MAT 322/523 ANALYSIS IN SEVERAL DIMENSIONS
HOMEWORK 5

DUE: MONDAY FEBRUARY 26, 12:00PM

- Each problem is worth 10 points.
- Submit the homework via Gradescope.
- Only submit problems 3 and 7.

1. (Munkres §11.1) Show that if A has measure zero in $\mathbb{R}^n$, the sets $\overline{A}$ and ∂A need not have measure zero.

2. (Munkres §11.6) Let $f\colon [a,b] \to \mathbb{R}$. The graph of f is the subset

$$G_f = \{(x,y) \mid y = f(x)\}$$

of $\mathbb{R}^2$. Show that f is continuous, G_f has measure zero in $\mathbb{R}^2$. [*Hint: Use uniform continuity of f.*]

3. (Munkres §11.9) Let Q be a rectangle in $\mathbb{R}^n$; let $f\colon Q \to \mathbb{R}$; assume f is integrable over Q.

(a) Show that if $f(\boldsymbol{x}) \geq 0$ for $\boldsymbol{x} \in Q$, then $\int_Q f \geq 0$.

(b) Show that if $f(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \in Q$, then $\int_Q f > 0$.

4. (Munkres §12.2) Let $I = [0,1]$; let $Q = I \times I$. Define $f\colon Q \to \mathbb{R}$ by letting $f(x,y) = 1/q$ if y is rational and $x = p/q$, where p and q are positive integers with no common factor; let $f(x,y) = 0$ otherwise.

(a) Show that $\int_Q f$ exists.

(b) Compute

$$\underline{\int}_{y\in I} f(x,y) \quad \text{and} \quad \overline{\int}_{y\in I} f(x,y).$$

(c) Verify Fubini's theorem.

5. (Munkres §12.3) Let $Q = A \times B$, where A is a rectangle in $\mathbb{R}^k$ and B is a rectangle in $\mathbb{R}^n$. Let $f\colon Q \to \mathbb{R}$ be a bounded function.

(a) Let g be a function such that

$$\underline{\int}_{y\in B} f(\boldsymbol{x},\boldsymbol{y}) \leq g(\boldsymbol{x}) \leq \overline{\int}_{y\in B} f(\boldsymbol{x},\boldsymbol{y})$$

for all $\boldsymbol{x} \in A$. Show that if f is integrable over Q, then g is integrable over A, and $\int_Q f = \int_A g$. [*Hint: Use Exercise 1 of §10.*]

(b) Give an example where $\int_Q f$ exists and one of the iterated integrals

$$\int_{\boldsymbol{x}\in A} \int_{\boldsymbol{y}\in B} f(\boldsymbol{x},\boldsymbol{y}) \quad \text{and} \quad \int_{\boldsymbol{y}\in B} \int_{\boldsymbol{x}\in A} f(\boldsymbol{x},\boldsymbol{y})$$

exists, but the other one does not.

*(c) Find an example where both the iterated integrals of (b) exist, but the integral $\int_Q f$ does not. [*Hint: One approach is to find a subset S of Q whose closure equals Q, such that S contains at most one point on each vertical line and at most one point on each horizontal line.*]

6. (Munkres §13.2) Let A be a rectangle in $\mathbb{R}^k$; let B be a rectangle in $\mathbb{R}^n$; let $Q = A \times B$. Let $f\colon Q \to \mathbb{R}$ be a bounded function. Show that if $\int_Q f$ exists, then

$$\int_{\boldsymbol{y}\in B} f(\boldsymbol{x}, \boldsymbol{y})$$

exists for $\boldsymbol{x} \in A - D$ where D is a set of measure zero in $\mathbb{R}^k$.

7. (Munkres §13.4) Let S_1 and S_2 be bounded sets in $\mathbb{R}^n$; let $f\colon S \to \mathbb{R}$ be a bounded function. Show that if f is integrable over S_1 and S_2, then f is integrable over $S_1 - S_2$ and

$$\int_{S_1 - S_2} f = \int_{S_1} f - \int_{S_1 \cap S_2} f.$$

***8.** (Munkres §13.7) Prove the following:

Theorem. *Let S be a bounded set in $\mathbb{R}^n$; let $f\colon S \to \mathbb{R}$ be a bounded function. Let D be the set of points of S at which f fails to be continuous. Let E be the set of points of ∂S at which the condition*

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0} f(\boldsymbol{x}) = 0$$

fails to hold. Then $\int_S f$ exists if and only if D and E have measure zero.

Proof. (a) Show that f_S is continuous at each point $\boldsymbol{x}_0 \notin D \cup E$.

(b) Let B be the set of isolated points of S; then $B \subset E$ because the limit cannot be defined if $\boldsymbol{x}_0$ is not a limit point of S. Show that if f_S is continuous at $\boldsymbol{x}_0$, then $\boldsymbol{x}_0 \notin D \cup (E - B)$.

(c) Show that B is countable.

(d) Complete the proof.

□