MAT 322/523 Lec 9 M Feb 19

<u>Recall</u>: $Q = \prod_{i=1}^{n}[a_i, b_i]$ rectangle

- $\underline{\int}_Q f = \sup_{P \text{ partition}} L(f,P) =$

$$= \sup_P \sum_{R \text{ subrectangle}} m_R(f)\,\mathrm{vol}(R)$$

$\inf_R f$ $\sup_R f$

- $\overline{\int}_Q f = \inf_P \sum_R M_R(f)\,\mathrm{vol}(R)$

- f integrable if $\underline{\int}_Q f = \overline{\int}_Q f$.

<u>Def</u>: Let $A \subset \mathbb{R}^n$. We say that A has <u>measure zero</u> if $\forall \varepsilon > 0$ if there are rectangles $\{Q_i\}_{i=1}^{\infty}$ such that $A \subset \bigcup_{i=1}^{\infty} Q_i$ and
$$\sum_{i=1}^{\infty} \mathrm{vol}(Q_i) < \varepsilon.$$

If A has measure zero we will write $\mu(A)=0$.

<u>Lma</u>: (a) If $B \subset A$, then
$\mu(A)=0 \Rightarrow \mu(B)=0$.

(b) If $A=\bigcup_{i=1}^{\infty} A_i$ and $\mu(A_i)=0$
$\forall i \in \mathbb{Z}_{\geq 1}$ then $\mu(A)=0$.

(c) $\mu(A)=0 \Leftrightarrow \forall \varepsilon>0$ there are <u>open</u> rectangles $\{\mathring{Q}_i\}_{i=1}^{\infty}$
: $A \subset \bigcup_{i=1}^{\infty} \mathring{Q}_i$ and $\sum_{i=1}^{\infty} \mathrm{Vol}(\mathring{Q}_i) < \varepsilon$.

(d) If Q is a rectangle, then
$\mu(\partial Q)=0$ but $\mu(Q) \neq 0$

(Q does not have measure zero)

<u>Proof</u>: (a) Clear from def.

(b) Since $\mu(A_i) = 0$, then $\forall i$

$\forall \varepsilon_i > 0$: $\{Q_{ij}\}_{j=1}^{\infty}$

$$A_i \subset \bigcup_{j=1}^{\infty} Q_{ij}$$

$$\sum_{j=1}^{\infty} \mathrm{Vol}(Q_{ij}) < \varepsilon_i .$$

so $A \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} Q_{ij}$. Let $\varepsilon > 0$

be arbitrary. Then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathrm{Vol}(Q_{ij}) < \sum_{i=1}^{\infty} \varepsilon_i \quad (*)$$

Since each $\varepsilon_i > 0$ is arbitrary we can assume $\varepsilon_i = \frac{\varepsilon}{2^i}$. Then

$(*)$ becomes

$$\sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon \sum_{i=1}^{\infty} \frac{1}{2^i} = \varepsilon \left(1 - \frac{1}{1-\frac{1}{2}}\right)$$

$$= \varepsilon$$

so $\sum_{i,j} Q_{ij} < \varepsilon$. This shows $\mu(A) < \varepsilon$.

$\square$

<u>Thm</u>: Let Q be a rectangle in $\mathbb{R}^n$, and let $f: Q \to \mathbb{R}$ be bounded. Let $D = \{x \in Q \mid f \text{ not continuous}\}$. Then $\int_Q f$ exists $\Leftrightarrow \mu(D) = 0$.

<u>Proof</u>: f bounded so, choose $M > 0$: $|f(x)| \leq M \ \forall x \in Q$.

$\boxed{\Leftarrow}$ Assume $\mu(D) = 0$. We will show that $\forall \varepsilon > 0 \ \exists$ partition P of Q :

$U(f,P) - L(f,P) < \varepsilon$.

To that end, let $\varepsilon > 0$.

