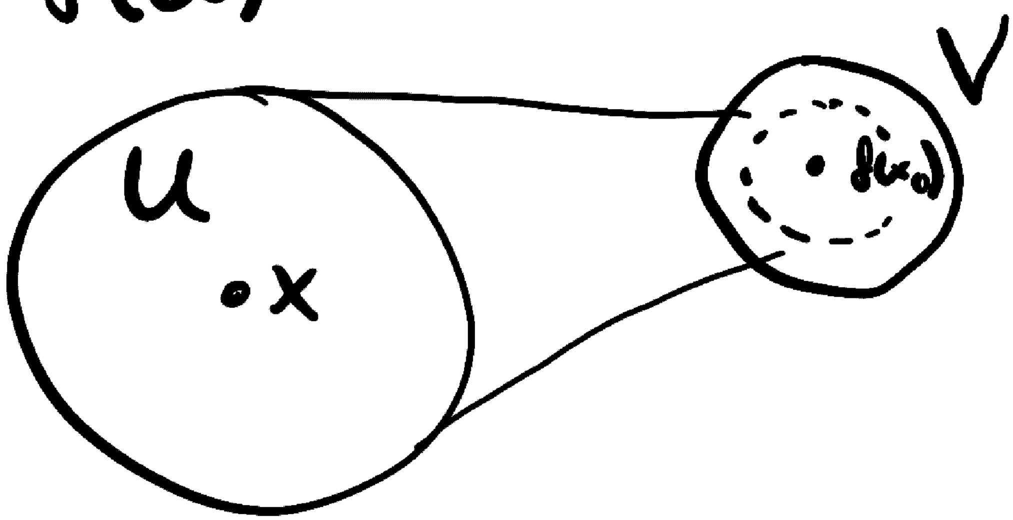
MAT 322/523 Lec 3, M Jan 29

<u>Recall</u>: $(X, d_X), (Y, d_Y)$ metric spaces

- $f: X \to Y$ is continuous at x_0 for every nghd $V \overset{\text{open}}{\subset} Y$ containing $f(x_0)$, there's some $U \overset{\text{open}}{\subset} X$: $f(U) \subset V$.



Now consider $(Y, d_Y) = (\mathbb{R}^n, d_{\text{standard}})$

<u>Thm</u>:

(1) $f: X \to \mathbb{R}^n$ function of the form $f(x) = (f_1(x), \dots, f_n(x))$ is continuous iff $f_i : X \to \mathbb{R}$ is continuous $\forall i \in \{1, \dots, n\}$.

(2) $f, g: X \to \mathbb{R}^n$ continuous. Then $f+g$, $f-g$, $f \cdot g$ are continuous. If $g(x_0) \neq 0$, f/g is continuous at x_0.

(3) The projection $\pi_i: \mathbb{R}^n \to \mathbb{R}$, $\vec{x} \mapsto x_i$ is continuous.

All functions formed from polynomials, trigonometric, exponential, logarithmic functions are continuous.

"All the usual functions are continuous."

Def $A \subset X$ subset.

$x_0 \in X$ limit pt of A. $f: A \to Y$

We say that $f(x)$ approaches y_0 as x approaches x_0

"$f(x) \to y_0$ as $x \to x_0$"

"$\lim_{x \to x_0} f(x) = y_0$"

If for every $V \overset{\text{open}}{\subset} Y$ containing y_0, there's $U \overset{\text{open}}{\subset} X$ containing x_0 such that $f(x) \in V$ whenever $x \in U \cap A$ and $x \neq x_0$.

Remark: Alternatively we can write it in terms of metrics:

$f(x) \to y_0$ as $x \to x_0$ iff

$\forall \varepsilon > 0 \ \exists \delta > 0$ such that
$d_Y(f(x), y_0) < \varepsilon$ whenever $0 < d_X(x, x_0) < \delta$

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we say $x \in (X, d)$ is <u>isolated</u> if
$\exists \delta > 0 : B(x; \delta) = \{x\}$.
Important connection w/ continuity:
<u>Thm</u>: If $x_0 \in (X, d)$ is not isolated, and $f : X \to Y$ is a function, then f is continuous at x_0 iff $\lim_{x \to x_0} f(x) = f(x_0)$

<u>§ Compactness</u> $\boxed{\text{§4 Munkres}}$

<u>Def</u>: Let X be a metric sp, and $Y \subset X$ a subspace.
An <u>open cover</u> of Y is a collection $\{U_i\}_{i \in I}$ of open subsets of X such that $Y \subset \bigcup_{i \in I} U_i$

Def A subspace $Y \subset X$ is called compact if for every open cover $\{U_i\}_{i \in I}$ of Y, we can cover Y with a finite number of them: $Y \subset \bigcup_{j=1}^{N} U_{i_j}$ for some $i_1, \dots, i_N \in I$.

