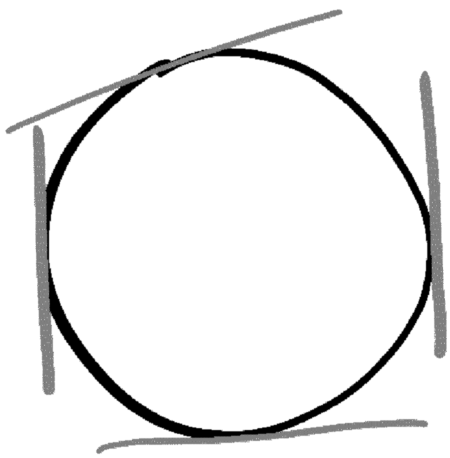
MAT 322/523 Lec 20 M Apr 8

Recall: • $T_{\vec{x}}M$ tangent space

$$TM = \bigcup_{\vec{x} \in M} T_{\vec{x}}M \quad \text{tangent bundle}$$



• A tangent vector field on M is a smooth function $F: M \to TM$ $: F(\vec{x}) \in T_{\vec{x}}M \ \forall \vec{x} \in M$.

• $A^k(T_{\vec{x}}M)$ alternating k-tensors on $T_{\vec{x}}M$.

$$\Omega^k(M) = \bigcup_{\vec{x} \in M} A^k(T_{\vec{x}}M)$$

bundle of alt. k-tensors.

• A k-form on M is a smooth function

$$\omega : M \to \Omega^k(M)$$

$: \omega(\vec{x}) \in A^k(T_{\vec{x}}M) \ \forall \vec{x} \in M.$

Ex: What is a 1-form? Vector fields are easy to visualize, but differentials are more difficult.

First consider the dual space $V^* = \mathcal{L}^1(V)$. If $f \in V^*$ and $\vec{v} \in V$, $\vec{v} = (v_1, \dots, v_n)$

$f = (f_1, \dots, f_n)$

$$f(\vec{v}) = (f_1 \varphi_1 + \dots + f_n \varphi_n)(\vec{v})$$

$$= \sum_{i=1}^{n} f_i \varphi_i(\vec{v}) = \sum_{i=1}^{n} f_i v_i$$

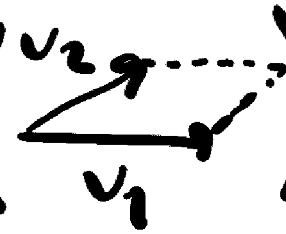
a kind of "length" measure of $\vec{v}$; f measures the "length" of $\vec{v}$.

Could imagine a 1-form on a manifold as a "ruler field" i.e. at every pt there is a ruler (dual vector) that measures the length of vectors at that pt.

Similarly 2-form: $f \in A^2(\mathbb{R}^2)$

$$f = c\,(\varphi_1 \wedge \varphi_2), \quad c \in \mathbb{R}$$

$$c\,(\varphi_1 \wedge \varphi_2)(\vec{v}_1, \vec{v}_2) = c \det\begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$$

$$= c \cdot \text{Area}\Big($$  $$\Big)$$

So $f \in A^2(V)$ is a "2d ruler" that measures "<u>areas</u>".

2-form on a manifold is a "2d ruler field".

etc ..

$\boxed{\S 30}$ Differential operator

<u>Def</u>: $A \overset{\text{open}}{\subset} \mathbb{R}^n$, $f : A \to \mathbb{R}$ smooth function (0-form). Define a 1-form by

$$df(\vec{x})(\bar{x}, \vec{v}) := Df(\vec{x}) \cdot \vec{v}.$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} v_i .$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i(\vec{v}).$$

so $\boxed{df := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i}$

Rmk The gradient of a fcn is

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \partial_{x_i}$$

so df is the "dual" to ∇f.

Thm: The function $f \mapsto df$ is linear.

Proof: $f, g : A \to \mathbb{R}$. $h := af + bg$

$$dh = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dx_i = \sum_{i=1}^{n} \frac{\partial(af+bg)}{\partial x_i} dx_i$$

$$= a \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i + b \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} dx_i$$

$$= a\,df + b\,dg$$

$\square$

Def: Let $A \overset{\text{open}}{\subset} \mathbb{R}^n$. Let

$$\Omega^k(A) = \{\text{smooth } k\text{-forms on } A\}$$

It's a vector space.

