MAT 322/523 Lec 21 M Apr 15

<u>Recall</u>. • A k-form ω on M is a smooth function

$$\omega : M \to \Omega^k(M)$$

$: \omega(\vec{x}) \in A^k(T_{\vec{x}}M)$ ↰ bundle of k-forms

$$A^k(M) = \bigcup_{\vec{x}\in M} A^k(T_{\vec{x}}M)$$

$\Omega^k(M)$ $= \{\text{k-forms on } M\}$

set of alt k-tensors on $T_{\vec{x}}M$

• Can take wedge & exterior derivatives:

$$\omega \in \Omega^k(M),\ \eta \in \Omega^\ell(M)$$

$$\omega \wedge \eta \in \Omega^{k+\ell}(M),\quad d\omega \in \Omega^{k+1}(M)$$

$\omega = \sum_{I \text{ ascending}} f_I \, dx_I$ then

$$d\omega = \sum_{I \text{ asc}} \sum_{i=1}^{n} \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I$$

Let's now generalize operations on vector/scalar fields in $\mathbb{R}^3$ to $\mathbb{R}^n$:

<u>Def</u>: $A \overset{\text{open}}{\subset} \mathbb{R}^n$, $f: A \to \mathbb{R}$ smooth. The gradient of f is

$$(\text{grad } f)(\vec{x}) = \nabla f(\vec{x})$$
$$= (\vec{x}, D_1 f(\vec{x}) e_1 + \cdots + D_n f(\vec{x}) e_n)$$

<u>Note</u>: This is a vector field & $\nabla f(\vec{x}) \in T_{\vec{x}} \mathbb{R}^n$ $\forall \vec{x} \in A$.

<u>Def</u>: $A \overset{\text{open}}{\subset} \mathbb{R}^n$: $G(\vec{x}) = (\vec{x}, g(\vec{x}))$ where $g: A \to \mathbb{R}^n$ is given by
$$g(\vec{x}) = (g_1(\vec{x}), \ldots, g_n(\vec{x})),$$ then

its divergence is

$$(\text{div } G)(\vec{x}) = (\nabla \cdot G)(\vec{x})$$

$= D_1 g_1(\vec{x}) + \cdots + D_n g_n(\vec{x})$.

<u>Note</u>: $\operatorname{div} G : A \to \mathbb{R}$ smooth function

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<u>Rmk</u>: There's <u>no</u> generalization of curl beyond $n=3$ in general because there's no "cross product" outside $n=3$.

[Actually there's something for $n=7$ but we ignore it.]

<u>Thm</u>: Let $A \overset{\text{open}}{\subset} \mathbb{R}^n$. There exists vector space isomorphisms α_i, β_j and diagrams

$$\begin{array}{ccc} C^\infty(A) & \xrightarrow{\alpha_0} & \Omega^0(A) \\ \downarrow \text{grad} & & \downarrow d \\ \text{vector fields in } A & \xrightarrow{\alpha_1} & \Omega^1(A) \end{array}$$

$$\begin{tikzcd}
\text{vector fields in } A \arrow[r, "\beta_{n-1}"] \arrow[d, "\mathrm{div}"] & \Omega^{n-1}(A) \arrow[d, "d"] \\
C^\infty(A) \arrow[r, "\beta_n"] & \Omega^n(A)
\end{tikzcd}$$

so that

$$\begin{cases} d\circ\alpha_0 = \alpha_1\circ \mathrm{grad} \\ d\circ\beta_{n-1} = \beta_n\circ \mathrm{div} \end{cases}$$

<u>Rmk</u>: Recall $\dim A^k(V) = \binom{n}{k}$ for an n-dim vector space V,

so $\dim A^k(V) = \binom{n}{k} = \binom{n}{n-k}$

$= \dim A^{n-k}(V)$

which means $A^k(V)$ and $A^{n-k}(V)$ are isomaphic as vector spaces.

This is the reason why

$$\Omega^k(A) \cong \Omega^{n-k}(A).$$

<u>Proof</u>: Let $f, g \in C^\infty(A)$

let

$$F(\vec{x}) = \left(\vec{x}, \sum_{i=1}^{n} f_i(\vec{x}) e_i\right)$$

$$G(\vec{x}) = \left(\vec{x}, \sum_{i=1}^{n} g_i(\vec{x}) e_i\right)$$

be vector fields.

