MAT 322/523 ANALYSIS IN SEVERAL DIMENSIONS
HOMEWORK 9

DUE: MONDAY APRIL 8, 12:00PM

- Each problem is worth 10 points.
- Submit the homework via Gradescope.
- Only submit problems 2, 3 and 6.

1. (Munkres §28.1) Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w} \in \mathbb{R}^5$. Let

$$F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = 2x_2y_2z_1 + x_1y_5z_4,$$
$$G(\boldsymbol{x}, \boldsymbol{y}) = x_1y_3 + x_3y_1,$$
$$h(\boldsymbol{w}) = w_1 - 2w_3.$$

Recall that the *alternatization* of a k-tensor is given by

$$AF = \sum_{\sigma \in S_k} (-1)^\sigma F.$$

(a) Write AF and AG in terms of elementary alternating tensors. [*Hint:* Write F and G in terms of elementary tensors and use Step 9 of the proof of Theorem 28.1 to compute $A\phi_I$.]
(b) Express $(AF) \wedge h$ in terms of elementary alternating tensors.
(c) Express $(AF)(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ as a function.

2. (Munkres §28.2) If G is symmetric, show that $AG = 0$. Does the converse hold?

3. (Munkres §28.4) Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ be vectors in $\mathbb{R}^n$; let X be the matrix $X = [\boldsymbol{x}_1 \cdots \boldsymbol{x}_k]$. If $I = (i_1, \ldots, i_k)$ is an *arbitrary* k-tuple from the set $\{1, \ldots, n\}$, show that

$$\phi_{i_1} \wedge \cdots \wedge \phi_{i_k}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) = \det X_I.$$

4. (Munkres §29.1) Let $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ be of class C^r. Show that the velocity vector of γ corresponding to the parameter value t is the vector $\gamma_*(t; \boldsymbol{e}_1)$.

5. (Munkres §29.3) Let M be a k-manifold of class C^r in $\mathbb{R}^n$. Let $\boldsymbol{p} \in M$. Show that the tangent space of M at $\boldsymbol{p}$ is well-defined. (Meaning that $T_{\boldsymbol{p}}M$ is independent of the choice of the coordinate chart.)

6. (Munkres §29.4) Let M be a k-manifold in $\mathbb{R}^n$ of class C^r. Let $\boldsymbol{p} \in M - \partial M$.
(a) Show that if $(\boldsymbol{p}, \boldsymbol{v})$ is a tangent vector to M, then there is a parametrized curve $\gamma\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ whose image set lies in M, such that $(\boldsymbol{p}, \boldsymbol{v})$ equals the velocity vector of γ corresponding to parameter value $t = 0$.

(b) Prove the converse. [*Hint:* Recall that for any coordinate chart α, the map α^{-1} is of class C^r.]