MAT 322/523 Lec 6 W Feb 7

- f diff'able if partial derivatives exist and are continuous.
- $f: \mathbb{R}^m \to \mathbb{R}^n$, $g: \mathbb{R}^n \to \mathbb{R}^p$ diff'able

then so is $g \circ f$ and

$$D(g \circ f)(\vec{x}) = Dg(f(\vec{x})) \cdot Df(\vec{x}).$$

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§ The inverse function theorem

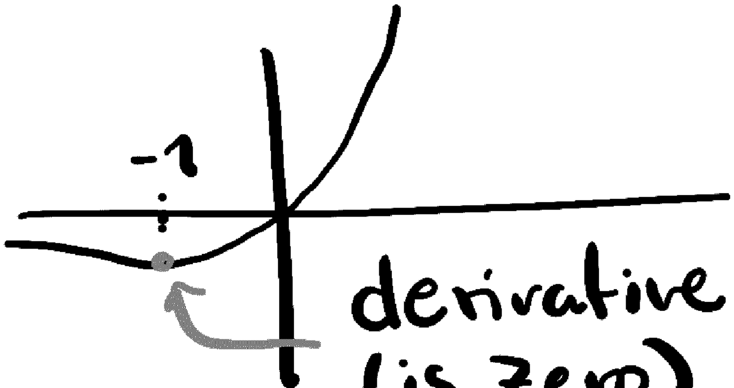
$\boxed{\text{§8 Munkres}}$

Idea: $f: \mathbb{R}^m \to \mathbb{R}^n$ of class C^1.

In order for f to have a diff'able inverse, it's necessary that $Df(\vec{x})$ is non-singular.

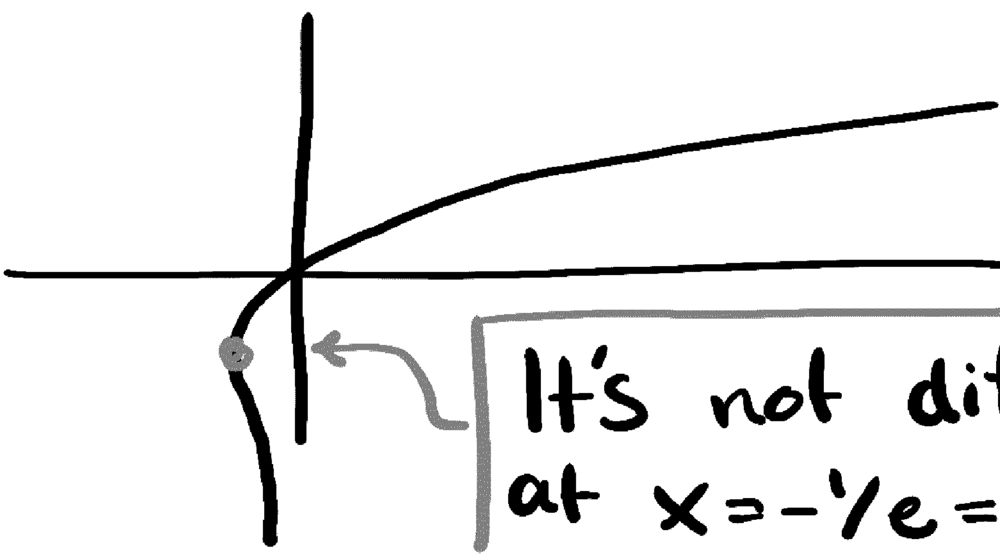
The inverse fcn thm will tell us that this is also sufficient to find a differentiable inverse Locally

Ex $f(x) = xe^x$



derivative singular (is zero) at $x=-1$.

The inverse is called Lambert's W function:



$\boxed{\text{It's not differentiable at } x = -1/e = f(-1)}$

Thm (The inverse fcn thm)

Let $A \overset{\text{open}}{\subset} \mathbb{R}^m$, $f : A \to \mathbb{R}^n$ class C^r.