Define
$$\begin{cases} \alpha_0 f := f \\ \alpha_1 F := \sum_{i=1}^{n} f_i \, dx_i \end{cases}$$

this indicates the factor that is omitted

$$\begin{cases} \beta_{n-1} G := \sum_{i=1}^{n} (-1)^{i-1} g_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\ \beta_n h := h \, dx_1 \wedge \cdots \wedge dx_n . \end{cases}$$

$$\alpha_1(\nabla f) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} dx_i$$

$$\| $$

$$d(\alpha_0 f) = df$$

Also $(\beta_n \circ \operatorname{div})G$

$$= \beta_n\left(\sum_{i=1}^{n} D_i g_i(\vec{x})\right)$$

$$= \left(\sum_{i=1}^{n} D_i\, g_i(\vec{x})\right) dx_1 \wedge \cdots \wedge dx_n$$

and

$$d(\beta_{n-1}G) = d\left[\sum_{i=1}^{n} (-1)^{i-1} g_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\right]$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial g_i}{\partial x_i} dx_i \wedge \left(dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\right)$$

$$= \left(\sum_{i=1}^{n} D_i g_i(\vec{x})\right) dx_1 \wedge \cdots \wedge dx_n$$

$\square$

<u>Def</u> $A \overset{\text{open}}{\subset} \mathbb{R}^3$, $F(\vec{x}) = \left(\vec{x}, \sum_{i=1}^{n} f_i(\vec{x}) e_i\right)$ vector field in A. Define another vector field in A by

$$(\operatorname{curl} F)(\vec{x}) = (\nabla \times F)(\vec{x})$$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ D_1 & D_2 & D_3 \\ f_1 & f_2 & f_3 \end{vmatrix}$$

For $n=3$ we can connect all three operations:

<u>Thm</u>:

$$\begin{tikzcd}
C^\infty(A) \arrow[r, "\text{grad}"] \arrow[d, "\alpha_0"'] & \text{v.f. in } A \arrow[r, "\text{curl}"] \arrow[d, "\alpha_1"'] & \text{v.f. in } A \arrow[r, "\text{div}"] \arrow[d, "\beta_2"'] & C^\infty(A) \arrow[d, "\beta_3"'] \\
\Omega^0(A) \arrow[r] & \Omega^1(A) \arrow[r] & \Omega^2(A) \arrow[r] & \Omega^3(A)
\end{tikzcd}$$

commutative diagram.

<u>Def</u>: $A \overset{\text{open}}{\subset} \mathbb{R}^k$, $\alpha: A \to \mathbb{R}^n$ smooth
$B \overset{\text{open}}{\subset} \mathbb{R}^n$: $\alpha(A) \subset B$ define
$\alpha^*: \Omega^l(B) \to \Omega^l(A)$ as follows:
if $\omega \in \Omega^l(B)$ then

$$(\alpha^*\omega)(\vec{x})\big((\vec{x},\vec{v}_1),\dots,(\vec{x},\vec{v}_\ell)\big)$$
$$=\omega(\alpha(\vec{x}))\big(\alpha_*(\vec{x},v_1),\dots,\alpha_*(\vec{x},v_\ell)\big)$$
$$=\omega(\alpha(\vec{x}))\big((\alpha(\vec{x}),D\alpha(\vec{x})\vec{v}_1),\dots,(\alpha(\vec{x}),D\alpha(\vec{x})\cdot\vec{v}_\ell)\big)$$

or without referring to the base-points:

$$(\alpha^*\omega)(v_1,\dots,v_\ell) = \omega(D\alpha\cdot v_1,\dots,D\alpha\cdot v_\ell)$$

the "pullback" of ω along α.

<u>Thm</u>: $A \overset{\text{open}}{\subset} \mathbb{R}^k$, $B \overset{\text{open}}{\subset} \mathbb{R}^m$

$\alpha: A \to \mathbb{R}^m$, $\alpha(A)\subset B$

$\beta: B \to \mathbb{R}^n$

$\omega,\eta \in \Omega^k(\mathbb{R}^n)$ $\quad \theta\in\Omega^\ell(\mathbb{R}^n)$,

