MAT 322/523 MIDTERM I

WEDNESDAY FEBRUARY 28, 2024
2:30–3:20PM

Name: ______________________________ ID: ______________

Instructions.

(1) Fill in your name and Stony Brook ID number and circle your lecture number at the top of this cover sheet.

(2) This exam is closed-book and closed-notes; no electronic devices.

(3) You have 80 minutes to complete this exam.

(4) You must justify all your answers and show all your work (unless the problem says otherwise). Even a correct answer without any justification will result in no credit.

1. Let A be an $(m\times n)$-matrix with the element a_{ij} at position (i,j). Define $\|A\| := \sqrt{\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2}$. Let $\boldsymbol{x} \in \mathbb{R}^n$.

(a) (10 pts) Show $\|A\boldsymbol{x}\| = \sqrt{\sum_{i=1}^m \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle^2}$, where $\boldsymbol{a}_i \in \mathbb{R}^n$ is the i-th row of A, and where $\langle -, -\rangle$ denotes the standard inner product on $\mathbb{R}^n$, and where $\|-\|$ denotes the standard Euclidean norm on $\mathbb{R}^n$.

Solution. By definition $A\boldsymbol{x}$ is the vector with i-th component equal to $\sum_{j=1}^n a_{ij}x_j$. Therefore $\|A\boldsymbol{x}\| = \sqrt{\sum_{i=1}^m \left(\sum_{j=1}^n a_{ij}x_j\right)^2}$. The i-th row of A is $\boldsymbol{a}_i = (a_{i1} \quad \cdots \quad a_{in})$, and we note that $\sum_{j=1}^n a_{ij}x_j = \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle$ by definition. Therefore $\|A\boldsymbol{x}\| = \sqrt{\sum_{i=1}^m \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle^2}$. □

(b) (10 pts) Show that $\|A\boldsymbol{x}\| \leq \|A\|\|\boldsymbol{x}\|$. [*Hint: Use Cauchy–Schwarz inequality.*]

Solution. By part (a) we have $\|A\boldsymbol{x}\| = \sqrt{\sum_{i=1}^m \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle^2}$. Applying the Cauchy–Schwarz inequality $\langle \boldsymbol{a}_i, \boldsymbol{x}\rangle^2 \leq \|\boldsymbol{a}_i\|^2\|\boldsymbol{x}\|^2$ yields

$$\begin{aligned}\|A\boldsymbol{x}\| &= \sqrt{\sum_{i=1}^m \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle^2} \leq \sqrt{\sum_{i=1}^m \|\boldsymbol{a}_i\|^2\|\boldsymbol{x}\|^2} \\ &= \sqrt{\sum_{i=1}^m \|\boldsymbol{a}_i\|^2}\|\boldsymbol{x}\| = \sqrt{\sum_{i=1}^m \left(\sum_{j=1}^n a_{ij}^2\right)}\|\boldsymbol{x}\| = \|A\|\|\boldsymbol{x}\|\,.\end{aligned}$$

□

2. (a) (5 pts) Write down the definition of what it means for a metric space (X, d) to be connected.

> *Solution.* A metric space X is connected if and only if $X = A \cup B$ for non-empty $A, B \overset{\text{open}}{\subset} X$ implies that $A \cap B \neq \varnothing$. □

(b) (15 pts) Suppose that (X, d) is a connected metric space. Show that if $A \subset X$ is both open and closed, then either $A = \varnothing$ or $A = X$.

> *Solution.* By hypothesis $A \overset{\text{open}}{\subset} X$. Since A is closed in X, it means by definition that $X \setminus A \overset{\text{open}}{\subset} X$. By construction $A \cap (X \setminus A) = \varnothing$. Therefore $X = A \cup (X \setminus A)$ is a separation of X, which by definition means that X is disconnected. This is a contradiction and therefore we must either have $A = \varnothing$ or $X \setminus A = \varnothing$. □

3. (a) (5 pts) Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function. Consider the following hypotheses on f:

(1) f is of class C^1

(2) all directional derivatives of f exist

(3) all partial derivatives of f exist

(4) f is differentiable

(5) f is of class C^∞

Without justification, rank the hypotheses by strength by filling in the blanks:

$$\_\_\_\_ \implies \_\_\_\_ \implies \_\_\_\_ \implies \_\_\_\_ \implies \_\_\_\_$$

Furthermore, which hypotheses implies that f is continuous? No justification needed.

