MAT 322/523 Lec 8 W Feb 14

Integral over a rectangle

$\boxed{\text{§10 Munkres}}$

Recall from single variable calc that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

Riemann sum

$$a = x_0 < x_1 < \cdots < x_n = b$$

$$\Delta x_i = x_i - x_{i-1}$$

$$x_i^* \in [x_{i-1}, x_i]$$

Will try to generalize this to any dimension.

<u>Def</u>: A <u>rectangle</u> is a product of closed intervals

$$Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

Its <u>volume</u> is

$$\mathrm{Vol}(Q) = \prod_{i=1}^{n} (b_i - a_i).$$

<u>Def</u>: A <u>partition</u> of $[a,b] \subset \mathbb{R}$ is a member of the set

$$P = \{(t_0, \dots, t_k) \in [a,b] \mid a = t_0 < t_1 < \dots < t_k = b\}$$

A <u>partition</u> of a rectangle $\prod_{i=1}^{n} [a_i, b_i] \subset \mathbb{R}^n$ is a member of the set

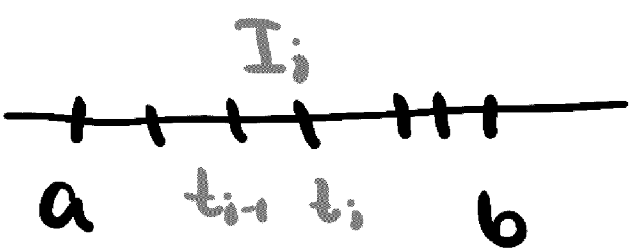
$$\mathcal{P} = \{(P_1, \dots, P_n) \mid P_i \text{ partition of } [a_i, b_i] \ \forall i \in \{1, \dots, n\}\}$$

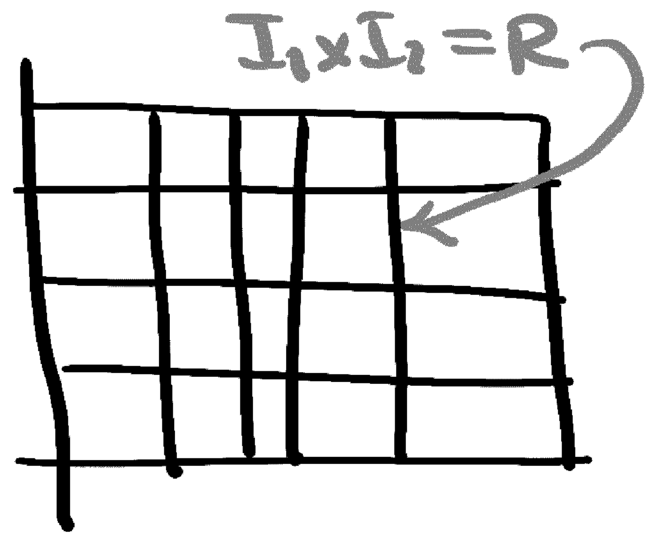
<u>Def</u>: P partition of $[a,b]$, then for some $(t_0, \dots, t_k) \in P$, $[t_{i-1}, t_i]$ is

called a <u>subinterval determined</u> by P.

P partition of $\prod_{i=1}^{n} [a_i, b_i]$, then

$R = \prod_{j=1}^{n} I_j$, where I_j is a subint. determined by P_j, is a <u>subrectangle determined by P</u>.





<u>Def</u>: Let $Q \subset \mathbb{R}^n$ be a rectangle.
Let $f : Q \to \mathbb{R}$ be bounded.
Let P be a partition of Q and let R be a subrectangle determined by P.

Define

$$m_R(f) := \inf\{f(\vec{x}) \mid \vec{x} \in R\}$$

$$M_R(f) := \sup\{f(\vec{x}) \mid \vec{x} \in R\}$$

Also define the <u>lower</u> & <u>upper</u> <u>sum</u>

$$\begin{cases} L(f,P) = \sum\limits_{\substack{R \\ \text{subrectangle}}} m_R(f) \cdot \mathrm{Vol}(R) \\ U(f,P) = \sum\limits_{R \text{ subrect}} M_R(f) \cdot \mathrm{Vol}(R) \end{cases}$$

<u>Note</u>: $L(f,P) \leq U(f,P)$ always true.

