

Recall:

$\omega \in \Omega^k(M)$  then

$d\omega \in \Omega^{k+1}(M)$

$\omega = \sum_{\substack{I \\ \text{ascending}}} f_I dx_I$

$d\omega = \sum_I df_I \wedge dx_I$   $\swarrow$  order of  $\omega$

•  $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$

•  $\alpha^*(\omega \wedge \eta) = \alpha^*\omega \wedge \alpha^*\eta$

•  $\alpha^*(d\omega) = d(\alpha^*\omega)$

§33 Integrating forms over manifolds

$A \subset \mathbb{R}^k$ ,  $\eta \in \Omega^k(A)$

Remember (e.g. from midterm) that  $A$  is then a  $k$ -manifold.  
( $A \xrightarrow{\text{id}} A$  is a coord chart)

we have

$$\eta = f dx_1 \wedge \dots \wedge dx_k$$

$$f: \mathbb{R}^k \rightarrow \mathbb{R} \text{ smooth.}$$

we define

$$\int_A \eta := \int_A f$$

provided that  $\int_A f$  exists.

---

Def.: Let  $A \subset \mathbb{R}^k$  open

$$\alpha: A \rightarrow \mathbb{R}^n \text{ smooth.}$$

Assume  $M$  is a  $k$ -manifold in  $\mathbb{R}^n$  so that  $M = \alpha(A)$ .

If  $\omega \in \Omega^k(\mathbb{R}^n)$  then

$$\int_{M, \alpha} \omega := \int_A \alpha^* \omega$$


---

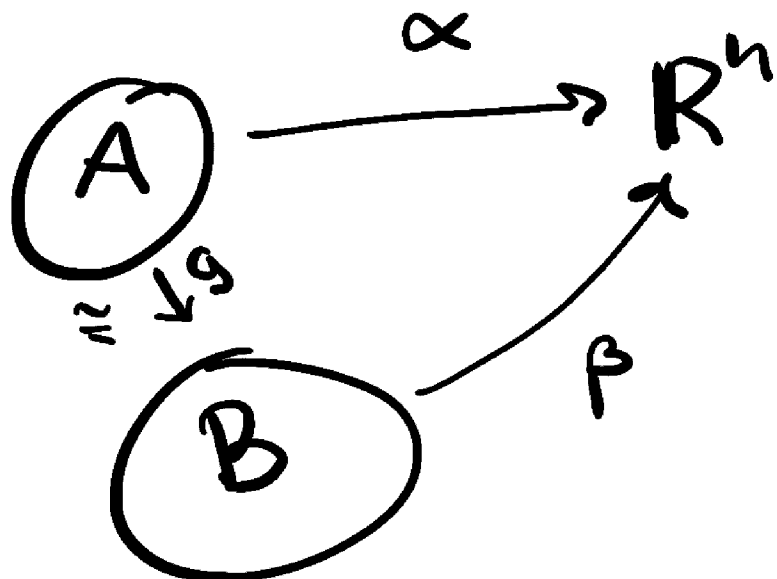
Thm  $A, B \subset \mathbb{R}^k$  open  $g: A \rightarrow B$   
 diffeo such that  $\det Dg$  has  
 the same sign on all of  $A$ .

$$\beta: B \rightarrow \mathbb{R}^n$$

$$\alpha = \beta \circ g$$

$$M := \beta(B)$$

$$\omega \in \Omega^k(\mathbb{R}^n)$$



$$\int_{M, \alpha} \omega = \pm \int_{M, \beta} \omega$$

Sign  $\det Dg$

Proof:  $x = (x_1, \dots, x_k) \in A \subset \mathbb{R}^k$   
 $y = (y_1, \dots, y_k) \in B \subset \mathbb{R}^k$

By def we want to show

$$\int_A \alpha^* \omega = \varepsilon \int_B \beta^* \omega$$

$$\varepsilon = \text{sgn det } Dg.$$

$\alpha = \beta \circ g$ . Letting  $\eta := \beta^* \omega$  the  
equ becomes

$$\int_A g^* \eta = \varepsilon \int_B \eta.$$

Write  $\eta = f \, dy_1 \wedge \dots \wedge dy_k$ , then

$$\begin{aligned} g^* \eta &= (f \circ g) g^*(dy_1 \wedge \dots \wedge dy_k) \\ &= (f \circ g) \cdot \det Dg \, dx_1 \wedge \dots \wedge dx_k \end{aligned}$$

