MAT 322/523 Lec 5, M Feb 5

- Directional derivative of $f:\mathbb{R}^m \to \mathbb{R}^n$ at $\vec{a} \in \mathbb{R}^m$ in direction $0 \neq \vec{u} \in \mathbb{R}^m$

$$f'(\vec{a};\vec{u}) = \lim_{t\to 0} \frac{f(\vec{a}+t\vec{u}) - f(\vec{a})}{t}$$

- $f:\mathbb{R}^m \to \mathbb{R}^n$ differentiable at $\vec{a} \in \mathbb{R}^m$ if there exists an $n\times m$ matrix B:

$$\lim_{\vec{h}\to \vec{0}} \frac{f(\vec{a}+\vec{h}) - f(\vec{a}) - B\cdot\vec{h}}{\|\vec{h}\|} = 0.$$

$B = Df(\vec{a})$ if limit exists.

- We proved: If f is diff'able at $\vec{a}$, then all directional derivatives exist at $\vec{a}$.

<u>Thm</u>: Let $A \subset \mathbb{R}^m$, $f: A \to \mathbb{R}^n$. If f is differentiable at $\vec{a}$, then f is continuous at $\vec{a}$.

<u>Proof</u>: Want to prove

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a}).$$ Equivalently:

$$\lim_{\vec{h}\to\vec{0}} f(\vec{a}+\vec{h}) - f(\vec{a}) = 0.$$

$$f(\vec{a}+\vec{h}) - f(\vec{a}) = \|\vec{h}\| \left[\frac{f(\vec{a}+\vec{h}) - f(\vec{a}) - B\cdot\vec{h}}{\|\vec{h}\|} \right] + \vec{B}\cdot\vec{h}$$

$$\xrightarrow{\vec{h}\to\vec{0}} 0. \qquad \square$$

How to compute $Df(\vec{a})$ if it exists?

<u>Def</u>: Let $A \subset \mathbb{R}^m$, $f: A \to \mathbb{R}^n$. For $1 \leq j \leq m$ the <u>j-th partial derivative</u> of f at $\vec{a}$ is

$$D_j f(\vec{a}) := f'(\vec{a}; \vec{e}_j)$$

where $\vec{e}_j = (0, \dots, 0, 1, 0, \dots, 0)$.

Position j

<u>Thm</u>: Let $A \subset \mathbb{R}^m$, $f : A \to \mathbb{R}$ $(n=1)$.
If f is diff'able at $\vec{a} \in A$, then

$$Df(\vec{a}) = \left(D_1 f(\vec{a}) \ \cdots\cdots \ D_m f(\vec{a})\right)$$

$(1 \times m)$ - matrix

<u>Proof</u>. By hypothesis $Df(\vec{a})$ exists & is an $(1 \times m)$ - matrix $(\lambda_1 \cdots \lambda_m)$.
By earlier thm $f'(\vec{a};\vec{u}) = Df(\vec{a}) \cdot \vec{u}$, So

$$D_j f(\vec{a}) \overset{\text{def}}{=} f'(\vec{a};\vec{e}_j) \overset{\text{thm}}{=} Df(\vec{a}) \cdot \vec{e}_j = \lambda_j$$

$\square$

<u>Thm</u>: Let $A \subset \mathbb{R}^m$, $f : A \to \mathbb{R}^n$.
Assume A contains a nghd of $\vec{a} \in A$.

$$\text{Let } f(\vec{x}) = \begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_n(\vec{x}) \end{pmatrix}$$

(a) $f(\vec{x})$ is differentiable at $\vec{a}$ iff each $f_i(\vec{x})$ is diff'able at $\vec{a}$.

(b) If $f(\vec{x})$ is diff'able at $\vec{a}$, then

$$Df(\vec{a}) = \begin{pmatrix} Df_1(\vec{a}) \\ \vdots \\ Df_n(\vec{a}) \end{pmatrix}$$

$$= \begin{pmatrix} D_1 f_1(\vec{a}) & \cdots & D_m f_1(\vec{a}) \\ \vdots & \ddots & \vdots \\ D_1 f_n(\vec{a}) & \cdots & D_m f_n(\vec{a}) \end{pmatrix}$$

This matrix is usually called the <u>Jacobian matrix</u> of f.

<u>Ex</u>: $f(x,y,z) = (\underbrace{x^2+z}_{f_1}, \underbrace{y^3-xz}_{f_2})$

$f_1(x,y,z) = x^2+z$

$f_2(y^3-xz) = y^3-xz$

$\vec{h} = (h_1, h_2, h_3)$.

Both f_1 & f_2 are differentiable:

① $\lim\limits_{\vec{h}\to\vec{0}} \dfrac{f_1(x+h_1, y+h_2, z+h_3) - f_1(x,y,z) - B\cdot\vec{h}}{\|\vec{h}\|}$ ← 1×3

$= \lim\limits_{\vec{h}\to\vec{0}} \dfrac{(x+h_1)^2 + z+h_3 - x^2 - z - B\cdot\vec{h}}{\|\vec{h}\|}$ ← 3×1

If $B = (2x \quad 0 \quad 1)$ then we see that the limit is zero.
So $Df_1(\vec{x}) = (2x, 0, 1)$. (also the gradient)

Similarly $Df_2(\vec{x}) = (-z, 3y^2, -x)$

So by the thm f is diff'able at every $\vec{x} \in \mathbb{R}^3$ and

$$Df(\vec{x}) = \begin{pmatrix} Df_1(\vec{x}) \\ Df_2(\vec{x}) \end{pmatrix} = \begin{pmatrix} 2x & 0 & 1 \\ -z & 3y^2 & -x \end{pmatrix}.$$

We have seen that existence of partial derivatives (or all directional derivatives for that matter) does <u>not</u> imply differentiability. $\boxed{\text{§6 Munkres}}$

Thm: Let $A \overset{\text{open}}{\subset} \mathbb{R}^m$. $f: A \to \mathbb{R}^n$.
$f(\vec{x}) = (f_1(\vec{x}), \dots, f_n(\vec{x}))$. If $D_j f_i(\vec{x})$ exists at each $\vec{x} \in A$ and are continuous on A, then f is differentiable at each $\vec{x} \in A$.