If $Df(\vec{x})$ is non-singular at $\vec{a} \in A$

there's a nghd U of $\vec{a}$ and $V \overset{\text{open}}{\subset} \mathbb{R}^n$

such that $f: \mathcal{U} \longrightarrow V$ is invertible and the inverse $f^{-1}: V \to \mathcal{U}$ is C^r.

We break the proof into 3 steps.

1. Find a nghd $\mathcal{U}$ of $\vec{a}$ so that $f|_{\mathcal{U}}$ is injective.
2. The inverse $f^{-1}: V \to \mathcal{U}$ is differentiable at each $\vec{v} \in V$
3. f^{-1} is of class C^r.

Lemma: $A \overset{\text{open}}{\subset} \mathbb{R}^n$. Let $f: A \to \mathbb{R}^n$ be of class C^1. If $Df(\vec{a})$ is non-singular then $\exists \alpha > 0$ such that

$$\|f(\vec{x}_0) - f(\vec{x}_1)\| \geq \frac{1}{2} \|\vec{x}_0 - \vec{x}_1\|$$

for all $\vec{x}_0, \vec{x}_1$ in some open ball $B(\vec{a}; \varepsilon)$ centered at $\vec{a}$.

<u>Proof</u>: f differentiable in an open ball $B(\vec{a}; \varepsilon)$ for some $\varepsilon > 0$.

Without loss of generality (wlog) assume $Df(\vec{a}) = I_m$. otherwise change f to $\tilde{f}(\vec{x}) = (Df(\vec{a}))^{-1} f(\vec{x})$.

Define $h(\vec{x}) = \vec{x} - f(\vec{x})$ and note.

$Dh(\vec{a}) = \vec{0}$. Wlog assume $\|Dh(\vec{x})\| < \frac{1}{2}$ for all $\vec{x} \in B(\vec{a}; \varepsilon)$ (else make ε smaller)

$= \sqrt{\sum_{i,j} (a_{ij})^2}$

MVT applied to h gives $\exists \vec{c} \in B(\vec{a}; \varepsilon)$ so that

$$h(\vec{x}_0) - h(\vec{x}_1) = Dh(\vec{c}) \cdot (\vec{x}_0 - \vec{x}_1).$$

Take norms:

$$\|h(\vec{x}_0) - h(\vec{x}_1)\| = \|Dh(\vec{c}) \cdot (\vec{x}_0 - \vec{x}_1)\|$$

$$\leq \|Dh(\vec{c})\| \cdot \|\vec{x}_0 - \vec{x}_1\|$$

Schwarz ineq

$< \frac{1}{2}\|\vec{x}_0 - \vec{x}_1\|$.

<u>LHS</u>: $\|h(\vec{x}_0) - h(\vec{x}_1)\|$

$= \|(\vec{x}_0 - \vec{x}_1) - (f(\vec{x}_0) - f(\vec{x}_1))\|$

$\geq \|\vec{x}_0 - \vec{x}_1\| - \|f(\vec{x}_0) - f(\vec{x}_1)\|$

[reverse Δ-ineq.

So

$\|\vec{x}_0 - \vec{x}_1\| - \|f(\vec{x}_0) - f(\vec{x}_1)\| \leq \frac{1}{2}\|\vec{x}_0 - \vec{x}_1\|$

$\iff \frac{1}{2}\|\vec{x}_0 - \vec{x}_1\| \leq \|f(\vec{x}_0) - f(\vec{x}_1)\|$

$\square$

The lemma implies <u>injectivity</u>:
if $f(x_0) = f(x_1)$ then $\frac{1}{2}\|\vec{x}_0 - \vec{x}_1\| \leq 0$
$\Rightarrow x_0 = x_1$. Step 1 done.

<u>Step 2</u>: The inverse $f^{-1}: V \to U$ is diff'able for all $\vec{v} \in V$.

