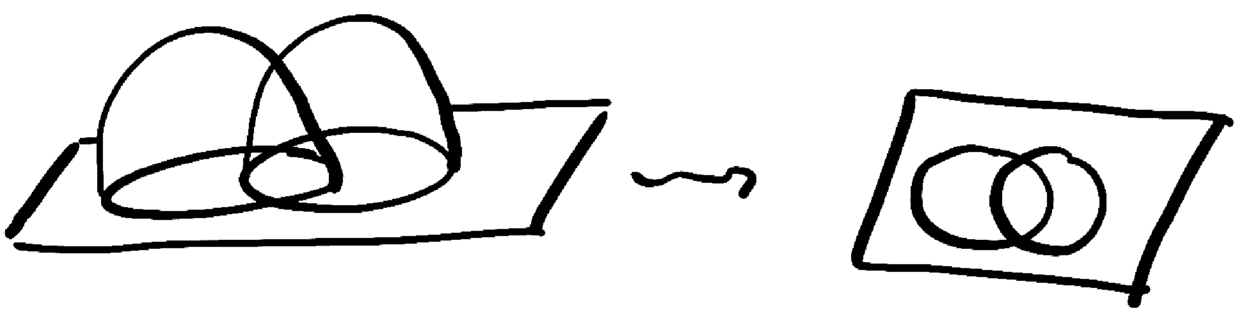
MAT 322/523 Lec 24 W Apr 24

Recall: • Orientation on M = k-mfld in $\mathbb{R}^n$ is collection of positively overlapping coord charts.

• M manifold w/ ∂

$\rightsquigarrow$ restricted orientation on ∂M by restricting coord charts:



The restricted orientation on ∂M is in fact not always what we prefer.

<u>Def</u>: Let M be a k-mfld in $\mathbb{R}^n$ w/ $\partial M \neq \emptyset$. Given an orientation on M, the <u>induced orientation</u> on ∂M is defined as follows:

- If k is even its the restricted orientation
- If k is odd its the <u>opposite</u> of the restricted orientation

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<u>Ex</u> Consider $B^3 \subset \mathbb{R}^3$ (the closed unit ball). Its boundary is $S^2 \subset \mathbb{R}^3$.

It turns out that the induced orientation on S^2 is the one corresponding to the <u>outwards</u> pointing normal vector field on S^2

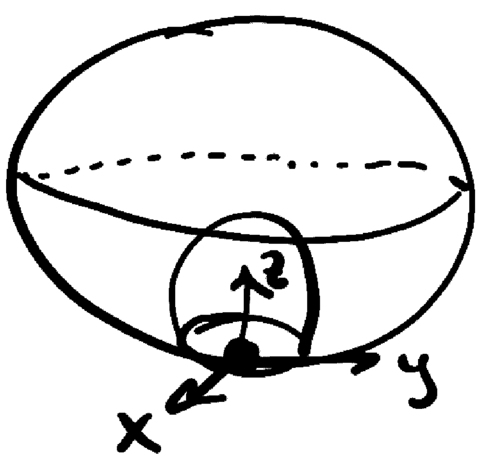
$\alpha: U \to V$ Coord chart around $x \in \partial M$. $U \subset \mathbb{H}^3$. The restricted coord chart is given by

$\alpha_\partial(x,y) = \alpha(x,y,0)$

Since $\dim M = 3$, the induced ori is the one opposite to the restricted one. The normal vf $p \mapsto (p, \vec{n})$ on ∂M that corresp to the induced ori is the one so that

$$\left(-\vec{n}, \frac{\partial \alpha}{\partial x}, \frac{\partial \alpha}{\partial y}\right) \text{ is}$$

a right-handed frame.



the restricted frame

$$\left(\frac{\partial \alpha}{\partial x}, \frac{\partial \alpha}{\partial y}, \frac{\partial \alpha}{\partial z}\right)$$

is right-handed

$\Rightarrow \left(\frac{\partial \alpha}{\partial z}, \frac{\partial \alpha}{\partial x}, \frac{\partial y}{\partial y}\right)$ is right-handed

and $\frac{\partial \alpha}{\partial z}$ is inwards normal

In general, one can describe the induced orientation on ∂M to be the one determined by

$\begin{pmatrix} \text{outwards} & \text{restricted} \\ \text{normal,} & \text{frame} \end{pmatrix}$ should be right-handed.