(1) $\beta^*(a\omega+b\eta) = a\beta^*\omega + b\beta^*\eta$

(2) $\beta^*(\omega\wedge\theta) = \beta^*\omega\wedge\beta^*\theta$

(3) $(\beta \circ \alpha)^* \omega = \alpha^*(\beta^* \omega)$

<u>Proof</u> Reformulations of earlier properties of the linear transformation

$$T^* : A^k(W) \longrightarrow A^k(V) \quad \text{where}$$

$$T : V \to W$$

$\square$

<u>Thm</u> $A \overset{\text{open}}{\subset} \mathbb{R}^k$, $\quad \alpha : A \longrightarrow \mathbb{R}^n$ smooth

$x = (x_1, \dots, x_k) \in \mathbb{R}^k \quad \alpha = (\alpha_1, \dots, \alpha_n)$

$y = (y_1, \dots, y_n) \in \mathbb{R}^n$

(a) $\alpha^*(dy_i) = d\alpha_i$

(b) $I = (i_1, \dots, i_k)$ ascending

$i_j \in \{1, \dots, n\} \ \forall j$, then

$$\alpha^*(dy_I) = \left(\det \frac{\partial \alpha_I}{\partial x}\right) dx_1 \wedge \dots \wedge dx_k$$

$$\frac{\partial \alpha_I}{\partial x} = \frac{\partial(\alpha_{i_1}, \dots, \alpha_{i_k})}{\partial(x_1, \dots, x_k)}$$

$$= \begin{pmatrix} \frac{\partial \alpha_{i_1}}{\partial x_1} & \cdots & \frac{\partial \alpha_{i_k}}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial \alpha_{i_k}}{\partial x_1} & \cdots & \frac{\partial \alpha_{i_k}}{\partial x_k} \end{pmatrix}$$

<u>Proof</u>: (a) By def if $(x,v) \in T_x A$

$$[\alpha^*(dy_i)]((x,v))$$

$$= dy_i((\alpha(x), D\alpha \cdot v)) = \text{i-th component of } D\alpha \cdot v$$

$$D\alpha \cdot v = \begin{pmatrix} \frac{\partial \alpha_1}{\partial x_1} & \cdots & \frac{\partial \alpha_1}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial \alpha_n}{\partial x_1} & \cdots & \frac{\partial \alpha_n}{\partial x_k} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{k} \frac{\partial \alpha_1}{\partial x_i} v_i \\ \vdots \\ \sum_{i=1}^{k} \frac{\partial \alpha_n}{\partial x_i} v_i \end{bmatrix} \quad \text{So}$$

$$dy_i((\alpha(x), D\alpha \cdot v)) = \sum_{j=1}^{k} \frac{\partial \alpha_i}{\partial x_j} v_j$$

$$= \sum_{j=1}^{k} \frac{\partial \alpha_i}{\partial x_j} dx_j((x,v))$$

$$= d\alpha_i(x,v)$$

(b) $\boxed{\alpha^*(dy_I)}$ $\quad I = (i_1, \dots, i_k)$

$i_1 < \dots < i_k \quad , i_j \in \{1, \dots, n\} \; \forall j$

is a k-form in $\mathbb{R}^k$ so it's of the form

$$\alpha^*(dy_I) = h \, dx_1 \wedge \dots \wedge dx_k$$

where $h: \mathbb{R}^k \to \mathbb{R}$ is smooth.

To find $\alpha^*(dy_I)$ we need to

calculate $h(x)$. Note

$$h(dx_1 \wedge \cdots \wedge dx_k)(\partial_{x_1}, \ldots, \partial_{x_k})$$
$$= h(x).$$ Evaluate

$\alpha^*(dy_I)$ on the same k-tuple:

$$\alpha^*(dy_I)(\partial_{x_1}, \ldots, \partial_{x_k})$$

$$= dy_I(D\alpha \cdot \partial_{x_1}, \ldots, D\alpha \cdot \partial_{x_k}) \quad (*)$$

$$\partial_{x_i} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$ Position i

$$D\alpha \cdot \partial_{x_i} = \begin{pmatrix} \partial\alpha_1/\partial x_1 & \cdots & \partial\alpha_1/\partial x_k \\ \vdots & \ddots & \vdots \\ \partial\alpha_n/\partial x_1 & \cdots & \partial\alpha_n/\partial x_k \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \partial\alpha_1/\partial x_i \\ \vdots \\ \partial\alpha_n/\partial x_i \end{pmatrix} = \sum_{j=1}^{n} \frac{\partial \alpha_j}{\partial x_i} \partial_{y_j}$$

