MAT 322/523 ANALYSIS IN SEVERAL DIMENSIONS
HOMEWORK 8

DUE: MONDAY APRIL 1, 12:00PM

- Each problem is worth 10 points.
- Submit the homework via Gradescope.
- Only submit problems 1, 3 and 5.

1. (Munkres §25.3(a)) Consider the torus T of Exercise 7 of §17, namely

$$T = \{(r, \theta, z) \mid (r-b)^2 + z^2 = a^2,\ 0 \leq \theta \leq 2\pi\}.$$

Find the area of this torus. [*Hint: The cylindrical coordinate transformation carries a cylinder onto T. Parametrize the cylinder using the fact that its cross-section are circles.*]

2. (Munkres §25.8) Let M and N be compact manifolds without boundary in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively.

(a) Let $f\colon M \to \mathbb{R}$ and $g\colon N \to \mathbb{R}$ be continuous. Show that

$$\int_{M \times N} fg\, dV = \left(\int_M f\, dV\right)\left(\int_N g\, dV\right).$$

[*Hint: Consider the case where the supports of f and g are contained in coordinate patches.*]

(b) Show that $\operatorname{vol}(M \times N) = \operatorname{vol}(M)\operatorname{vol}(N)$.

(c) Find the area of the 2-manifold $S^1 \times S^1$ in $\mathbb{R}^4$.

3. (Munkres §26.2)

(a) Check that if f and g are multilinear, so is $f \otimes g$.

(b) Check the basic properties of the tensor product (Theorem 26.4). Namely

(i) $f \otimes (g \otimes h) = (f \otimes g) \otimes h$

(ii) $(cf) \otimes g = c(f \otimes g) = f \otimes (cg)$

(iii) Suppose f and g have the same order. Then:

$$(f+g) \otimes h = f \otimes h + g \otimes h$$
$$h \otimes (f+g) = h \otimes f + h \otimes g$$

(iv) Given a basis $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ for V, the corresponding elementary tensors ϕ_I satisfy the equation

$$\phi_I = \phi_{i_1} \otimes \phi_{i_2} \otimes \cdots \otimes \phi_{i_k},$$

where $I = (i_1, \dots, i_k)$.

4. (Munkres §26.7) Show that the four properties stated in Theorem 26.4 characterize the tensor product uniquely, for finite-dimensional vector spaces V.

5. (Munkres §27.1) Which of the following are alternating tensors in $\mathbb{R}^4$?

$$f(\boldsymbol{x}, \boldsymbol{y}) = x_1 y_2 - x_2 y_1 + x_1 y_1$$
$$g(\boldsymbol{x}, \boldsymbol{y}) = x_1 y_3 - x_3 y_2$$
$$h(\boldsymbol{x}, \boldsymbol{y}) = (x_1)^3 (y_2)^3 - (x_2)^3 (y_1)^3.$$

6. (Munkres §27.2) Let $\sigma \in S_6$ be the permutation such that

$$(\sigma(1), \sigma(2), \sigma(3), \sigma(4), \sigma(5)) = (3, 1, 4, 5, 2).$$

Use the procedure given in the proof of Lemma 27.1 to write σ as a composite of elementary permutations.