<u>Def</u>: M compact, oriented k-mfd in $\mathbb{R}^n$

$M \subset A \overset{\text{open}}{\subset} \mathbb{R}^n$. $\omega \in \Omega^k(A)$

$C := M \cap (\operatorname{supp} \omega) \leftarrow$ this is compact

Suppose $\exists$ coord chart $\alpha : \mathcal{U} \to V$ belonging to the given Ori on M such that $C \subset V$. Define

$$\boxed{\int_{M,\alpha} \omega := \int_{\operatorname{int} \mathcal{U}} \alpha^* \omega}$$

$$\operatorname{int} \mathcal{U} = \begin{cases} \mathcal{U} & \text{if } \mathcal{U} \overset{\text{open}}{\subset} \mathbb{R}^k \\ \mathcal{U} \cap \mathbb{H}^k_+ & \text{if } \mathcal{U} \overset{\text{open}}{\subset} \mathbb{H}^k \end{cases}.$$

<u>Rmk</u>: $\int_{M,\alpha} \omega$ is independent of coord chart, as long as its chosen from the given orientation on M!

Recall

$$\int_{M,\alpha} \omega = \varepsilon \int_{M,\beta} \omega$$

from 2 lectures ago; the sign $\varepsilon = \det D(\text{transition fcn})$ is $+1$

if α, β both belong to the same ori on M, by definition.

Motivated by this we write $\int_M \omega$ instead of $\int_{M,\alpha} \omega$.

In general we now use a partition of unity:

<u>Def</u> M compact, oriented k-manifold in $\mathbb{R}^n$. Choose partition of unity $\{\phi_i\}_{i=1}^N$ wrt the cover of M determined by coord charts in the given orientation on M.

Def
$$\boxed{\int_M \omega = \sum_{i=1}^N \int_M \phi_i \omega}$$

Rmk

① This agrees with the previous definition when the support of ω is covered by a single coord chart.

② The definition is independent of choice of partition of unity. The proof is the same as previously seen for $\int_M f\, dV$

Thm: M cpt oriented k-mfd in $\mathbb{R}^n$.

$\omega, \eta \in \Omega^k(A)$, $M \subset A \overset{\text{open}}{\subset} \mathbb{R}^n$

then

① $\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta$

② If $-M$ denotes M with the opposite orientation, then

$$\int_{-M} \omega = -\int_{M} \omega.$$

<u>Proof</u>: ② $\alpha : U \to V$ and the opposite coord chart

$$\alpha \circ r : U \longrightarrow V$$

$$r : \mathbb{R}^k \longrightarrow \mathbb{R}^k$$
$$(x_1, \dots, x_k) \mapsto (-x_1, x_2, \dots, x_k)$$

overlap negatively, so

$$\boxed{\int_{M,\alpha} \omega = -\int_{M,\alpha \circ r} \omega.}$$

If α belongs to one ori of M $\alpha \circ r$ always belongs to the other.

