MAT 322/523 ANALYSIS IN SEVERAL DIMENSIONS
HOMEWORK 6

DUE: MONDAY MARCH 11, 12:00PM

- Each problem is worth 10 points.
- Submit the homework via Gradescope.
- Only submit problems 1 and 4.

1. (Munkres §16.1) Prove that the function f defined by $f(x) = e^{-1/x}$ if $x > 0$ and $f(x) = 0$ otherwise, is of class C^∞ as follows: Given any integer $n \geq 0$, define $f_n \colon \mathbb{R} \to \mathbb{R}$ by the equation

$$f_n(x) = \begin{cases} \left(e^{-1/x}\right)/x^n, & \text{for } x > 0, \\ 0, & \text{for } x \leq 0. \end{cases}$$

(a) Show that f_n is continuous at 0 [*Hint: Show that* $a < e^a$ *for all* a. *Then set* $a = t/2n$ *to conclude that*

$$\frac{t^n}{e^t} < \frac{(2n)^n}{e^{t/2}}.$$

Set $t = 1/x$ *and let* x *approach* 0 *through positive values.*]

(b) Show that f_n is differentiable at 0.

(c) Show that $f_n'(x) = f_{n+2}(x) - nf_{n+1}(x)$ for all x.

(d) Show that f_n is of class C^∞.

2. (Munkres §16.3)

(a) Let S be an arbitrary subset of $\mathbb{R}^n$; let $\boldsymbol{x}_0 \in S$. We say that the function $f \colon S \to \mathbb{R}$ is **differentiable** at $\boldsymbol{x}_0$, of class C^r, provided there is a C^r function $g \colon U \to \mathbb{R}$ defined in a neighborhood U of $\boldsymbol{x}_0$ in $\mathbb{R}^n$, such that g agrees with f on the set $U \cap S$. In this case, show that if $\phi \colon \mathbb{R}^n \to \mathbb{R}$ is a C^r function whose support lies in U, then the function

$$h(\boldsymbol{x}) = \begin{cases} \phi(\boldsymbol{x})g(\boldsymbol{x}), & \text{for } \boldsymbol{x} \in U, \\ 0, & \text{for } \boldsymbol{x} \notin \operatorname{Support} \phi, \end{cases}$$

is well-defined and of class C^r on $\mathbb{R}^n$.

(b) Prove the following:

Theorem. *If* $f \colon S \to R$ *and* f *is differentiable of class* C^r *at each point* $\boldsymbol{x}_0$ *of* S, *then* f *may be extended to a* C^r *function* $h \colon A \to \mathbb{R}$ *that is defined on an open set* A *of* $\mathbb{R}^n$ *containing* S.

[*Hint: Cover* S *by appropriately chosen neighborhoods, let* A *be their union, and take a* C^∞ *partition of unity on* A *dominated by this collection of neighborhoods.*]

3. (Munkres §17.3) Let U be the open set in $\mathbb{R}^2$ consisting of all $\boldsymbol{x}$ with $\|\boldsymbol{x}\| < 1$. Let $f(x, y) = 1/(x^2 + y^2)$ for $(x, y) \neq \mathbf{0}$. Determine whether f is integrable over $U - \mathbf{0}$ and over $\mathbb{R}^2 - \overline{U}$; if so, evaluate.

4. (Munkres §17.4)

(a) Show that
$$\int_{\mathbb{R}^2} e^{-(x^2+y^2)} = \left[\int_{\mathbb{R}} e^{-x^2}\right]^2,$$
provided the first of these integrals exists.

(b) Show the first of these integrals exists and evaluate it.

5. (Munkres §17.5) Let B the portion of the first quadrant in $\mathbb{R}^2$ lying between the hyperbolas $xy = 1$ and $xy = 2$ and the two straight lines $y = x$ and $y = 4x$. Evaluate $\int_B x^2y^3$. [*Hint: Set* $x = u/v$ *and* $y = uv$.]