We call a function satisfying the hypotheses a continuously differentiable fcn, or a fcn of class C^1.

Similarly if the partial derivatives of order $\leq r$ exist and are continuous, then f is of class C^r.
A fcn of class C^∞ is also called smooth.

Thm: $A \overset{\text{open}}{\subset} \mathbb{R}^m$, $f: A \to \mathbb{R}^n$ is of class C^2, then $\forall \vec{a} \in A$

$$D_k D_j f(\vec{a}) = D_j D_k f(\vec{a}).$$

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§Chain rule $\boxed{\text{§7 Munkres}}$

Thm $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^n$

$f: A \to \mathbb{R}^n$, $g: B \to \mathbb{R}^p$

$f(A) \subset B$ and suppose $f(\vec{a}) = \vec{b}$.

If f is diff'able at $\vec{a}$ and if g is diff'able at $\vec{b}$ then $g \circ f$ is diff'able at $\vec{a}$ and

$$D(g \circ f)(\vec{a}) = \underset{p \times n}{Dg(\vec{b})} \cdot \underset{n \times m}{Df(\vec{a})}$$

Proof idea: $g \circ f : \mathbb{R}^m \to \mathbb{R}^p$.

(1) $$\frac{g(\vec{y}+\vec{h}) - g(\vec{y})}{\|\vec{h}\|} - \frac{Dg(\vec{y}) \cdot \vec{h}}{\|\vec{h}\|} \longrightarrow 0$$

(2) $$\frac{f(\vec{x}+\vec{h}) - f(\vec{x})}{\|\vec{h}\|} - \frac{Df(\vec{x}) \cdot \vec{h}}{\|\vec{h}\|} \longrightarrow 0$$

Want to prove

(*) $$\frac{g(f(\vec{x}+\vec{h})) - g(f(\vec{x})) - Dg(f(\vec{x})) \cdot Df(\vec{x}) \cdot \vec{h}}{\|\vec{h}\|} \longrightarrow 0$$

(2): $f(\vec{x}+\vec{h})-f(\vec{x}) \approx Df(\vec{x})\cdot\vec{h}$

Here is where it stops being rigorous

$g(f(\vec{x}+\vec{h}))-g(f(\vec{x}))-D(g\circ f)(\vec{x})\cdot\vec{h}$

$\approx B(\vec{x})\cdot\vec{h}$

matrix roughly of norm $\|\vec{h}\|^2$.

roughly norm $\|\vec{h}\|$

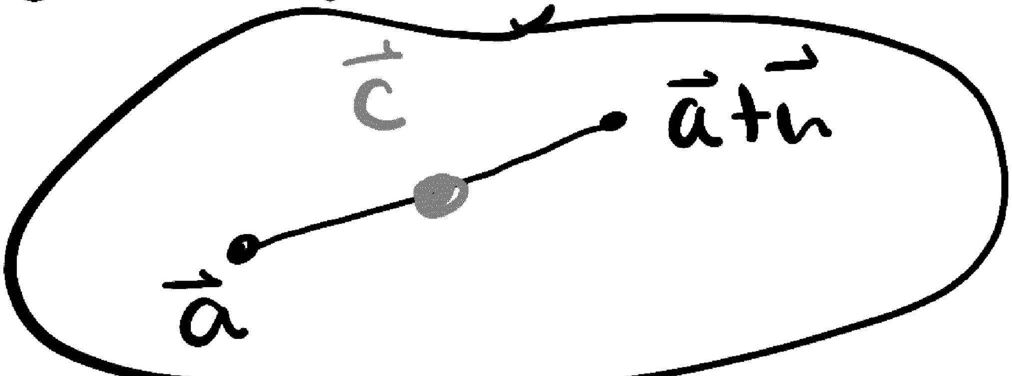
So (*) $\approx \dfrac{\boxed{B(\vec{x})}\cdot\vec{h}}{\|\vec{h}\|^2} \xrightarrow[\vec{h}\to 0]{} 0$ $\square$

<u>Thm</u> (Mean-value thm)

Let $A \overset{\text{open}}{\subset} \mathbb{R}^m$, $f: A \to \mathbb{R}^n$ diff'able on A.

If A contains the line segment between $\vec{a}$ and $\vec{a}+\vec{h}$, then there's a point $\vec{c}=\vec{a}+t_0\vec{h}$, $0<t_0<1$ such that

$$f(\vec{a}+\vec{h})-f(\vec{a}) = Df(\vec{c})\cdot\vec{h}.$$



A.

Thm: $f:\mathbb{R}^m \to \mathbb{R}^n$; $g:\mathbb{R}^n \to \mathbb{R}^m$ are differentiable and $g(f(\vec{x})) = \vec{x}$ then

$$Dg(f(\vec{x})) = \left(Df(\vec{x})\right)^{-1}.$$

Proof: By assumption

$$D(g\circ f)(\vec{x}) = I_m$$

derivative of $\vec{x}$

The chain rule gives

$$Dg(f(\vec{x}))\cdot Df(\vec{x}) = I_m$$

$\square$