$$\int_M \omega = \sum_{i=1}^{N} \int_{M,\alpha_i} \phi_i \omega = \sum_{i=1}^{N} - \int_{M,\alpha_i \circ r} \phi_i \omega$$

$$= - \int_{-M} \omega$$

$\square$

Stokes' theorem $\boxed{\S 37}$

$I^k := [0,1]^k \subset \mathbb{R}^k$ unit k-cube

$\operatorname{int} I^k = (0,1)^k \subset \mathbb{R}^k$

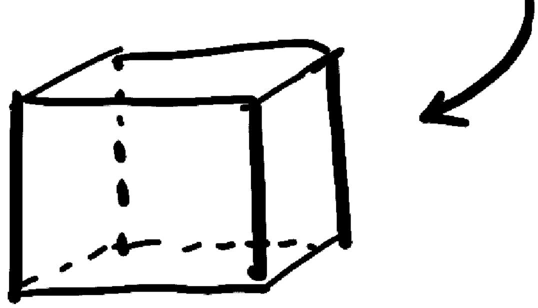
$\partial I^k = I^k \setminus \operatorname{int} I^k$.

Lma: Let $k > 1$. $I^k \subset U \overset{\text{open}}{\subset} \mathbb{R}^k$

$\eta \in \Omega^{k-1}(U)$ and assume

$\eta(\vec{x}) = 0$ at all $\vec{x} \in \partial I^k$ except possibly at points of

$(\operatorname{int} I^{k-1}) \times \{0\}$ ← interior of one particular boundary face



Thm

$$\boxed{\int_{\text{int } I^k} d\eta = (-1)^k \int_{\text{int } I^{k-1}} b^*\eta}$$

Where $b: I^{k-1} \longrightarrow I^k$

$(x_1,\dots,x_{k-1}) \mapsto (x_1,\dots,x_{k-1},0)$

<u>Proof</u>: $\vec{x} \in \mathbb{R}^k$, $\bar{u} \in \mathbb{R}^{k-1}$

$j \in \{1,\dots,k\}$

$I_j := (1,\dots,\hat{j},\dots,k) = (1,\dots,j-1,j+1,\dots,k)$

Basis for $A^{k-1}(T_{\vec{x}}\mathbb{R}^k)$ is given by

$$\{dx_{I_j} = dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k\}_{j=1}^{k}$$

We prove it for $\eta = f\,dx_{I_j}$ for some $j \in \{1,\dots,k\}$.

Step 1: Let's compute $\int_{\text{int } I^k} d\eta$.

$$d\eta = df \wedge dx_{I_j}$$
$$= \sum_{i=1}^{k} \frac{\partial f}{\partial x_i} dx_i \wedge dx_{I_j}$$
$$= \frac{\partial f}{\partial x_j} dx_j \wedge dx_{I_j}$$
$$= (-1)^{j-1} \frac{\partial f}{\partial x_j} dx_1 \wedge \cdots \wedge dx_k.$$

$$\int_{\text{int } I^k} d\eta = \int_{\text{int } I^k} (-1)^{j-1} \frac{\partial f}{\partial x_j} = (-1)^{j-1} \int_{I^k} \frac{\partial f}{\partial x_j}$$

$$\underset{\text{Fubini}}{=} (-1)^{j-1} \int_{I^{k-1}} \int_{x_j \in I} \frac{\partial f}{\partial x_j}(x_1, \ldots, x_k)$$

Position j

$$\underset{\substack{\text{Fund.}\\ \text{thm}\\ \text{of}\\ \text{Calculus}}}{=} (-1)^{j-1} \int_{I^{k-1}} \Big[f(x_1, \ldots, 1, \ldots, x_k) - f(x_1, \ldots, 0, \ldots, x_k) \Big](t)$$

This $I^{k-1} \ni (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k)$

Recall assumption that

$\eta(\vec{x}) = f(\vec{x})\, dx_{I_j} = 0$

unless $\vec{x} \in (\text{int } I^{k-1}) \times \{0\}$.

It follows that

$(+) = 0$ if $j<k$.

If $j=k$

$$(+) = (-1)^{k-1} \int_{I^{k-1}} -f(x_1, \ldots, x_{k-1}, 0)$$

CLAIM

$$= (-1)^k \int_{I^{k-1}} f \circ b \overset{?}{=} (-1)^k \int_{I^{k-1}} b^* \eta$$

We'll continue w/ the proof of the claim next time.