$$(*) = dy_I\left(\left(\sum_{j=1}^{n} \frac{\partial \alpha_j}{\partial x_1} \partial y_j, \dots, \sum_{j=1}^{n} \frac{\partial \alpha_j}{\partial x_k} \partial y_j\right)\right)$$

$$= \det \begin{bmatrix} \frac{\partial \alpha_{i_1}}{\partial x_1} & \cdots & \frac{\partial \alpha_{i_1}}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \alpha_{i_k}}{\partial x_k} & \cdots & \frac{\partial \alpha_{i_k}}{\partial x_k} \end{bmatrix}$$

$$= \det \frac{\partial \alpha_I}{\partial x} = h(x)$$

$\square$

<u>Thm</u>: $\alpha : \mathbb{R}^k \to \mathbb{R}^n$ smooth
$\omega \in \Omega^l(\mathbb{R}^n)$ then

$$\alpha^*(d\omega) = d(\alpha^*\omega)$$

<u>Proof</u>: $x = (x_1, \dots, x_k) \in \mathbb{R}^k$
$y = (y_1, \dots, y_n) \in \mathbb{R}^n$

<u>Step 1</u>: Let's prove it for $l = 0$ (so $f : \mathbb{R}^n \to \mathbb{R}$ smooth)

$$\alpha^*(df) = \alpha^*\left(\sum_{j=1}^{n} \frac{\partial f}{\partial y_j} dy_j\right)$$

$$= \sum_{j=1}^{n} \frac{\partial f}{\partial y_j} \alpha^*(dy_j) = \sum_{j=1}^{n} \frac{\partial f}{\partial y_j} d\alpha_j$$

$$= \sum_{j=1}^{n} \frac{\partial f}{\partial y_j} \sum_{i=1}^{k} \frac{\partial \alpha_j}{\partial x_i} dx_i$$

$$d(\alpha^* f) = d(f \circ \alpha)$$

$$= \sum_{i=1}^{k} D_i(f \circ \alpha) \, dx_i \quad (*)$$

$$D(f \circ \alpha) = Df(y) \cdot D\alpha(x)$$

$$= \left(\frac{\partial f}{\partial y_1} \cdots \frac{\partial f}{\partial y_n}\right) \begin{pmatrix} \frac{\partial \alpha_1}{\partial x_1} & \cdots & \frac{\partial \alpha_1}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial \alpha_n}{\partial x_1} & \cdots & \frac{\partial \alpha_n}{\partial x_k} \end{pmatrix}$$

$$= \left(\sum_{j=1}^{n} \frac{\partial f}{\partial y_j} \cdot \frac{\partial \alpha_j}{\partial x_1} \;\cdots\; \sum_{j=1}^{n} \frac{\partial f}{\partial y_j} \cdot \frac{\partial \alpha_j}{\partial x_k}\right)$$

$$D_i(f \circ \alpha) = \sum_{j=1}^{n} \frac{\partial f}{\partial y_j} \cdot \frac{\partial \alpha_j}{\partial x_i}$$

$$(*) = \sum_{i=1}^{k} \sum_{j=1}^{n} \frac{\partial f}{\partial y_j} \cdot \frac{\partial \alpha_j}{\partial x_i} dx_i$$

$$= \alpha^*(df)$$

<u>Step 2</u>: Suffices to prove it for $\omega = f\, dy_I$, $I = \{i_1, \dots, i_l\}$

$i_j \in \{1, \dots, n\} \ \forall j$

for any $l > 0$.

$$d(\alpha^* \omega) = d\alpha^*(f dy_I)$$

$$= d\left(\alpha^*(f \wedge dy_I)\right)$$

$$= d\left(\alpha^* f \wedge \alpha^* dy_I\right)$$

$$= d(\alpha^* f) \wedge \alpha^* dy_I + \underbrace{(-1)^0 \alpha^* f \wedge d(\alpha^* dy_J)}_{=0}$$

since $d(\alpha^* dy_I)$

$$= \alpha^*(\underbrace{d(dy_I)}_{=0}) = 0$$

so $d(\alpha^*\omega) = d(\alpha^* f) \wedge \alpha^* dy_I$

$$= \alpha^*(df \wedge dy_I)$$

$$= \alpha^*(d(f\,dy_I)) = \alpha^* d\omega.$$

$\square$
