MAT 322/523 Lec 23 M Apr 22

<u>Recall</u>: • M k-mfd in $\mathbb{R}^n$

such that $\exists A \overset{\text{open}}{\subset} \mathbb{R}^k$ and

$\alpha : A \to \mathbb{R}^n$, $\alpha(A) = M$

$\omega \in \Omega^k(M)$

$$\boxed{\int_{M,\alpha} \omega := \int_A \alpha^* \omega}$$

• $A, B \overset{\text{open}}{\subset} \mathbb{R}^k$, $g : A \to B$ diffeo

$\alpha : A \to \mathbb{R}^n$, $\alpha = \beta \circ g$

$\beta : B \to \mathbb{R}^n$

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "g", "\cong"'] & \mathbb{R}^n \\ B \arrow[ur, "\beta"'] & \end{tikzcd}$$

$$\boxed{\int_{M,\alpha} \omega = \varepsilon \int_{M,\beta} \omega}$$

$\varepsilon = \text{sign} \det Dg$

• g is <u>orientation-preserving</u> if $\varepsilon = +1$

• g is <u>orientation-reversing</u> if $\varepsilon = -1$

Def: Let M be a k-mfd in $\mathbb{R}^n$.
We say that M is orientable if for any two coord charts

$$\alpha_i : U_i \longrightarrow V_i, \quad U_i \overset{\text{open}}{\subset} \mathbb{R}^k, \quad V_i \overset{\text{open}}{\subset} M$$

$i = 0, 1$

with $V_0 \cap V_1 \neq \emptyset$, the coordinate change

$$\alpha_1^{-1} \circ \alpha_0 : U_0 \longrightarrow U_1$$

is orientation-preserving.
If not, M is non-orientable.

An orientation on M is a choice of a collection of coord charts $\{\alpha_i : U_i \longrightarrow V_i\}_{i \in J}$ covering M w/ all ori-preserving coord. changes.

We now describe the def of an orientation for k-mfds where $k = 1, n-1, n$.

<u>Def</u>: Let M be an oriented 1-manifold in $\mathbb{R}^n$.
Define a unit tangent as follows: choose a coord chart
$\alpha : U \to V$ around $p \in M$
belonging to the chosen orientation

$$T(p) := \left(p, \frac{D\alpha(t_0)}{\|D\alpha(t_0)\|} \right) \in T_pM$$

t_0 - parameter st $\alpha(t_0) = p$.

<u>Rmk</u> • This is just the normalization of the velocity tangent vector field

- An orientation on a 1-mfld is the same as the choice of a <u>direction</u>.
- Any 1-manifold is <u>orientable</u>.

It's clear that there are at <u>least</u> two "different" orientations:

exactly if connected

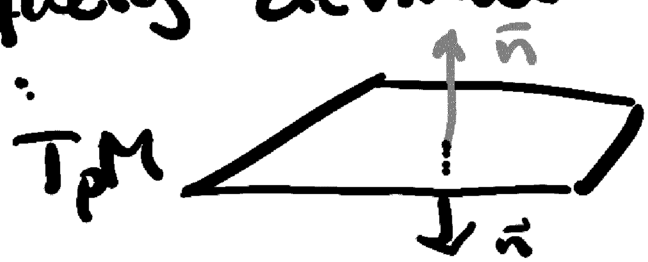
$\longrightarrow$ or $\longleftarrow$

For $(n-1)$-manifolds we instead define a unit <u>normal</u> vector field

<u>Def</u> $M = (n-1)$-mfd in $\mathbb{R}^n$ pick $(p, \vec{n}) \in T_p\mathbb{R}^n$ unit vector orthogonal to the $(n-1)$-dim linear subspace

$$T_pM \subset T_p\mathbb{R}^n$$

[$(p,\vec{n})$ is uniquely defined up to sign:



]

Pick an orientation on M & a coord chart $\alpha : U \to V$ around $p \in M$, $x \in U$: $\alpha(x) = p$

A basis for T_pM is

$$\left\{ \frac{\partial \alpha}{\partial x_1}, \ldots, \frac{\partial \alpha}{\partial x_{n-1}} \right\}.$$

Specify a sign for $\vec{n}$ by declaring

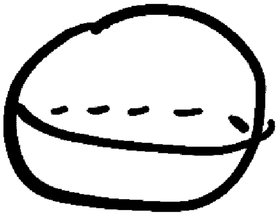
$$\left(\vec{n}, \frac{\partial \alpha}{\partial x_1}, \ldots, \frac{\partial \alpha}{\partial x_{n-1}} \right)$$ to

be right-handed.

