MAT 322/523 ANALYSIS IN SEVERAL DIMENSIONS
HOMEWORK 2

DUE: MONDAY FEBRUARY 5, 12:00PM

- Each problem is worth 10 points.
- Submit the homework via Gradescope.
- Only submit problems 1, 2 and 3.

1. (Munkres §4.3) Let $\mathbb{R}^\infty$ be the set of all "infinite-tuples" $\boldsymbol{x} = (x_1, x_2, \ldots)$ of real numbers that end in an infinite string of 0's. (See the exercises of §1.) Define an inner product on $\mathbb{R}^\infty$ by the rule $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{\infty} x_i y_i$. (This is a finite sum, since all but finitely many terms vanish.) Let $\|\boldsymbol{x} - \boldsymbol{y}\|$ be the corresponding metric on $\mathbb{R}^\infty$ (where $\|\boldsymbol{z}\|$ is defined as $\sqrt{\langle \boldsymbol{z}, \boldsymbol{z} \rangle}$). Define

$$\boldsymbol{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0, \ldots),$$

where 1 appear in the i-th place. Then the $\boldsymbol{e}_i$ form a basis for $\mathbb{R}^\infty$. Let X be the set of all points $\boldsymbol{e}_i$.

(a) Show that X is bounded in $\mathbb{R}^\infty$.
(b) Show that X is closed in $\mathbb{R}^\infty$. [*Hint:* How would you prove that a single point in $\mathbb{R}^2$ is closed?]
(c) Show that X is non-compact in $\mathbb{R}^\infty$.

2. (Munkres §5.1) Let $A \subset \mathbb{R}^m$; let $f\colon A \to \mathbb{R}^n$. Show that if $f'(\boldsymbol{a}; \boldsymbol{u})$ exists, then $f'(\boldsymbol{a}; c\boldsymbol{u})$ exists and equals $cf'(\boldsymbol{a}; \boldsymbol{u})$.

3. (Munkres §5.2) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by setting $f(\mathbf{0}) = 0$ and

$$f(x, y) = \frac{xy}{x^2 + y^2}, \quad \text{if } (x, y) \neq \mathbf{0}.$$

(a) For which vectors $\boldsymbol{u} \neq \mathbf{0}$ does $f'(\mathbf{0}; \boldsymbol{u})$ exist? Evaluate it when it exists.
(b) Do $D_1 f$ and $D_2 f$ exist at $\mathbf{0}$?
(c) Is f differentiable at $\mathbf{0}$?
(d) Is f continuous at $\mathbf{0}$?

4. (Munkres §5.3) Repeat Problem 3 (Munkres §5.2) for the function f defined by setting $f(\mathbf{0}) = 0$ and

$$f(x, y) = \frac{x^2 y^2}{x^2 y^2 + (y - x)^2}, \quad \text{if } (x, y) \neq \mathbf{0}.$$

5. (Munkres §5.4) Repeat Problem 3 (Munkres §5.2) for the function f defined by setting $f(\mathbf{0}) = 0$ and

$$f(x, y) = \frac{x^3}{x^2 + y^2}, \quad \text{if } (x, y) \neq \mathbf{0}.$$