<u>Def</u>: If $P_1 = (t_0, \dots, t_k)$ $(a = t_0 < \dots < t_k = b)$

$P_2 = (s_0, \dots, s_\ell)$ $(a = s_0 < \dots < s_\ell = b)$

We say that P_2 is a <u>refinement</u> of P_1, written $P_1 \leq P_2$, if

$$\{t_0, \dots, t_k\} \subset \{s_0, \dots, s_\ell\}$$

If $P=(P_1,\dots,P_n)$, $P'=(P_1',\dots,P_n')$ are partitions of a rectangle we say $P \le P'$ (P' is a refinement of P) if $P_i \le P_i'$ $\forall i \in \{1,\dots,n\}$.

The partition $P''=(P_1 \cup P_1', \dots, P_n \cup P_n')$ refines both P and P'; it's called their <u>common refinement</u>.

<u>Lma</u>: Let P be a partition of a rectangle Q and $f: Q \to \mathbb{R}$ a bounded function.

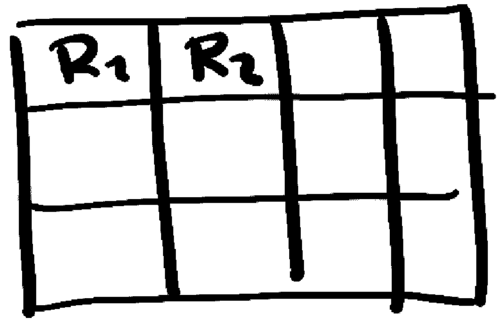
If $P \le P'$ then

$$L(f,P) \le L(f,P')$$

$$U(f,P) \ge U(f,P').$$

Proof sketch: Each subrectangle of P

R



$R = R_1 \cup \cdots \cup R_k$ where R_i are subrectangles of P'.

For each $i \in \{1, \ldots, k\}$:

$$m_R(f) = \min_{i \in \{1, \ldots, k\}} m_{R_i}(f) \leq m_{R_i}(f)$$

then

$$m_R(f) \cdot Vol(R) = \sum_{i=1}^{k} m_R(f) \cdot Vol(R_i)$$

$$\leq \sum_{i=1}^{k} m_{R_i}(f) \, Vol(R_i)$$

so summing over all R:

$$L(f, P) = \sum_{R \text{ in } P} m_R(f) \, Vol(R)$$

$$\leq \sum_{R \text{ in } P} \sum_{i} m_{R_i}(f) \, Vol(R_i)$$

$$= \sum_{R' \text{ in } P'} m_{R'}(f) \, Vol(R')$$

$= L(f,P')$.

The proof for $U(f,P)$ is similar & relies on the observation

$$M_R(f) = \max_i M_{R_i}(f) \geq M_{R_i}(f)$$

$\square$

<u>Lma 1</u>: If P, P' are any partitions of a rectangle Q, $f: Q \to \mathbb{R}$ bounded.

$$L(f,P) \leq U(f,P')$$

<u>Proof</u>: Let $P'' = (P_1 \cup P_1', \ldots, P_n \cup P_n')$ be the common refinement of P and P'. Then

$$L(f,P) \leq L(f,P'') \leq U(f,P'') \leq U(f,P')$$

$\square$

<u>Def</u>: Let Q be a rectangle, and $f: Q \to \mathbb{R}$ bounded.

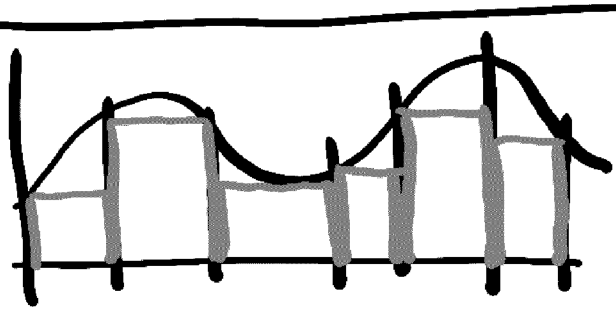
$$\underline{\int_Q} f := \sup_P L(f,P) \quad \text{lower integral}$$