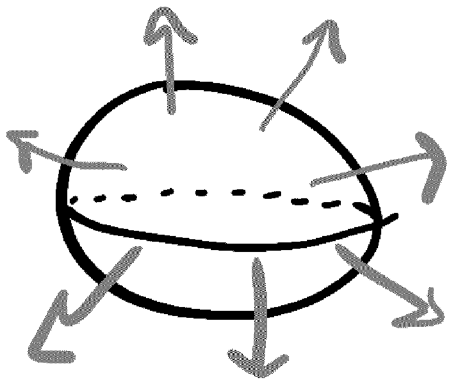
Meaning that the matrix

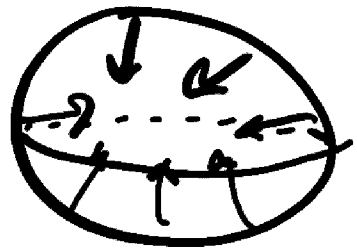
$$\left(\vec{n} \;\; D\alpha(\bar{x}) \right)$$ should

have positive determinant.

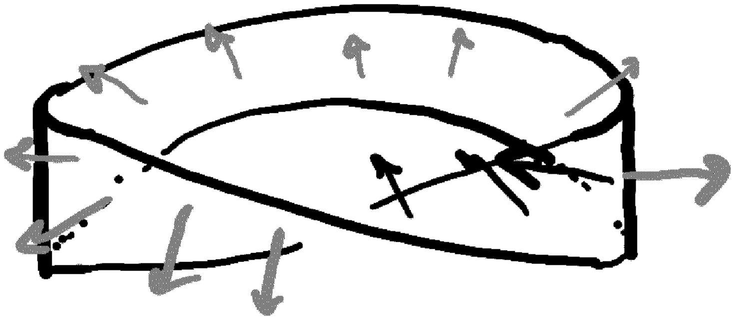
Ex:  admits <u>two</u> orientations

$S^2 \subset \mathbb{R}^3$



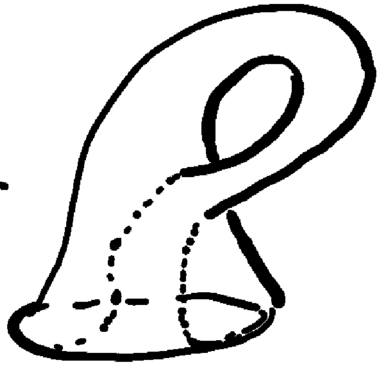


Ex: Not every $(n-1)$-manifold is orientable.



Möbius strip is non-orientable because we can not define a unit normal vector field consistently.

Ex: The Klein bottle is also non-orientable for the same reason.

$K =$ 

Namely, choosing a normal direction to K at $x \in K$ can not be done consistently.

Def: Let M be an n-manifold in $\mathbb{R}^n$. Given a collection of coord charts $A = \{\alpha_i : U_i \to V_i\}_i$ covering M, the natural orientation wrt A is given by

$$\{\alpha_i : U_i \to V_i \mid \det D\alpha_i > 0\}$$

$$\cup \{\alpha_i \circ r : U_i \to V_i \mid \det D\alpha_i < 0\}$$

where
$$r : \mathbb{R}^n \to \mathbb{R}^n$$
$$(x_1, x_2, \dots, x_n) \mapsto (-x_1, x_2, \dots, x_n)$$

Rmk: ① Note that $\det D(\alpha_i \circ r) > 0$ since $\det Dr = \det \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = -1$

$$\det D(\alpha_i \circ r) = (\det D\alpha_i)(\det Dr) > 0$$

② The map r is the reflection in the first coordinate and used to "reverse" the orientation of those coord charts w/ the "wrong" orientation.

Def: M k-mfd in $\mathbb{R}^n$ with orientation $\{\alpha_i : U_i \to V_i\}$.