Solution. The correct answer is

$$(5) \implies (1) \implies (4) \implies (2) \implies (3).$$

The hypotheses (4) (and hence (1) and (5)) implies that f is continuous. We have seen an example in the lectures of a discontinuous function f such that all directional derivatives exist. □

(b) (15 pts) A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called *Lipschitz continuous* if there exists a constant $K \geq 0$ such that for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ we have

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leq K\|\boldsymbol{x} - \boldsymbol{y}\|\,.$$

(i) Show that if f is Lipschitz continuous, then f is continuous.

(ii) Show that if f is differentiable such that Df is bounded, then it is Lipschitz continuous [*Hint: Use the mean value theorem and the result of Problem 1(b).*]

Solution. (i) Continuity means that by definition $\forall\varepsilon > 0\ \exists\delta > 0$ such that $\|\boldsymbol{x} - \boldsymbol{y}\| < \delta \Rightarrow |f(\boldsymbol{x}) - f(\boldsymbol{y})| < \varepsilon$. Letting $\varepsilon > 0$ be arbitrary and assuming $\|\boldsymbol{x} - \boldsymbol{y}\| < \delta$ we have by the assumption of Lipschitz continuity that $|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leq K\|\boldsymbol{x} - \boldsymbol{y}\| < K\delta$, which means that we can choose $\delta = \frac{\varepsilon}{K}$ to achieve $|f(\boldsymbol{x}) - f(\boldsymbol{y})| < \varepsilon$.

(ii) The mean value theorem says that if $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, then there is some point $\boldsymbol{c}$ that lies on the straight line connecting $\boldsymbol{x}$ and $\boldsymbol{y}$, such that $f(\boldsymbol{x}) - f(\boldsymbol{y}) = Df(\boldsymbol{c}) \cdot (\boldsymbol{x} - \boldsymbol{y})$. Taking norms on both sides gives

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| = |Df(\boldsymbol{c}) \cdot (\boldsymbol{x} - \boldsymbol{y})| \overset{\text{Problem 1(b)}}{\leq} \|Df(\boldsymbol{c})\|\|\boldsymbol{x} - \boldsymbol{y}\| \leq M\|\boldsymbol{x} - \boldsymbol{y}\|\,,$$

which shows that f is Lipschitz continuous with $K = M$, which is the bound on the derivative of f.

□

4. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(0,y) = 0$ for all $y \in \mathbb{R}$, $f(x,0) = 0$ for all $x \in \mathbb{R}$, and

$$f(x,y) = \frac{x}{y} + \frac{y}{x} \text{ for } xy \neq 0.$$

(a) (5 pts) Compute the partial derivatives $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.

Solution. We use the definition. Namely,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = 0$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = 0,$$

since the numerators are constantly equal to zero. □

(b) (15 pts) Compute the directional derivative $f'((0,0);(1,1))$. Is f differentiable at the origin?

Solution. We use the definition. Namely,

$$f'((0,0);(1,1)) = \lim_{t\to 0} \frac{f(t,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t}{t} + \frac{t}{t}}{t} = \lim_{t\to 0} \frac{2}{t} = \infty.$$

Since this limit does not exist, we conclude that the directional derivative $f'((0,0);(1,1))$ does not exist. It follows that f can not be differentiable at the origin, because differentiability implies existence of every directional derivative. □

5. Let $Q \subset \mathbb{R}^n$ be a rectangle.

(a) (5 pts) For any $\boldsymbol{a} \in Q$, show that $\{\boldsymbol{a}\} \subset Q$ is a bounded set of measure zero.

Solution. By definition $\{\boldsymbol{a}\}$ is bounded because it is contained in a ball of any (finite) radius centered at $\boldsymbol{a}$. Let $\varepsilon > 0$. We cover $\{\boldsymbol{a}\}$ by a single rectangle $Q_\varepsilon := \prod_{j=1}^n \left[a_j - \frac{1}{2}\sqrt[n]{\frac{\varepsilon}{2}}, a_j + \frac{1}{2}\sqrt[n]{\frac{\varepsilon}{2}}\right]$ where a_j is the j-th component of $\boldsymbol{a} \in \mathbb{R}^n$. This rectangle clearly contains $\{\boldsymbol{a}\}$ and it has volume

