MAT 322/523 ANALYSIS IN SEVERAL DIMENSIONS
HOMEWORK 1

DUE: MONDAY JANUARY 29, 12:00PM

- Each problem is worth 10 points.
- Submit the homework via Gradescope.
- Only submit problems 2, 5 and 6.

1. (Munkres §1.1) Let V be a vector space with inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ and norm $||x|| = \langle \boldsymbol{x}, \boldsymbol{x} \rangle^{1/2}$.

(a) Prove the Cauchy–Schwarz inequality $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq \|\boldsymbol{x}\| \|\boldsymbol{y}\|$. [*Hint:* If $\boldsymbol{x}, \boldsymbol{y} \neq \boldsymbol{0}$, set $c = 1/\|\boldsymbol{x}\|$ and $d = 1/\|\boldsymbol{y}\|$ and use the fact that $\|c\boldsymbol{x} \pm d\boldsymbol{y}\| \geq 0$.]

(b) Prove that $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$. [*Hint:* Compute $\langle \boldsymbol{x} + \boldsymbol{y}, \boldsymbol{x} + \boldsymbol{y} \rangle$ and apply (a).]

(c) Prove that $\|\boldsymbol{x} - \boldsymbol{y}\| \geq \|\boldsymbol{x}\| - \|\boldsymbol{y}\|$.

2. (Munkres §1.4)

(a) If $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$, show that the function

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

is an inner product on $\mathbb{R}^2$.

(b) Show that the function

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

is an inner product on $\mathbb{R}^2$ if and only if $b^2 - ac < 0$ and $a > 0$.

3. (Munkres §2.1) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

(a) Find two different left inverses for A.

(b) Show that A has no right inverse.

4. (Munkres §2.4) Let A be an n by m matrix with $n \neq m$.

(a) If $\operatorname{rank} A = m$, show there exists a matrix D that is a product of elementary matrices such that

$$D \cdot A = \begin{bmatrix} I_m \\ 0 \end{bmatrix}.$$

(b) Show that A has a left inverse if and only if $\operatorname{rank} A = m$.

(c) Show that A has a right inverse if and only if $\operatorname{rank} A = n$.

5. (Munkres §3.2) Let $Y \subset X$. Give an example where A is open in Y but not open in X. Give an example where A is closed in Y but not closed in X.

6. (Munkres §3.6) Let $X = A \cup B$, where A and B are subspaces of X. Let $f\colon X \to Y$; suppose that the restricted functions

$$f|_A \colon A \longrightarrow Y \quad \text{and} \quad f|_B \colon B \longrightarrow Y$$

are continuous. Show that if A and B are closed in X, then f is continuous.

7. (Munkres §3.8) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) = \sin x$ if x is rational, and $f(x) = 0$ otherwise. At what points is f continuous?