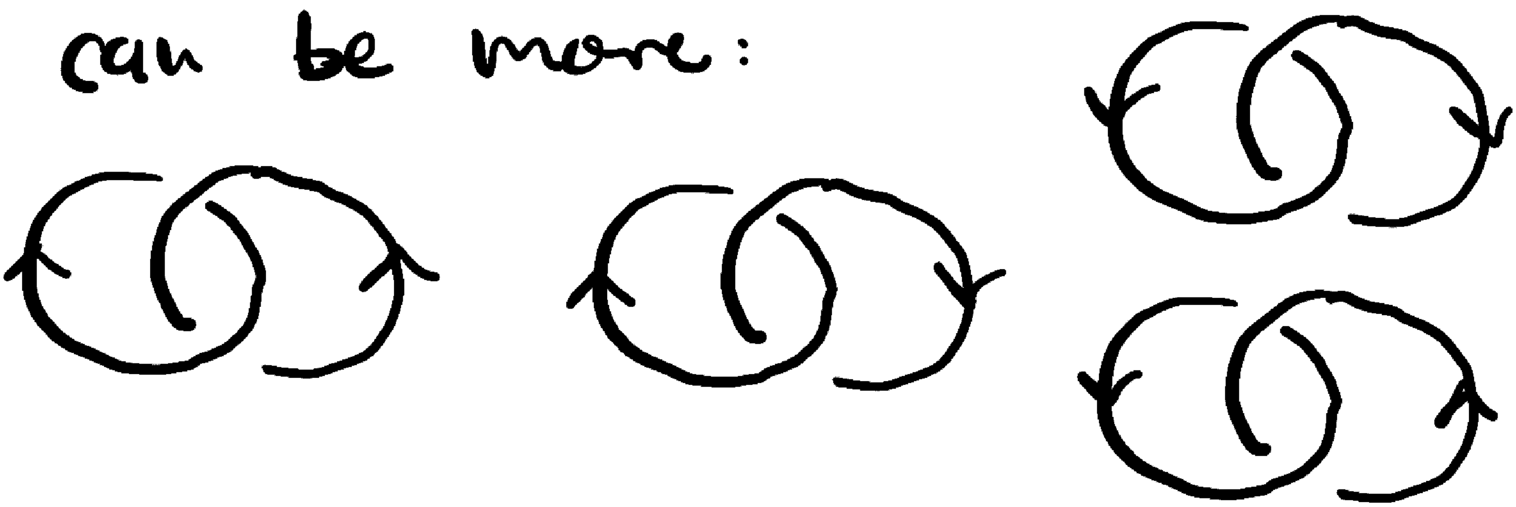
The reverse orientation is given by $\{\alpha_i \circ r : U_i \to V_i\}$ where

$$r : \mathbb{R}^k \to \mathbb{R}^k$$
$$(x_1, x_2, \dots, x_k) \mapsto (-x_1, x_2, \dots, x_k)$$

Rmk: α_i overlaps negatively w/ $\alpha_i \circ r$, but if α, β overlaps positively then so does $\alpha \circ r$ and $\beta \circ r$. (Note $r^{-1} = r$.)

A consequence of this is that there are at least two orientations on any k-manifold in $\mathbb{R}^n$, since reversing a given one gives a "new" orientation.

If M is connected there's exactly two. If M is disconnected there can be more:



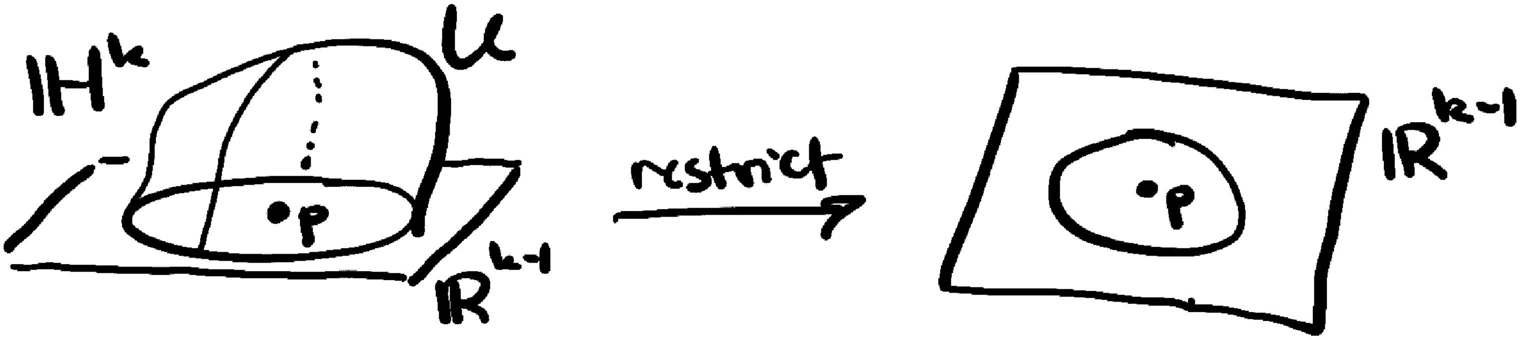
orientations

$$= 2^{\#\text{conn. components}}$$

Manifolds with boundary

Thm: Let $k>1$. If M is an orientable k-manifold with $\partial M \neq \emptyset$, then ∂M is orientable.

