MAT 322/523 Lec 7 M Feb 12

Recall: Inverse function thm:

If $f: \mathbb{R}^n \to \mathbb{R}^n$ diff'able, and $Df(\vec{a})$ non-singular, there's a nghd U of $\vec{a}$ and $V \overset{\text{open}}{\subset} \mathbb{R}^n$ such that $f: U \to V$ injective and $f^{-1}: V \to U$ is differentiable.

Implicit function theorem $\boxed{\text{§9 Munkres}}$

Given the eq $x^3y + 2e^{xy} = 0$ it determines y as a function of x, we can calculate $\frac{dy}{dx}$ by implicit differentiation.

In general if $f(x,y) = 0$ determines y as a fcn of x, then we can calculate

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

so its clearly necessary that $\frac{\partial f}{\partial y} \neq 0$. In fact, if $\frac{\partial f}{\partial y} \neq 0$ at $(x,y) = (a,b)$, then y can be described as a fcn in x, near $(x,y) = (a,b)$.

This is what the implicit fcn thm says.

Def $f: \mathbb{R}^m \longrightarrow \mathbb{R}^n$. $f = (f_1, \dots, f_n)$

we will use the notation

$$Df = \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_m)} = \frac{\partial f}{\partial \vec{x}}$$

or more generally $\frac{\partial(f_{i_1}, \dots, f_{i_k})}{\partial(x_{j_1}, \dots, x_{j_l})}$.

the $(k\times l)$-matrix with entry in position (p,q) being $\dfrac{\partial f_{ip}}{\partial x_{jq}}$.

<u>Thm</u>: Let $f:\mathbb{R}^{k+n}\to\mathbb{R}^n$ and write it as $f=f(\vec{x},\vec{y})$, $(\vec{x},\vec{y})\in\mathbb{R}^k\times\mathbb{R}^n$.
If there is a diff'able fcn $g:\mathbb{R}^k\to\mathbb{R}^n$ such that $f(\vec{x},g(\vec{x}))=0$, then

$$\frac{\partial f}{\partial\vec{x}}(\vec{x},g(\vec{x}))+\frac{\partial f}{\partial\vec{y}}(\vec{x},g(\vec{x}))\cdot Dg(\vec{x})=0$$

Note: this implies

$$Dg(\vec{x})=-\left(\frac{\partial f}{\partial\vec{y}}(\vec{x},g(\vec{x}))\right)^{-1}\frac{\partial f}{\partial\vec{x}}(\vec{x},g(\vec{x}))$$

so the $n\times n$ matrix $\frac{\partial f}{\partial\vec{y}}$ must be non-singular at $(\vec{x}, g(\vec{x}))$.

<u>Proof</u>: Define $h(\vec{x}) = (\vec{x}, g(\vec{x}))$

and

$$H(\vec{x}) := f(h(\vec{x})) = f(\vec{x}, g(\vec{x})) = \vec{0}$$

for all $\vec{x} \in \mathbb{R}^n$.

Apply the chain rule

(exercise to carry it out)

$\square$

<u>Thm</u> (Implicit function thm)

Let $f : \mathbb{R}^{k+n} \longrightarrow \mathbb{R}^n$ be of class C^r,

and write $f = f(\vec{x}, \vec{y})$, $(\vec{x}, \vec{y}) \in \mathbb{R}^k \times \mathbb{R}^n$

If $(\vec{a}, \vec{b}) \in \mathbb{R}^{k+n}$ is so that

$f(\vec{a}, \vec{b}) = 0$ and $\det \frac{\partial f}{\partial \vec{y}}(\vec{a}, \vec{b}) \neq 0$

then there's a nghd $B \subset \mathbb{R}^k$

of $\vec{a}$ and a unique continuous

function $g: B \to \mathbb{R}^n$: $g(\vec{a}) = \vec{b}$ and $f(\vec{x}, g(\vec{x})) = 0$ $\forall x \in B$. Moreover g is in fact of class C^r.

<u>Proof</u>: We construct a fcn $F: \mathbb{R}^{k+n} \longrightarrow \mathbb{R}^{k+n}$ to which we can apply the inverse function thm. Namely $F(\vec{x}, \vec{y}) = (\vec{x}, f(\vec{x}, \vec{y}))$. To apply inverse function thm, we need to check that DF is non-singular at $(\vec{a}, \vec{b}) \in \mathbb{R}^{k+n}$.

$$DF(\vec{a}, \vec{b}) = \begin{pmatrix} I_k & 0 \\ \frac{\partial f}{\partial \vec{x}}(\vec{a}, \vec{b}) & \frac{\partial f}{\partial \vec{y}}(\vec{a}, \vec{b}) \end{pmatrix}$$

So by Lemm 2.12 in Munkres

$$\det DF(\vec{a}, \vec{b}) = \det \frac{\partial f}{\partial \vec{y}}(\vec{b}) \neq 0.$$

Also note $F(\vec{a},\vec{b}) = (\vec{a},\vec{0})$.

