Recall:

$\omega \in \Omega^k(M)$ then

$d\omega \in \Omega^{k+1}(M)$

$$\omega = \sum_{\substack{I \\ \text{ascending}}} f_I \, dx_I$$

$$d\omega = \sum_I df_I \wedge dx_I$$

order of ω

- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
- $\alpha^*(\omega \wedge \eta) = \alpha^*\omega \wedge \alpha^*\eta$
- $\alpha^*(d\omega) = d(\alpha^*\omega)$

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§33 Integrating forms over manifolds

$A \overset{\text{open}}{\subset} \mathbb{R}^k$, $\eta \in \Omega^k(A)$

Remember (e.g. from midterm) that A is then a k-manifold.
($A \xrightarrow{id} A$ is a coord chart)

We have

$$\eta = f\, dx_1 \wedge \cdots \wedge dx_k$$

$f: \mathbb{R}^k \to \mathbb{R}$ smooth.

We define

$$\int_A \eta := \int_A f$$

provided that $\int_A f$ exists.

<u>Def</u>: Let $A \subset \mathbb{R}^k$ open
$\alpha: A \to \mathbb{R}^n$ smooth.
Assume M is a k-manifold in $\mathbb{R}^n$ so that $M = \alpha(A)$.
If $\omega \in \Omega^k(\mathbb{R}^n)$ then

$$\int_{M,\alpha} \omega := \int_A \alpha^* \omega$$

<u>Thm</u> $A, B \overset{\text{open}}{\subset} \mathbb{R}^k$ $g: A \to B$ diffeo such that $\det Dg$ has the same sign on all of A.

$\beta : B \to \mathbb{R}^n$

$\alpha = \beta \circ g$

$$\begin{tikzcd}
A \arrow[rr, "\alpha"] \arrow[d, "g"', "\cong"] & & \mathbb{R}^n \\
B \arrow[urr, bend right, "\beta"'] & &
\end{tikzcd}$$

$M := \beta(B)$

$\omega \in \Omega^k(\mathbb{R}^n)$

$$\int_{M,\alpha} \omega = \pm \int_{M,\beta} \omega$$

$\boxed{\text{sign} \det Dg}$

Proof: $x=(x_1,\dots,x_k)\in A\subset\mathbb{R}^k$
$y=(y_1,\dots,y_k)\in B\subset\mathbb{R}^k$

By def we want to show

$$\int_A \alpha^*\omega = \varepsilon\int_B \beta^*\omega$$

$\varepsilon = \text{sign}\det Dg.$

$\alpha=\beta\circ g$. Letting $\eta:=\beta^*\omega$ then eqn becomes

$$\int_A g^*\eta = \varepsilon\int_B \eta.$$

Write $\eta = f\, dy_1\wedge\cdots\wedge dy_k$, then

$$g^*\eta = (f\circ g)\, g^*(dy_1\wedge\cdots\wedge dy_k)$$
$$= (f\circ g)\cdot\det Dg\; dx_1\wedge\cdots\wedge dx_k$$

So

$$\int_A g^*\eta = \int_A (f\circ g)\det Dg$$

$$= \int_A (f\circ g)\,\varepsilon\,|\det Dg| = \int_B f = \int_B \eta$$

Change of variables

$\square$

For computations:

<u>Thm</u>: $A \overset{\text{open}}{\subset} \mathbb{R}^k$ $\quad \alpha : A \to \mathbb{R}^n$ smooth

$M = \alpha(A)$. $\quad x = (x_1, \dots, x_k) \in A$

$z = (z_1, \dots, z_n) \in \mathbb{R}^n$

If $\omega = f\, dz_I$, $I = (i_1, \dots, i_k)$, $i_j \in \{1, \dots, n\}$

$$\int_{M,\alpha} \omega = \int_A (f\circ\alpha)\det \frac{\partial \alpha_I}{\partial x}$$

<u>Proof</u>:

$$\alpha^*\omega = \alpha^*(f\, dz_I)$$

$$= (f\circ\alpha)\det\frac{\partial\alpha_I}{\partial x}\,dx_I$$

$$\int_{M,\alpha}\omega = \int_A \alpha^*\omega = \int_A (f\circ\alpha)\det\frac{\partial\alpha_I}{\partial x}$$

$\square$

<u>Rmk</u> <u>Calculus</u> • $A=(a,b)$

$\eta = f\,dx$ then

$$\int_A \eta = \int_A f = \int_a^b f.$$

$P, Q, R : \mathbb{R}^3 \to \mathbb{R}$

• $\eta = P\,dx + Q\,dy + R\,dz \in \Omega^1(\mathbb{R}^3)$

$\gamma : (a,b) \to \mathbb{R}^3$ smooth curve

$C := \gamma(a,b)$

$$\int_{C,\gamma} P\,dx + Q\,dy + R\,dz$$

$$= \int_{(a,b)} \gamma^*(P\,dx + Q\,dy + R\,dz)$$

$$= \int_a^b (P\circ\gamma)\,d\gamma_1 + (Q\circ\gamma)\,d\gamma_2 + (R\circ\gamma)\,d\gamma_3$$

$$= \int_a^b (P\circ\gamma)\frac{d\gamma_1}{dt}dt + (Q\circ\gamma)\frac{d\gamma_2}{dt}dt + (R\circ\gamma)\frac{d\gamma_3}{dt}dt$$

$$= \int_a^b P(\gamma(t))\frac{d\gamma_1}{dt} + Q(\gamma(t))\frac{d\gamma_2}{dt} + R(\gamma(t))\frac{d\gamma_3}{dt}$$

Remember

$$\int_{M,\alpha} \omega = \varepsilon \int_{M,\beta} \omega$$

$$\varepsilon = \text{sign}\det Dg$$

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "g", "\cong"'] & \mathbb{R}^n \\ B \arrow[ur, "\beta"'] & \end{tikzcd}$$

<u>Def</u>: $g: A \to B$ diffeo $A, B \overset{\text{open}}{\subset} \mathbb{R}^k$

- g is <u>orientation-preserving</u> if $\varepsilon = +1$
- g is <u>orientation-reversing</u> if $\varepsilon = -1$