By the assumption that $\mu(D) = 0$ we can cover D w/ open rects $\{\mathring{Q}_i\}_{i=1}^{\infty}$, $D \subset \bigcup_{i=1}^{\infty} \mathring{Q}_i$ so that

$$\sum_{i=1}^{\infty} \mathrm{vol}(\mathring{Q}_i) < \varepsilon.$$

For all $\vec{a} \in Q \setminus D$, we can choose an open rectangle $\mathring{Q}_a \ni \vec{a}$ so that

$$|f(\vec{x}) - f(\vec{a})| < \varepsilon \quad \forall \vec{x} \in Q_a \cap Q$$

(by continuity of f at a).

Then $\{\mathring{Q}_i\}_{i=1}^{\infty} \cup \{\mathring{Q}_a\}_{a \in Q \setminus D}$ is an open cover of Q.

Compactness of Q means that we can find a finite subcover.

$\{\mathring{Q}_{i_j}\}_{j=1}^{k} \cup \{\mathring{Q}_{a_j}\}_{j=1}^{\ell}$ that still cover Q.

Note that $\forall \vec{x}, \vec{y} \in Q_{a_j} \cap Q$

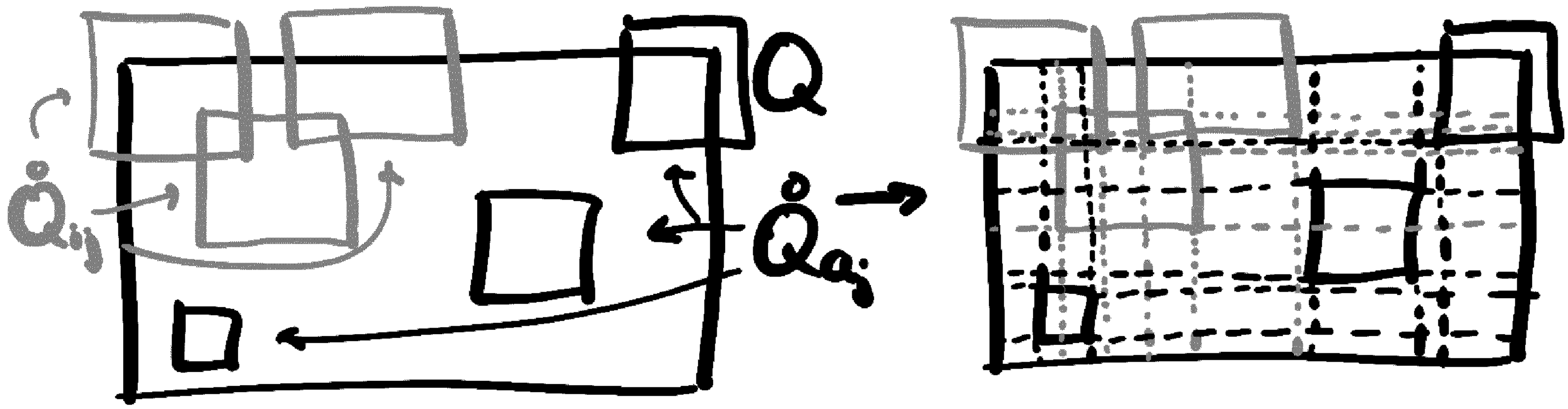
$$|f(\vec{x}) - f(\vec{y})| = |f(\vec{x}) - f(\vec{a}) + f(\vec{a}) - f(\vec{y})|$$

$$\leq \underbrace{|f(\vec{x}) - f(\vec{a})|}_{<\varepsilon} + \underbrace{|f(\vec{a}) - f(\vec{y})|}_{<\varepsilon} < 2\varepsilon.$$

Let $\begin{cases} Q_j^a := \mathring{Q}_{a_j} \cap Q & \forall j \in \{1, \dots, \ell\} \\ Q_j^D := \mathring{Q}_{i_j} \cap Q & \forall j \in \{1, \dots, k\} \end{cases}$

Each Q_j^a, Q_j^D is a rectangle that is a subset of Q.

the endpoints of the component intervals of Q_j^a and Q_j^D determine a partition of Q:



By construction each Q_j^a and Q_j^D is a union of subrectangles of P.