Let $E = Df(\vec{a})$. We want to

$(\vec{b} = f(\vec{a}))$

Show that

$$\frac{f^{-1}(\vec{b}+\vec{k}) - f^{-1}(\vec{b}) - E^{-1}\cdot\vec{k}}{\|\vec{k}\|} \xrightarrow{\vec{k}\to\vec{0}} \vec{0}$$

By lemma $\|f(\vec{x}_0) - f(\vec{x}_1)\| \geq \frac{1}{2}\|\vec{x}_0 - \vec{x}_1\|$

So at $\vec{x}_0 = f^{-1}(\vec{b}+\vec{k})$ and $\vec{x}_1 = f^{-1}(\vec{b})$

we get $\|\vec{k}\| \geq \frac{1}{2}\|f^{-1}(\vec{b}+\vec{k}) - f^{-1}(\vec{b})\|$

$$\Leftrightarrow \frac{\|f^{-1}(\vec{b}+\vec{k}) - f^{-1}(\vec{b})\|}{\|\vec{k}\|} \leq 2$$

THIS IMPLIES f^{-1} IS CONTINUOUS

so

$$\frac{f^{-1}(\vec{b}+\vec{k}) - f^{-1}(\vec{b}) - E^{-1}\cdot\vec{k}}{\|\vec{k}\|} \cdot \frac{\|f^{-1}(\vec{b}+\vec{k}) - f^{-1}(\vec{b})\|}{\|f^{-1}(\vec{b}+k) - f^{-1}(\vec{b})\|}$$

$$\leq \frac{f^{-1}(\vec{b}+\vec{k}) - f^{-1}(\vec{b}) - E^{-1}\cdot\vec{k}}{\|f^{-1}(\vec{b}+k) - f^{-1}(\vec{b})\|} \cdot 2$$

$$= 2\,\frac{E^{-1}\left(E\left(f^{-1}(\vec{b}+\vec{k}) - f^{-1}(\vec{b})\right) - \vec{k}\right)}{\|f^{-1}(\vec{b}+k) - f^{-1}(\vec{b})\|} \quad (*)$$

Let $\Delta(\vec{k}) := f^{-1}(\vec{b}+\vec{k}) - f^{-1}(\vec{b})$.

Then (*) is

$$-2E^{-1}\left[\underbrace{\frac{\vec{k} - E\cdot\Delta(\vec{k})}{\|\Delta(\vec{k})\|}}_{(**)}\right].$$

Note

$$\begin{cases} \vec{b}+\vec{k} = f(f^{-1}(\vec{b}+\vec{k})) = f(\vec{a}+\Delta(\vec{k})) \\ \vec{b} = f(f^{-1}(\vec{b})) = f(\vec{a}) \end{cases}$$

So (**) becomes

$$\frac{f(\vec{a}+\Delta(\vec{k})) - f(\vec{a}) - E\Delta(\vec{k})}{\|\Delta(\vec{k})\|} \quad (***)$$

Since f^{-1} is continuous

$$\Delta(\vec{k}) = f^{-1}(\vec{b}+\vec{k}) - f^{-1}(\vec{b}) \xrightarrow[\vec{k}\to\vec{0}]{} 0$$

so $(***) \longrightarrow 0$

as $k \to 0$ because f is diff'able.

Going back to (*) it means

$$\frac{f^{-1}(\vec{b}+\vec{k})-f^{-1}(\vec{b})-E^{-1}\cdot\vec{k}}{\|\vec{k}\|} \longrightarrow 0$$

$\square$

<u>Step 3</u>: We show that f^{-1} is of class C^r.

<u>Induction on r</u>:

<u>Base case $r=1$</u>: We know f^{-1} is diff'able, but need to prove that Df^{-1} is continuous:
We know $Df^{-1}(\vec{b}) = (Df(\vec{a}))^{-1}$ and Df is continuous, and so is the inverse.
Induction hypothesis: Assume thm holds for fcns of class C^{r-1}.
Let f be of class C^r, then Df is of class C^{r-1}, so taking

inverses gives that Df^{-1} is of class C^{r-1}.

(taking inverses is a smooth operation) $\square$