$$\overline{\int_Q} f := \inf_P U(f,P). \quad \text{upper integral.}$$

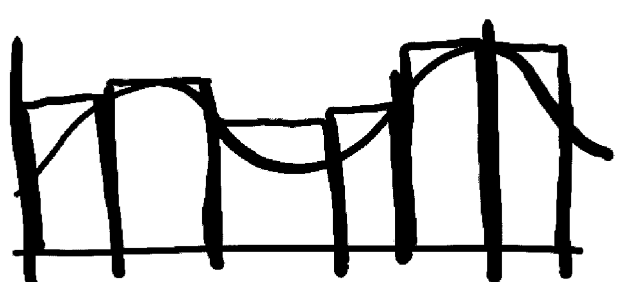
<u>Def</u>: Q a rectangle, $f: Q \to \mathbb{R}$ bounded. We say that f is <u>integrable</u> if

$$\underline{\int_Q} f = \overline{\int_Q} f,$$

and in this case we write $\int_Q f$ for the integral of f over Q.



L(f,P)



U(f,P)

Thm: Q rectangle, $f: Q \to \mathbb{R}$ bounded

then $\underline{\int_Q} f \leq \overline{\int_Q} f$

w/ equality $\Leftrightarrow$ $\forall \varepsilon > 0$ $\exists$ partition P

such that $U(f,P) - L(f,P) < \varepsilon$.

Proof: $\underline{\int_Q} f \leq \overline{\int_Q} f$ follows from

lemma 1. The rest is exercise

$\square$

Thm. Constant functions are integrable

and $\int_Q c = c \cdot \mathrm{Vol}(Q)$.

Proof: For any partition P of Q

$m_R(f) = M_R(f) = c$ $\forall$ subrectangles R of P.

so $L(f,P) = c \sum_R \mathrm{Vol}(R) = c \mathrm{Vol}\, Q$

$U(f,P) = $ — " — $= c \mathrm{Vol}\, Q$ $\square$

Ex: Let $f:[0,1] \to \mathbb{R}$ be def by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Let P be any partition of $[0,1]$

Fact: Any interval $[a,b]$ contains both rational and irrational numbers.

Therefore $m_R(f) = 0$, $M_R(f) = 1$ for any subinterval of P.

so

$$L(f,P) = \sum_R m_R(f) \cdot \mathrm{Vol}(R) = 0$$

$$U(f,P) = \sum_R M_R(f) \cdot \mathrm{Vol}(R)$$

$$= \sum_R \mathrm{Vol}(R) = \mathrm{Vol}\,[0,1] = 1.$$

$$\Rightarrow \underline{\int}_{[0,1]} f = 0, \quad \overline{\int}_{[0,1]} f = 1.$$ so f not integrable

Existence of integrals $\boxed{\S 11}$

The main question we deal w/ next is: When does $\int_Q f$ exist?

- If f is continuous, then $\int_Q f$ exists (easy to see from the def)

But continuity is not necessary! We can allow f to be discontinuous on a set of "small size".

<u>Def</u>. Let $A \subset \mathbb{R}^n$. We say that A has <u>measure zero</u> if $\forall \varepsilon > 0$ if there are rectangles $\{Q_i\}_{i=1}^{\infty}$ such that $A \subset \bigcup_{i=1}^{\infty} Q_i$ and

$$\sum_{i=1}^{\infty} \mathrm{vol}(Q_i) < \varepsilon.$$

If A has measure zero we will write $\mu(A)=0$.

Lma: (a) If $B \subset A$, then
$\mu(A)=0 \Rightarrow \mu(B)=0$.

(b) If $A=\bigcup_{i=1}^{\infty} A_i$ and $\mu(A_i)=0$ $\forall i \in \mathbb{Z}_{\geq 1}$ then $\mu(A)=0$.

(c) $\mu(A)=0 \Leftrightarrow \forall \varepsilon>0$ there are open rectangles $\{\mathring{Q}_i\}_{i=1}^{\infty}$: $A \subset \bigcup_{i=1}^{\infty} \mathring{Q}_i$ and $\sum_{i=1}^{\infty} \mathrm{vol}(\mathring{Q}_i) < \varepsilon$.

(d) If Q is a rectangle, then
$\mu(\partial Q)=0$ but $\mu(Q) \neq 0$

(Q does not have measure zero)
