MAT 322/523 Lec 4, W Jan 31

Recall:

- $A \subset \mathbb{R}^n$ compact $\Leftrightarrow$ A closed and bounded
- Extreme value thm: continuous $f: C \longrightarrow \mathbb{R}^n$ admits max & min value
  compact
- Intermediate value thm: continuous $f: X \longrightarrow Y$ then $f(X) \subset Y$ connected.
  connected

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§Differentiation $\boxed{\text{§5 Munkres}}$

$\phi: \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ if

$$\lim_{t \to 0} \frac{\phi(a+t) - \phi(a)}{t} \text{ exists.}$$

For functions $\phi: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ we must specify a direction along which we compute the derivative:

<u>Def</u>: Let $A \subset \mathbb{R}^m$, $f: A \to \mathbb{R}^n$.
Assume A contains a nghd of $\vec{a} \in A$.
Given $\vec{0} \neq \vec{u} \in \mathbb{R}^m$ define the directional derivative of f at $\vec{a}$ in the direction $\vec{u}$ as:

$$f'(\vec{a};\vec{u}) := \lim_{t \to 0} \frac{f(\vec{a}+t\vec{u}) - f(\vec{a})}{t}$$

if the limit exists.

<u>Ex</u>: $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = xy$.

$$f'((a,b);(1,0)) =$$

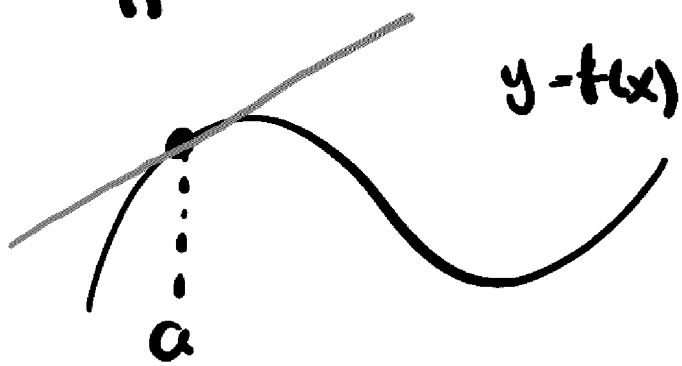
$$= \lim_{t \to 0} \frac{f((a,b) + t(1,0)) - f(a,b)}{t}$$

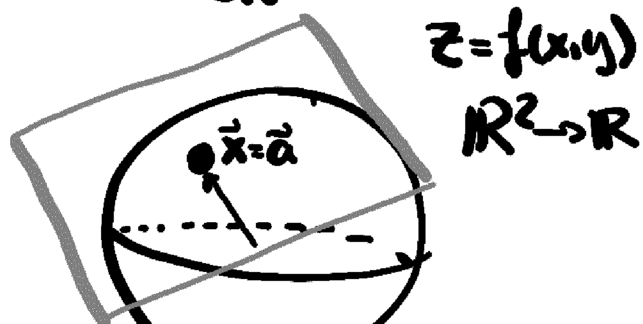
$$= \lim_{t\to 0} \frac{f(a+t,b)-f(a,b)}{t}$$

$$= \lim_{t\to 0} \frac{(a+t)b-ab}{t} = b.$$

Directional derivatives are useful, but NOT the correct definition of differentiation. For instence, all diretional derivatives can exist, but the function could not be continuous.

The derivative should be a linear approximation of the function.





$f(x) \approx f(a) + f'(a)(x-a)$

near $x=a$

$f(\vec{x}) \approx f(\vec{a}) + B \cdot (\vec{x}-\vec{a})$

B is a (1×2)-matrix

Def: Let $A \subset \mathbb{R}^m$, $f: A \to \mathbb{R}^n$. Suppose A contains a nghd of $\vec{a} \in A$. We say that f is differentiable at $\vec{a}$ if there's an $(n \times m)$-matrix B such that

$$\lim_{\vec{h} \to \vec{0}} \frac{f(\vec{a}+\vec{h}) - f(\vec{a}) - B \cdot \vec{h}}{\|\vec{h}\|} = 0.$$

If the limit exists, B is unique and is called the derivative of f at $\vec{a}$, denoted by $Df(\vec{a})$.