So

$$\int_A g^* \eta = \int_A (f \circ g) \det Dg$$
$$= \int_A (f \circ g) \varepsilon |\det Dg| = \int_B f = \int_B \eta$$

Change of  
variables

□

---

For Computations:

Thm:  $A \subset \mathbb{R}^k$  open  $\alpha: A \rightarrow \mathbb{R}^n$  smooth

$M = \alpha(A)$ .  $x = (x_1, \dots, x_k) \in A$

$z = (z_1, \dots, z_n) \in \mathbb{R}^n$

If  $\omega = f dz_I$ ,  $I = (i_1, \dots, i_k)$ ,  $i_j \in \{1, \dots, n\}$

$$\int_{M, \alpha} \omega = \int_A (f \circ \alpha) \det \frac{\partial \alpha_I}{\partial x}$$

---

Proof:

$$\alpha^* \omega = \alpha^* (f dz_I)$$

$$= (f \circ \alpha) \det \frac{\partial x_I}{\partial x} dx_I$$

$$\int_{M, \alpha} \omega = \int_A \alpha^* \omega = \int_A (f \circ \alpha) \det \frac{\partial x_I}{\partial x}$$

□

Rmk Calculus •  $A = (a, b)$

$\eta = f dx$  then

$$\int_A \eta = \int_A f = \int_a^b f.$$

$P, Q, R: \mathbb{R}^3 \rightarrow \mathbb{R}$

•  $\eta = P dx + Q dy + R dz \in \Omega^1(\mathbb{R}^3)$

$\gamma: (a, b) \rightarrow \mathbb{R}^3$  smooth curve

$C := \gamma(a, b)$

$$\int_{C, \gamma} P dx + Q dy + R dz$$

$$= \int_{(a, b)} \gamma^* (P dx + Q dy + R dz)$$

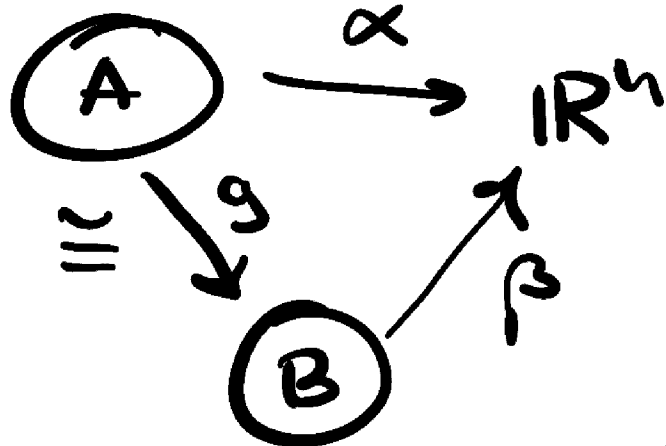
$$\begin{aligned}
&= \int_a^b (P_0 \delta) d\delta_1 + (Q_0 \delta) d\delta_2 + (R_0 \delta) d\delta_3 \\
&= \int_a^b (P_0 \delta) \frac{d\delta_1}{dt} dt + (Q_0 \delta) \frac{d\delta_2}{dt} dt \\
&\quad + (R_0 \delta) \frac{d\delta_3}{dt} dt \\
&= \int_a^b P(\delta(t)) \frac{d\delta_1}{dt} + Q(\delta(t)) \frac{d\delta_2}{dt} \\
&\quad + R(\delta(t)) \frac{d\delta_3}{dt}
\end{aligned}$$


---

Remember

$$\int_{M, \alpha} \omega = \epsilon \int_{M, \beta} \omega$$

$$\epsilon = \text{Sign det } Dg$$



Def:  $g: A \rightarrow B$  diffeo  $A, B \subset \mathbb{R}^k$  <sup>open</sup>

•  $g$  is orientation-preserving if  $\varepsilon = +1$

•  $g$  is orientation-reversing if  $\varepsilon = -1$