MAT 322/523 ANALYSIS IN SEVERAL DIMENSIONS
HOMEWORK 10

DUE: MONDAY APRIL 22, 12:00PM

- Each problem is worth 10 points.
- Submit the homework via Gradescope.
- Only submit problems 1, 3 and 4.

1. (Munkres §30.2) Consider the forms

$$\omega = xydx + 3dy - yzdz$$
$$\eta = xdx - yz^2dy + 2xdz$$

in $\mathbb{R}^3$. Verify by direct computation that

$$d(d\omega) = 0 \quad \text{and} \quad d(\omega \wedge \eta) = (d\omega) \wedge \eta - \omega \wedge d\eta.$$

2. (Munkres §30.5) Prove the following:

Theorem. *Let $A = \mathbb{R}^2 \setminus \{0\}$; let*

$$\omega = \frac{-ydx + xdy}{x^2 + y^2}$$

in A. Then ω is closed but not exact, in A.

Proof. (a) Show ω is closed.

(b) Let B consist of $\mathbb{R}^2$ with the non-negative x-axis deleted. Show that for each $(x, y) \in B$, there is a unique $0 < t < 2\pi$ such that

$$x = \sqrt{x^2 + y^2}\cos t \quad \text{and} \quad y = \sqrt{x^2 + y^2}\sin t;$$

denote this value of t by $\phi(x, y)$.

(c) Show that ϕ is of class C^∞. [*Hint:* The inverse sine and inverse cosine functions are of class C^∞ on the intervals $(-\pi/2, \pi/2)$ and $(0, \pi)$, respectively.]

(d) Show that $\omega = d\phi$ in B. [*Hint:* We have $\tan\phi = y/x$ if $x \neq 0$ and $\cot\phi = x/y$ if $y \neq 0$.]

(e) Show that if g is a closed 0-form in B, then g is constant in B. [*Hint:* Use the mean-value theorem to show that if $\boldsymbol{a}$ is the point $(-1, 0)$ of $\mathbb{R}^2$, then $g(\boldsymbol{x}) = g(\boldsymbol{a})$ for all $\boldsymbol{x} \in B$.]

(f) Show that ω is not exact in A. [*Hint:* If $\omega = df$ in A, then $f - \phi$ is constant in B. Evaluate the limit of $f(1, y)$ as y approaches 0 through positive and negative values.] □

3. (Munkres §31.3) Let A be an open set in $\mathbb{R}^3$.

(a) Translate the equation $d(d\omega) = 0$ into two theorems about vector and scalar fields in $\mathbb{R}^3$.

(b) A is called *homologically trivial of dimension* k if every closed k on A is exact on A. Translate the condition that A is homologically trivial in dimension k into a statement about vector and scalar fields in A. Consider the cases $k = 0, 1, 2$.

4. (Munkres §32.3) In $\mathbb{R}^3$, let

$$\omega = xydx + 2zdy - ydz.$$

Let $\alpha\colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by the equation

$$\alpha(u,v) = (uv, u^2, 3u+v).$$

Calculate $d\omega$ and $\alpha^*\omega$ and $d(\alpha^*\omega)$ directly.