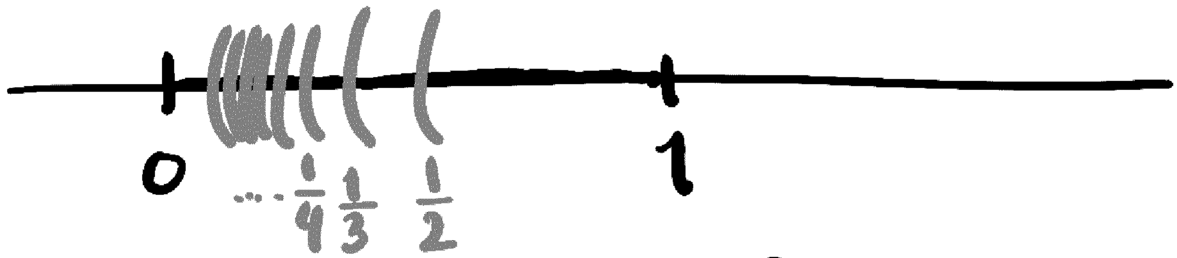
$$\boxed{\text{SLOGAN: } Y \subset X \text{ is compact iff every open cover admits a finite subcover}}$$

Ex: $(0,1] \subset \mathbb{R}$ is not compact.

We can check that the infinite cover

$$\left\{\left(\frac{1}{n}, 2\right)\right\}_{n=1}^{\infty}$$

Does not admit a finite subcover.



only finite number of such will miss elements that are close to 0.

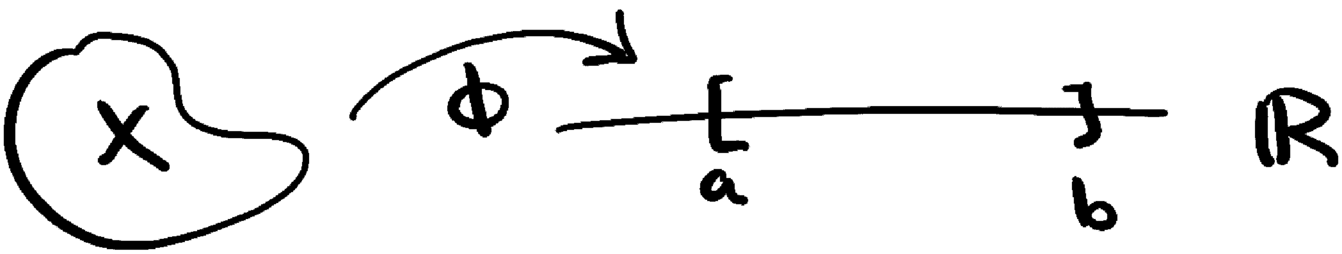
Thm: $[a,b] \subset \mathbb{R}$ is compact.

Def A subset $A \subset \mathbb{R}^n$ is bounded if there's some $M>0$ such that $\|\vec{x}\| \leq M$ for all $\vec{x} \in A$.

Thm (Heine-Borel) $A \subset \mathbb{R}^n$ is compact if and only if A is closed and bounded.

Rmk: The "$\Leftarrow$"-direction is not true for general metric spaces.

Thm (Extreme value theorem)
Let $X \subset \mathbb{R}^m$ be compact.
If $f: X \to \mathbb{R}^n$ is continuous, then $f(X)$ is compact in $\mathbb{R}^n$.
In particular if $\phi: X \to \mathbb{R}$ is continuous then ϕ has a maximum and a minimum value.



Thm. Let $X \subset \mathbb{R}^m$ be compact and let $f: X \to \mathbb{R}^n$ be continuous.
Then f is <u>uniformly continuous</u>

In other words: Given $\varepsilon > 0$, there is a $\delta > 0$ such that when $\vec{x}, \vec{y} \in X$,

$$\|\vec{x} - \vec{y}\| < \delta \implies \|f(\vec{x}) - f(\vec{y})\| < \varepsilon.$$

<u>Thm</u>: Every "box"
$[a_1,b_1]\times\cdots\times[a_n,b_n]\subset\mathbb{R}^n$ is compact.

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§ <u>Connectedness</u> (X,d) metric sp

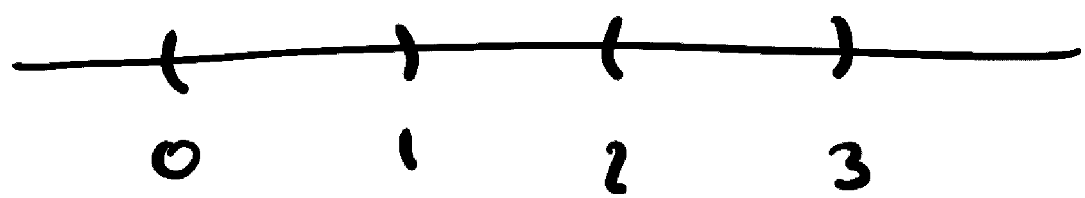
<u>Def</u>: X is <u>disconnected</u> if

$X = A \cup B$ where

- $A, B \overset{\text{open}}{\subset} X$
- $A \cap B = \varnothing$

X is <u>connected</u> if it's not disconnected.

<u>Ex</u>: $X = (0,1)\cup(2,3)\subset\mathbb{R}$ is disconnected



$(0,1)\subset X$ is an example of a clopen subset (closed and open)

Thm: Any interval (a,b), $(a,b]$, $[a,b)$, $[a,b] \subset \mathbb{R}$ is connected.

Thm (Intermediate value theorem)
X, Y metric spaces
X connected.
If $f: X \to Y$ is continuous then $f(X) \subset Y$ is connected.

Exercise: Convince yourself that this gives the usual version of the IVT for continuous functions $\phi: X \to \mathbb{R}$.