Ex*: Let $f: \mathbb{R}^3 \to \mathbb{R}^2$, $B = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

and

$$f(x,y,z) = B \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \vec{b}$$

$$= \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} x-y \\ 2x+y+z \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} x-y+2 \\ 2x+y+z-1 \end{pmatrix}.$$

For some $\vec{0} \neq \vec{h} \in \mathbb{R}^3$ we have

$$f(\vec{x}+\vec{h}) - f(\vec{x}) = \left(B(\vec{x}+\vec{h})+\vec{b}\right) - \left(B\vec{x}+\vec{b}\right)$$

$$= B\cdot\vec{h}.$$

So $$\lim_{\vec{h}\to\vec{0}} \frac{\overbrace{f(\vec{x}+\vec{h}) - f(\vec{x})}^{=B\cdot\vec{h}} - B\cdot\vec{h}}{\|\vec{h}\|} = 0$$

$$Df(\vec{x}) = B = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \text{ for all } \vec{x} \in \mathbb{R}^3.$$

We will perform a number of sanity checks to ensure that our def is sound:

(1) Differentiable functions are continuous

(2) Composites of diff fcns are diff.

(3) Differentiability at $\vec{a}$ implies that all directional derivatives exist at $\vec{a}$.

<u>Thm</u>: Let $A \subset \mathbb{R}^m$, $f: A \to \mathbb{R}^n$. If f is diff'able at $\vec{a} \in A$, then all the directional derivatives of f at $\vec{a}$ exist, and

$$Df(\vec{a};\vec{u}) = Df(\vec{a}) \cdot \vec{u}$$

<u>Proof</u>: Set $\vec{h} = t\vec{u}$, $\vec{u} \neq \vec{0}$

By assumption there's some B $(n \times m)$

$$(*) \quad \frac{f(\vec{a}+t\vec{u}) - f(\vec{a}) - B \cdot t\vec{u}}{\|t\vec{u}\|} \xrightarrow{t \to 0} 0$$

If $t > 0$, multiply $(*)$ by $\|\vec{u}\|$ to get

$$\frac{f(\vec{a}+t\vec{u}) - f(\vec{a})}{t} - B \cdot \vec{u} \xrightarrow{t \to 0} 0$$

If $t < 0$ multiply $(*)$ with $-\|\vec{u}\|$ to reach the same conclusion. $\square$

Ex: $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$ is def by $f(0)=0$ and

$$f(x,y)=\frac{x^2y}{x^4+y^2}\ , \quad (x,y)\neq(0,0)$$

① All directional derivatives exist at $(x,y)=(0,0)$:

$\vec{u}=(u_1,u_2)$

$$\frac{f(tu_1,tu_2)-f(0,0)}{t}=\frac{\frac{(tu_1)^2\,tu_2}{(tu_1)^4+(tu_2)^2}-0}{t}$$

$$=\frac{t^3u_1^2u_2}{t(t^4u_1^4+t^2u_2^2)}=\frac{u_1^2u_2}{t^2u_1^4+u_2^2}$$

$$\longrightarrow \frac{u_1^2}{u_2} \text{ as } t\to 0$$

$$f'(\vec{0};(u_1,u_2))=\begin{cases} u_1^2/u_2\ , & u_2\neq 0 \\ 0\ , & u_2=0 \end{cases}$$

then $f(tu_1,0)=0$

② f is not differentiable at $(0,0)$:

If it would, $Df(\vec{0}) = (a \ b)$ (1×2) — matrix

$\vec{h} = \begin{pmatrix} h \\ k \end{pmatrix}$, $\|\vec{h}\| = \sqrt{h^2+k^2}$

$$\frac{f(\vec{h}) - f(\vec{0}) - (a\ b)\vec{h}}{\|\vec{h}\|} = \frac{\frac{h^2k}{h^4+k^2} - 0 - (ah+bk)}{\sqrt{h^2+k^2}}$$

$$= \frac{h^2k - ah - bk}{(h^4+k^2)\sqrt{h^2+k^2}}$$ the limit as $(h,k) \to (0,0)$ doesn't exist.* Look at behavior along curve $k = h^2$ as $h \to 0^+$ (that is $h > 0$ and $h \to 0$)

$$\frac{h^4 - ah - bh^2}{2h^4\sqrt{2h^2}} = \frac{h^4 - ah - bh^2}{2\sqrt{2}\, h^5} \xrightarrow{h \to 0^+} \infty$$

③ In fact f is not even* continuous at $(0,0)$!

Look at $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ along the curve (t, t^2) for $t \to 0$, and we get

$$f(t, t^2) = \frac{t^4}{t^4 + t^4} = \frac{1}{2} \xrightarrow{t \to 0} \frac{1}{2} \neq 0.$$