Inverse function thm

$\Longrightarrow$ there are $U \overset{\text{open}}{\subset} \mathbb{R}^k$, $V \overset{\text{open}}{\subset} \mathbb{R}^n$

so that $U \times V \overset{\text{open}}{\subset} \mathbb{R}^k \times \mathbb{R}^n$ nghd

of $(\vec{a},\vec{b}) \in \mathbb{R}^k \times \mathbb{R}^n$

<u>and</u> $W \overset{\text{open}}{\subset} \mathbb{R}^{k+n}$ nghd of $(\vec{a},0)$

(1) $F: U \times V \longrightarrow W$ injective

(2) $F^{-1}: W \to U \times V$ is of class C^r.

so $F^{-1}(\vec{x}, f(\vec{x},\vec{y})) = (\vec{x},\vec{y})$. The inverse is of the form

$F^{-1}(\vec{x},\vec{z}) = (\vec{x}, h(\vec{x},\vec{z}))$, where h is of class C^r.

<u>Claim</u>: Our sought function

is $g(\vec{x}) := h(\vec{x}, 0)$.

① We show $f(\vec{x}, g(\vec{x})) = 0$:

We have $F^{-1}(\vec{x}, \vec{0}) = (\vec{x}, h(\vec{x}, 0))$

so $(\vec{x}, \vec{0}) = F(\vec{x}, h(\vec{x}, \vec{0}))$

$= (\vec{x}, f(\vec{x}, h(\vec{x}, \vec{0})))$

and hence $f(\vec{x}, g(\vec{x})) = 0$.

② We show $g(\vec{a}) = \vec{b}$.

$F(\vec{a}, \vec{b}) = (\vec{a}, \vec{0})$ so

$(\vec{a}, \vec{b}) = F^{-1}(\vec{a}, \vec{0}) = (\vec{a}, h(\vec{a}, 0))$

$= (\vec{a}, g(\vec{a}))$

$\Rightarrow g(\vec{a}) = \vec{b}$.

③ We show that g is unique.

Assume $g_0 : \mathbb{R}^k \longrightarrow \mathbb{R}^n$ is another function w/ the same properties as g. In particular $g(\vec{a}) = \vec{b} = g_0(\vec{a})$.

Next Step: Show that g and g_0 agree in a nghd of $\vec{a} \in \mathbb{R}^k$.

g_0 continuous, so $\exists$ nghd $B_0 \overset{\text{open}}{\subset} \mathbb{R}^k$ of $\vec{a}$ such that $g_0(B_0) \subset V \overset{\text{open}}{\subset} \mathbb{R}^n$

nghd of $\vec{b}$.

Then in B_0 we have

$f(\vec{x}, g_0(\vec{x})) = \vec{0}$ so

$F(\vec{x}, g_0(\vec{x})) = (\vec{x}, \vec{0}) \Leftrightarrow$

$$(\vec{x}, g_0(\vec{x})) = F^{-1}(\vec{x}, \vec{0}) = (\vec{x}, \underbrace{h(\vec{x}, \vec{0})}_{= g(\vec{x})})$$

So $g_0(\vec{x}) = g(\vec{x}) \quad \forall \vec{x} \in B_0$.

Next we show that $g_0 = g$ in U

We just showed that

$\{\vec{x} \mid |g(\vec{x}) - g_0(\vec{x})| = 0\} \overset{\text{open}}{\subset} U$

It's also closed since g, g_0 and the norm $|\cdot|$ are continuous fcns.

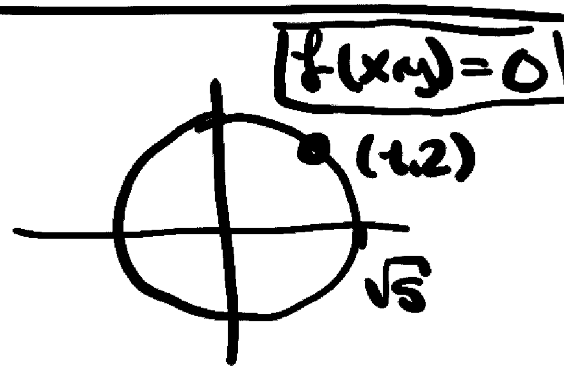
$\Rightarrow \{\vec{x} \in B \mid |g(\vec{x}) - g_0(\vec{x})| = 0\}$ is clopen in U. Since U is connected, the only clopen subsets are $\emptyset$ and U, so

$$\{\vec{x} \in B \mid |g(\vec{x}) - g_0(\vec{x})| = 0\} = U$$

$\square$

<u>Ex</u>: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x,y) \mapsto x^2 - y^2 - 5$



For $(x,y) = (1,2)$ $f(1,2) = 0$

Also $\dfrac{\partial f}{\partial x} = 2x$, $\dfrac{\partial f}{\partial y} = -2y$ both of which are non-zero at $(1,2)$ so can solve $x^2 - y^2 - 5 = 0$ for either x or y near $(x,y) = (1,2)$.

e.g. $y = g(x) = \sqrt{5 - x^2}$.