MAT 322/523 ANALYSIS IN SEVERAL DIMENSIONS
HOMEWORK 7

DUE: MONDAY MARCH 25, 12:00PM

- Each problem is worth 10 points.
- Submit the homework via Gradescope.
- Only submit problems 2, 3 and 4.

1. (Munkres §23.1) Let $\alpha\colon \mathbb{R} \to \mathbb{R}^2$ be the map $\alpha(x) = (x, x^2)$; let M be the image set of α. Show that M is a 1-manifold in $\mathbb{R}^2$ covered by a single coordinate patch α.

2. (Munkres §23.3)
(a) Show that the unit circle S^1 is a 1-manifold in $\mathbb{R}^2$.
(b) Show that the function $\alpha\colon [0,1) \to S^1$ given by

$$\alpha(t) = (\cos 2\pi t, \sin 2\pi t)$$

is not a coordinate patch on S^1.

3. (Munkres §24.4) Show that the upper hemisphere of $S^{n-1}(a)$, defined by the equation

$$E_+^{n-1}(a) = S^{n-1}(a) \cap \mathbb{H}^n,$$

is an $(n-1)$-manifold. What is its boundary?

4. (Munkres §24.5) Let $O(3)$ denote the set of all orthogonal 3×3 matrices, considered as a subspace of $\mathbb{R}^9$.
(a) Define a C^∞ function $f\colon \mathbb{R}^9 \to \mathbb{R}^6$ such that $O(3)$ is the solution set of the equation $f(\boldsymbol{x}) = \mathbf{0}$.
(b) Show that $O(3)$ is a compact 3-manifold in $\mathbb{R}^9$ without boundary. [*Hint: Show the rows of $Df(\boldsymbol{x})$ are independent if $\boldsymbol{x} \in O(3)$.*]

5. (Munkres §21.1) Let

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}.$$

(a) Find $X^T \cdot X$.
(b) Find $V(X)$.

6. (Munkres §21.2) Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k$ be vectors in $\mathbb{R}^n$. Show that

$$V(\boldsymbol{x}_1, \dots, \lambda \boldsymbol{x}_i, \dots, \boldsymbol{x}_k) = |\lambda| V(\boldsymbol{x}_1, \dots, \boldsymbol{x}_k).$$