$$\operatorname{vol} Q_\varepsilon = \prod_{j=1}^n \left(a_j + \frac{1}{2}\sqrt[n]{\frac{\varepsilon}{2}}\right) - \left(a_j - \frac{1}{2}\sqrt[n]{\frac{\varepsilon}{2}}\right) = \prod_{j=1}^n \sqrt[n]{\frac{\varepsilon}{2}} = \frac{\varepsilon}{2} < \varepsilon,$$

for any $\varepsilon > 0$. Consequently $\{\boldsymbol{a}\}$ has measure zero. □

(b) (15 pts) Let $f\colon Q \to \mathbb{R}$ be an integrable function. For any $\boldsymbol{a} \in \mathbb{R}^n$, show that f is integrable over $\{\boldsymbol{a}\}$ and $\int_{\{\boldsymbol{a}\}} f = 0$. [*Hint: Pick a partition of Q that has $\prod_{i=1}^n [a_i - \varepsilon, a_i + \varepsilon]$ as a subrectangle.*]

Solution. By part (a), $\{\boldsymbol{a}\} \subset Q$ is a bounded subset of Q of measure zero. We show that f is integrable (and we compute the integral) by using the definition. We define

$$f_{\{\boldsymbol{a}\}}(\boldsymbol{x}) = \begin{cases} f(\boldsymbol{a}), & \boldsymbol{x} = \boldsymbol{a} \\ 0, & \text{otherwise} \end{cases}.$$

Then by definition $\int_{\{\boldsymbol{a}\}} f = \int_Q f_{\{\boldsymbol{a}\}}$ if it exists. We therefore need to check that $f_{\{\boldsymbol{a}\}}$ is integrable on Q, provided that f is. We will show that for any $\varepsilon > 0$ there exists a partition P of Q such that $U(f, P) - L(f, P) < \varepsilon$. To that end let P be any partition of Q that has Q_ε as in part (a) as a subrectangle. For any subrectangle R determined by P that is different from Q_ε we have $f_{\{\boldsymbol{a}\}}(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in R$ so $m_R(f_{\{\boldsymbol{a}\}}) = M_R(f_{\{\boldsymbol{a}\}})$. For $R = Q_\varepsilon$ we have

$$M_{Q_\varepsilon}(f_{\{\boldsymbol{a}\}}) - m_{Q_\varepsilon}(f_{\{\boldsymbol{a}\}}) = |f(\boldsymbol{a})|,$$

(this we see by looking at two cases: we conclude the above in either of the two cases $f(\boldsymbol{a}) \geq 0$ and $f(\boldsymbol{a}) \leq 0$) and consequently

$$U(f_{\{\boldsymbol{a}\}}, P) - L(f_{\{\boldsymbol{a}\}}, P) = \sum_R (M_R(f_{\{\boldsymbol{a}\}}) - m_R(f_{\{\boldsymbol{a}\}})) \operatorname{vol} R = |f(\boldsymbol{a})| \operatorname{vol} Q_\varepsilon < |f(\boldsymbol{a})|\varepsilon.$$

This shows that f is integrable on $\{\boldsymbol{a}\}$ (more precisely, repeat the above with $Q_{\frac{\varepsilon}{|f(\boldsymbol{a})|}}$).

Finally, because f is integrable we have $\int_{\{\boldsymbol{a}\}} f = \overline{\int}_{\{\boldsymbol{a}\}} f = \underline{\int}_{\{\boldsymbol{a}\}} f$. Let P be a partition as above. We have two cases:

$f(\boldsymbol{a}) \geq 0$: Then $m_R(f_{\{\boldsymbol{a}\}}) = 0$ for any R determined by P (if R does not contain $\boldsymbol{a}$ the value of $f_{\{\boldsymbol{a}\}}$ is constantly equal to zero and if R contains $\boldsymbol{a}$, the minimum of $f_{\{\boldsymbol{a}\}}$ over R is 0 since $f(\boldsymbol{a}) \geq 0$).

In this case we obtain $\int_{\{\boldsymbol{a}\}} f = \underline{\int}_{\{\boldsymbol{a}\}} f = 0$.

$f(\boldsymbol{a}) \leq 0$: Similar to the above we have $M_R(f_{\{\boldsymbol{a}\}}) = 0$ for any R determined by P. Therefore we obtain $\int_{\{\boldsymbol{a}\}} f = \underline{\int}_{\{\boldsymbol{a}\}} f = 0$.

□