Proof: Recall the proof of the thm that ∂M is a $(k-1)$-mfd: Take an atlas of M (collection of coord charts covering M) and restrict charts at points $x \in \partial M$



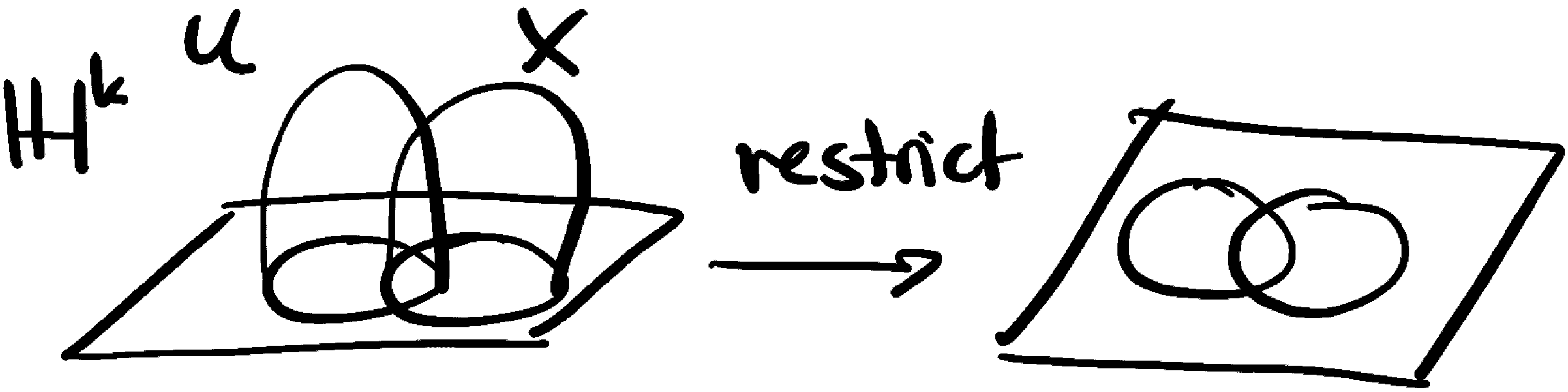
$x = \alpha(p) \in \partial M$

Now consider $\beta: X \longrightarrow Y$ that overlaps positively w/ $\alpha: U \to V$ (both are coord charts at $x \in \partial M$)

meaning the transition function

$$g := \beta^{-1} \circ \alpha : U \cap X \longrightarrow U \cap X$$

satisfies $\det Dg > 0$.



$$\mathbb{H}^k = \{(x_1, \dots, x_k) \mid x_k \geq 0\}$$

$$\partial \mathbb{H}^k = \{(x_1, \dots, x_{k-1}, 0)\} \cong \mathbb{R}^{k-1}$$

If $p \in \partial \mathbb{H}^k$ then

$$g = (g_1, \dots, g_k) : U \cap X \longrightarrow U \cap X$$

Note since $g(U \cap X \cap \partial \mathbb{H}^k)$

$$\subset \partial M$$

it follows that $g_k = g_k(x_k)$

is independent of $x_1, \dots, x_{k-1}$.

Also $\frac{\partial g_k}{\partial x_k} > 0$. Now observe

$$Dg = \left(\begin{array}{c|c} \frac{\partial(g_1,\dots,g_{k-1})}{\partial(x_1,\dots,x_{k-1})} & * \\ \hline 0 & \frac{\partial g_k}{\partial x_k} \end{array}\right)$$

$$\Rightarrow \det Dg = \left(\det \frac{\partial(g_1,\dots,g_{k-1})}{\partial(x_1,\dots,x_{k-1})}\right) \underbrace{\frac{\partial g_k}{\partial x_k}}_{>0}$$

$$\overset{>0}{\Rightarrow} \det \frac{\partial(g_1,\dots,g_{k-1})}{\partial(x_1,\dots,x_{k-1})} > 0$$

which is the det of the Jacobian of the transition function of the restricted coord charts (= those coord charts in ∂M)

$\square$