrt{\det\left(\begin{pmatrix} 1 & 0 & \frac{\partial F}{\partial x} \\ 0 & 1 & \frac{\partial F}{\partial y} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{pmatrix}\right)}$$

$$= \sqrt{\det \begin{pmatrix} 1 + \left(\frac{\partial F}{\partial x}\right)^2 & -\left(\frac{\partial F}{\partial x}\right)\left(\frac{\partial F}{\partial y}\right) \\ -\left(\frac{\partial F}{\partial x}\right)\left(\frac{\partial F}{\partial y}\right) & 1 + \left(\frac{\partial F}{\partial y}\right)^2 \end{pmatrix}}$$

$$= \sqrt{\left(1 + \left(\frac{\partial F}{\partial x}\right)^2\right)\left(1 + \left(\frac{\partial F}{\partial y}\right)^2\right) - \left(\frac{\partial F}{\partial x}\right)^2 \left(\frac{\partial F}{\partial y}\right)^2}$$

$$= \sqrt{1 + \left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial x}\right)^2}$$

$\square$

3. Consider the following subsets of $\mathbb{R}^2$:

$$\begin{aligned} A &= \{x = 0\} \\ B &= \{x = y \mid x \geq 0\} \cup \{x = -y \mid x \leq 0\} \\ C &= \{x = 0\} \cup \{y = 0\} \\ D &= \{(t, t^3) \mid t \in \mathbb{R}\} \end{aligned}$$

(a) (10 pts) Which of the subsets A, B, C and D are C^0 manifolds in $\mathbb{R}^2$? Which are not? You have to explain your answer, but you do not have to give a full proof.

Solution. C is not a C^0 manifold, and the other subsets are. First C is not a C^0 manifold because there is no neighborhood $V \overset{\text{open}}{\subset} D$ of the origin $(0,0) \in C$ that is homeomorphic (meaning that it is continuous and has a continuous inverse) to an open subset of $\mathbb{R}$.

- A is a C^0 manifold with a single coordinate chart $\alpha\colon \mathbb{R} \to A$, $\alpha(x) = (0, x)$. This is clearly a homeomorphism.
- B is a C^0 manifold with a single coordinate chart $\beta\colon \mathbb{R} \to A$, $\beta(x) = (x,|x|)$. This is clearly a homeomorphism.
- D is a C^0 manifold with a single coordinate chart $\gamma\colon \mathbb{R} \to A$, $\gamma(x) = (x, x^3)$. This is clearly a homeomorphism.

□

(b) (10 pts) Which of the subsets A, B, C and D are smooth (C^∞) manifolds in $\mathbb{R}^2$? Which are not? You have to explain your answer, but you do not have to give a full proof.

Solution. The answer is the same as in part (a), except that B is not a smooth manifold; there is no neighborhood of the origin $(0,0) \in B$ that is diffeomorphic to an open subset of $\mathbb{R}$, because of the corner. (The coordinate chart β is only a homeomorphism but not a diffeomorphism. It is not even a C^1 diffeomorphism in fact.) It is clear that α and γ are smooth and hence diffeomorphisms (they define global coordinate charts). □

4. Let $\Omega \in \mathcal{A}^k(V)$ be an alternating k-tensor on a vector space.

(a) (10 pts) If $v_1, \dots, v_k \in V$ are vectors so that there are two distinct indices i and j such that $v_i = v_j$, show that $\Omega(v_1, \dots, v_k) = 0$.

Solution. If $\sigma \in S_k$ is a permutation, then by definition Ω is alternating if $\Omega^\sigma = -\Omega$, where

$$\Omega^\sigma(v_1, \dots, v_k) := \Omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

Now suppose $v_1, \dots, v_k$ is a collection of vectors so that $v_i = v_j$ for two different indices $i \neq j \in \{1, \dots, k\}$. Then let $\sigma_{ij} \in S_k$ be the permutation with $\sigma_{ij}(i) = j$, $\sigma_{ij}(j) = i$ and $\sigma_{ij}(k) = k$ for $k \neq i, j$. Then $\Omega^\sigma(v_1, \dots, v_k) = \Omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \Omega(v_1, \dots, v_k)$. But by definition of being alternating we conclude $\Omega(v_1, \dots, v_k) = -\Omega(v_1, \dots, v_k)$ which implies $\Omega(v_1, \dots, v_k) = 0$. $\square$

(b) (10 pts) Let $W \subset V$ be a subspace of dimension $< k$. Show that $\Omega(w_1, \dots, w_k) = 0$ for any $w_1, \dots, w_k \in W$. [*Hint:* You may use part (a) even if you have not solved it.]

Solution. Since W has dimension $< k$, any tuple of k vectors $w_1, \dots, w_k$ are linearly dependent. Therefore we can for instance write $w_k = \sum_{i=1}^{k-1} c_i w_i$, which gives

$$\Omega(w_1, \dots, w_k) = \Omega\left(w_1, \dots, \sum_{i=1}^{k-1} c_i w_i\right) = \sum_{i=1}^{k-1} c_i \Omega(w_1, \dots, w_{k-1}, w_i).$$

Each term in this sum is zero by part (a). This is because the tuple $(w_1, \dots, w_{k-1}, w_i)$ of k vectors in W always contain at one pair of duplicates. $\square$

5. (a) (5 pts) Show that any open set in $\mathbb{R}^n$ is a smooth n-manifold in $\mathbb{R}^n$.

Solution. Let $U \overset{\text{open}}{\subset} \mathbb{R}^n$. The smooth function $\mathrm{id}\colon U \to U$ is a coordinate chart (meaning a diffeomorphism) at every point $x \in U$, and hence U is an n-manifold in $\mathbb{R}^n$. $\square$

(b) (10 pts) Let $\mathbb{R}^{n\times n}$ denote the set of $n \times n$-matrices; we fix some identification between $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n^2}$. Show that the determinant as a function $\det\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is smooth.

Solution. If $A \in \mathbb{R}^{n\times n}$ is identified with the tuple $(x_1, \ldots, x_{n^2}) \in \mathbb{R}^{n^2}$, then $\det A$ is a polynomial function in the variables $x_1, \ldots, x_{n^2}$, and polynomials are smooth. $\square$

(c) (10 pts) Let $\mathrm{GL}(n;\mathbb{R}) \subset \mathbb{R}^{n\times n}$ denote the set of all invertible $n \times n$-matrices. Show that $\mathrm{GL}(n;\mathbb{R})$ is a smooth n^2-manifold in $\mathbb{R}^{n^2}$.

Solution. By definition a matrix $A \in \mathbb{R}^{n\times n}$ is invertible if and only if $\det A \neq 0$, so

$$\mathrm{GL}(n;\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\} = \det^{-1}(\mathbb{R} \smallsetminus \{0\}).$$

Now $\mathbb{R} \smallsetminus \{0\} \overset{\text{open}}{\subset} \mathbb{R}$, and because det is smooth by part (b) (and therefore continuous). Therefore $\det^{-1}(\mathbb{R} \smallsetminus \{0\}) = \mathrm{GL}(n;\mathbb{R}) \overset{\text{open}}{\subset} \mathbb{R}^{n^2}$ and is hence a smooth n^2-manifold in $\mathbb{R}^{n^2}$ by part (a). $\square$