MAT 322/523 ANALYSIS IN SEVERAL DIMENSIONS
HOMEWORK 4

DUE: MONDAY FEBRUARY 19, 12:00PM

- Each problem is worth 10 points.
- Submit the homework via Gradescope.
- Only submit problems 1, 5 and 7.

1. (Munkres §8.1) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the equation

$$f(x,y) = (x^2 - y^2, 2xy).$$

(a) Show that f is one-to-one on the set A consisting of all (x,y) with $x > 0$. [*Hint: If* $f(x,y) = f(a,b)$, *then* $\|f(x,y)\| = \|f(a,b)\|$.]
(b) What is the set $B = f(A)$?
(c) If g is the inverse function, find $Dg(0,1)$.

2. (Munkres §8.5) Let A be open in $\mathbb{R}^n$; let $f\colon A \to \mathbb{R}^n$ be of class C^r; assume $Df(\boldsymbol{x})$ is non-singular for $\boldsymbol{x} \in A$. Show that even if f is not one-to-one on A, the set $B = f(A)$ is open in $\mathbb{R}^n$.

3. (Munkres §9.1) Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be of class C^1; write f in the form $f(x, y_1, y_2)$. Assume that $f(3,-1,2) = \mathbf{0}$, and

$$Df(3,-1,2) = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

(a) Show that there is a function $g\colon B \to \mathbb{R}^2$ of class C^1 defined on an open set B in $\mathbb{R}$ such that

$$f(x, g_1(x), g_2(x)) = 0$$

for $x \in B$ and $g(3) \in (-1,2)$.
(b) Find $Dg(3)$.
(c) Discuss the problem of solving the equation $f(x,y_1,y_2) = \mathbf{0}$ for an arbitrary pair of the unknowns in the terms of the third, near the point $(3,-1,2)$.

4. (Munkres §9.4) Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be of class C^2, with $F(0,0) = 0$ and $DF(0,0) = [2 \quad 3]$. Let $G\colon \mathbb{R}^3 \to \mathbb{R}$ be defined by the equation

$$G(x,y,z) = F(x + 2y + 3z - 1, x^3 + y^2 - z^2).$$

(a) Note that $G(-2,3,-1) = F(0,0) = 0$. Show that one can solve the equation $G(x,y,z) = 0$ for z, say $z = g(x,y)$, for (x,y) in a neighborhood B of $(-2,3)$, such that $g(-2,3) = -1$.
(b) Find $Dg(-2,3)$.
*(c) If $D_1D_1F = 0$ and $D_1D_2F = -1$ and $D_2D_2F = 5$ at $(0,0)$, find $D_2D_1g(-2,3)$.

5. (Munkres §9.6) Let $f\colon \mathbb{R}^{k+n} \to \mathbb{R}^n$ be of class C^1; suppose that $f(\boldsymbol{a}) = \boldsymbol{0}$ and that $Df(\boldsymbol{a})$ has rank n. Show that if $\boldsymbol{c}$ is a point of $\mathbb{R}^n$ sufficiently close to $\boldsymbol{0}$, then the equation $f(\boldsymbol{x}) = \boldsymbol{c}$ has a solution.

6. (Munkres §10.1) Let $f, g\colon Q \to \mathbb{R}$ be bounded functions such that $f(\boldsymbol{x}) \leq g(\boldsymbol{x})$ for $\boldsymbol{x} \in Q$. Show that $\underline{\int}_Q f \leq \underline{\int}_Q g$ and $\overline{\int}_Q f \leq \overline{\int}_Q g$.

7. (Munkres §10.5) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) = 1/q$ if $x = p/q$, where p and q are positive integers with no common factor, and $f(x) = 0$ otherwise. Show f is integrable over $[0, 1]$.