Def: We define the <u>exterior derivative</u> as the function

$$d : \Omega^k(A) \longrightarrow \Omega^{k+1}(A)$$

defined as follows

$$\omega = \sum_{I \text{ asc.}} f_I \, dx_I$$

$$d\omega := \sum_{I} df_I \wedge dx_I$$

$$= \sum_I \left(\sum_{i=1}^n \frac{\partial f_I}{\partial x_i} dx_i \right) \wedge dx_I$$

$$= \sum_I \sum_{i=1}^n \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I .$$

<u>Thm.</u> The function

$$d : \Omega^k(A) \longrightarrow \Omega^{k+1}(A)$$

satisfies the following:

(1) $d(a\omega + b\eta) = a\,d\omega + b\,d\eta$

(2) $\omega \in \Omega^k(A)$, $\theta \in \Omega^\ell(A)$

$$d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^k \omega \wedge d\theta$$

("Leibniz rule")

(3) $d(d\omega) = 0$

<u>Proof</u>: (1) Easy.

(2) $\omega = \sum_I f_I \, dx_I$, $\theta = \sum_J g_J \, dx_J$

$$\omega \wedge \theta = \sum_{I,J} f_I g_J \, dx_I \wedge dx_J$$

$$d(\omega \wedge \theta)$$

$$= \sum_{I,J} \sum_{i=1}^{n} \frac{\partial (f_I g_J)}{\partial x_i} dx_i \wedge dx_I \wedge dx_J$$

$$= \sum_{I,J} \sum_{i=1}^{n} \left[\left(\frac{\partial f_I}{\partial x_i}\right) g_J + f_I \left(\frac{\partial g_I}{\partial x_i}\right) \right] dx_i \wedge dx_I \wedge dx_J$$

$$= \sum_{I,J} \underbrace{\left(\sum_{i=1}^{n} \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I \right)}_{d\omega} \wedge \underbrace{(g_J dx_J)}_{\theta}$$

$$+ \sum_{I,J} \sum_{i=1}^{n} \left(f_I \left(\frac{\partial g_J}{\partial x_i}\right) dx_i \wedge dx_I \wedge dx_J \right) \quad (*)$$

I = k-tuple, J = l-tuple

$dx_i \wedge dx_J$

$$= dx_i \wedge (dx_{i_2} \wedge \cdots \wedge dx_{i_k})$$

$$= -(dx_{i_1} \wedge dx_i \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k})$$

$$= (-1)^k dx_I \wedge dx_i$$

So

$$(*) = d\omega \wedge \theta + \sum_{I,J} \sum_{i=1}^{n} f_I \frac{\partial g_J}{\partial x_i} dx_i \wedge dx_I \wedge dx_J$$

$$= d\omega \wedge \theta + (-1)^k \sum_{I,J} \sum_{i=1}^{n} f_I \frac{\partial g_J}{\partial x_i} dx_I \wedge dx_i \wedge dx_J$$

$$= d\omega \wedge \theta + (-1)^k \sum_{I,J} \Big[\underbrace{(f_I dx_I)}_{\omega} \wedge \underbrace{\Big(\sum_{i=1}^{n} \frac{\partial g_J}{\partial x_i} dx_i \wedge dx_J \Big)}_{d\theta} \Big]$$

$$= d\omega \wedge \theta + (-1)^k \omega \wedge d\theta.$$

(3) $\omega = \sum_I f_I dx_I$

$$d\omega = \sum_I \sum_{i=1}^{n} \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I$$

$$d(d\omega) = \sum_I \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f_I}{\partial x_i \partial x_j} dx_j \wedge dx_i \wedge dx_I$$

$$= \sum_I \sum_{j<i} \underbrace{\left(\frac{\partial^2 f_I}{\partial x_i \partial x_j} - \frac{\partial^2 f_I}{\partial x_j \partial x_i} \right)}_{=0} dx_j \wedge dx_i \wedge dx_I$$

$$= 0. \qquad \square$$

<u>Def</u>: $A \overset{\text{open}}{\subset} \mathbb{R}^n$

$\omega \in \Omega^k(A)$ is

- <u>closed</u> if $d\omega = 0$
- <u>exact</u> if $\exists \eta \in \Omega^{k-1}(A) : d\eta = \omega$.

<u>Rmk</u>: Exact $\Rightarrow$ closed since $d\omega = d(d\eta) = 0$.

Midterm II review

- Same format as midterm I.

5 questions, §18-27

- L Diffeomorphisms & volumes
- L Integral on manifold
- L Manifolds (2 problems)
- L Alternating tensors

Important topics:

§18: Diffeomorphisms in $\mathbb{R}^n$

§21: k-dim volume

§23: Manifolds in $\mathbb{R}^n$

§24: Boundary of a manifold

§25: Integrating scalar functions on manifolds

§26-27: Multilinear algebra and alternating tensors.