$\overline{R} = \{\text{subrectangles } R \text{ of } P\}$.

Write $\overline{R} = \overline{R}^D \cup \overline{R}^a$

$$\begin{cases} R^D = \{R \in \overline{R} \mid R \subset Q_j^D \text{ for some } j\} \\ R^a = \{R \in \overline{R} \mid R \subset Q_j^a \text{ for some } j\}. \end{cases}$$

Then because

$$|f(\vec{x}) - f(\vec{y})| \le |f(\vec{x})| + |f(\vec{y})| \le 2M$$
$$\forall \vec{x}, \vec{y} \in Q$$

we have

$$M_R(f) - m_R(f) \le 2M$$ for any subrect R of P.

Similarly, $|f(\vec{x})-f(\vec{y})| \leq 2\varepsilon'$
$\forall \vec{x},\vec{y} \in Q \setminus D$

implies $M_R(f)-m_R(f) \leq 2\varepsilon'$ for any subrectangle $R \in \bar{R}^a$.

Therefore

$$U(f,P)-L(f,P) = \sum_R (M_R(f)-m_R(f))\,\mathrm{vol}\,R$$

$$= \sum_{R\in \bar{R}^D} (M_R(f)-m_R(f))\,\mathrm{vol}(R)$$

$$+ \sum_{R\in \bar{R}^a} (M_R(f)-m_R(f))\,\mathrm{vol}(R)$$

$$\leq 2M \sum_{R\in\bar{R}^D} \mathrm{vol}(R) + 2\varepsilon \underbrace{\sum_{R\in\bar{R}^a} \mathrm{vol}(R)}_{\leq \mathrm{vol}(Q)}$$

$$\leq 2M \sum_{j=1}^{k} \underbrace{\sum_{R\subset Q_i^D} \mathrm{vol}(R)}_{\leq \mathrm{vol}(Q_i^D)} + 2\varepsilon\,\mathrm{vol}(Q)$$

$$\leq 2M \underbrace{\sum_{j=1}^{k} \mathrm{Vol}(Q_i^D)}_{<\varepsilon} + 2\varepsilon\, \mathrm{Vol}(Q)$$

$$< 2\varepsilon\,(M + \mathrm{Vol}(Q))$$

can be made arbitrarily small by choosing $\varepsilon > 0$ small enough.

This shows

$\mu(D) = 0 \Rightarrow \int_Q f$ exists.

$\boxed{\Longrightarrow}$ see the book

$\square$

__Def__: If a function $f: A \to \mathbb{R}^n$ satisfies a property on all of A except on a set of measure zero, we say that the property holds "almost everywhere" in A.

(a.e.)

Thm: Let Q be a rectangle and let $f: Q \to \mathbb{R}$ be integrable.

(a) If $f=0$ a.e. then $\int_Q f = 0$

(b) If $f \geq 0$ and $\int_Q f = 0$, then $f = 0$ a.e.

Proof: (a) Let P be a partition of Q. Let $E = \{\vec{x} \in Q \mid f(\vec{x}) \neq 0\}$. We know $\mu(E) = 0$. If R is a subrect determined by P then $R \not\subset E$. meaning $\exists \vec{x} \in R : f(\vec{x}) = 0$.

$$\Rightarrow \begin{cases} m_R(f) \leq 0 \\ M_R(f) \geq 0 \end{cases} \Rightarrow \begin{cases} L(f,P) \leq 0 \\ U(f,P) \geq 0 \end{cases}$$

$$\Rightarrow \underline{\int}_Q f \leq 0 \text{ and } \overline{\int}_Q f \geq 0.$$

$\int_Q f$ exists by hypothesis so

$$\int_Q f = \underline{\int}_Q f = \overline{\int}_Q f \Rightarrow \int_Q f = 0$$