MAT 322/523 Lec 2, W Jan 24

§ Determinants

Assigns to a square matrix A a real number $\det A$.

Thm: Let A be an $n \times n$ matrix.
$\operatorname{rank} A = n \Leftrightarrow \det A \neq 0$.

In other words, a square matrix is invertible iff its determinant is non-zero.

Cofactor expansion:

Let A be $n \times n$, and $i \in \{1, \ldots, n\}$ then

$$\det A = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} \cdot \det A_{ik}$$

- A_{ik} is the (ik)-minor, which is the $(n-1) \times (n-1)$ matrix obtained from A be deleting row i and col k.

ex: $\begin{vmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ -2 & -4 & 1 \end{vmatrix} = (-1)^{2+1} \cdot 2 \cdot \begin{vmatrix} 1 & -1 \\ -4 & 1 \end{vmatrix}$

$+ (-1)^{2+3} \cdot 3 \cdot \begin{vmatrix} 1 & 1 \\ -2 & -4 \end{vmatrix} = 12$

§ Metric spaces (§3 in Munkres)

Def: A metric space is a pair (X, d) consisting of a set X and a function $d: X \times X \to \mathbb{R}$ (called a metric, or distance) such that $\forall x, y, z \in X$

(1) $d(x,y) = d(y,x)$ (Symmetry)

(2) $d(x,y) \geq 0$ w/ equality iff $x=y$ (non-degeneracy)

(3) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality)

If (X,d) metric sp, $x_0 \in X$ and $\varepsilon > 0$ we call the set

$$B(x_0;\varepsilon) := \{x \in X \mid d(x,x_0) < \varepsilon\}$$

the <u>ε-neighborhood of x_0</u> or the <u>ball w/ radius ε centered at x_0</u>

<u>Ex</u>: (1) The <u>Euclidean</u> (or <u>Standard</u>) metric on $\mathbb{R}^n$ is given by

$$d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\| = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}.$$

<u>$n=2$</u> $B(x_0;\varepsilon) =$ 

<u>$n=3$</u> $B(x_0;\varepsilon) =$ 

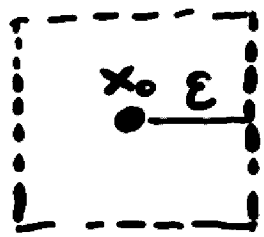
(2) Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and define

$$d(\vec{x},\vec{y}) = \max_{i\in\{1,\dots,n\}} |x_i - y_i|$$

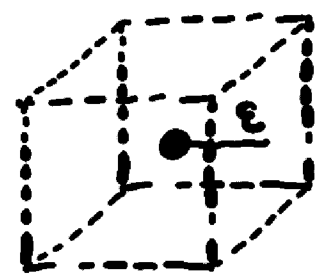
then d is a metric on $\mathbb{R}^n$ called the sup metric (sup is short for supremum)

$\underline{n=2}$

$B(x_0;\varepsilon) =$



$\underline{n=3}$



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§ open and closed sets

Def: Let $A \subset (X,d)$ be a subset of a metric space.

- A is open if for each $x \in A$ there is some $\varepsilon > 0$ such that $B(x;\varepsilon) \subset A$
- A is closed if $X \setminus A$ is open.

We will write $A \overset{\text{open}}{\subset} X$, $A \overset{\text{closed}}{\subset} X$.

<u>Remark</u>: $\boxed{\text{sets are not doors!}}$

open and closed are not opposites. A set can be open and closed at the same time, and it could be neither.

<u>Exercise</u>: Show: (some indexing set)

(1) If A_i is open for $i \in I$ then $\bigcup_{i \in I} A_i$ is open.

(2) If $A_1, \dots, A_n$ are open then $A_1 \cap \cdots \cap A_n$ is open.

Given $Y \subset (X, d)$ the metric restricts to Y, giving an induced metric sp $(Y, d|_Y)$. (<u>metric subspace</u>)

<u>Thm</u>: Let X be a metric sp, and Y a subspace.

- $A \overset{\text{open}}{\subset} Y$ iff $A = U \cap Y$ where $U \overset{\text{open}}{\subset} X$.

- $A \overset{\text{closed}}{\subset} Y$ iff $A = C \cap Y$ where $C \overset{\text{closed}}{\subset} X$.

<u>Def</u>: A point $x \in (X,d)$ is called a <u>limit point</u> of the subset $A \subset X$ if every neighborhood of x intersects A in at least one pt different from x.

<u>Rmk</u>: More formally we have that $x \in X$ is a limit pt iff

$$(B(x,\varepsilon) \setminus \{x\}) \cap A \neq \emptyset \quad \forall \varepsilon > 0$$

Def: The set

$$\bar{A} := A \cup \{x \in X \mid x \text{ limit pt of } A\}$$

is called the Closure of A.

Note that by construction

$A \subset \bar{A}$ is always true.

Thm: If $A \subset X$, then

① $\bar{A} \subset X$ is the smallest closed set containing A.

② $A \subset X$ is closed if and only if $A = \bar{A}$.

Proof*: Exercise.

Note: "Smallest" means that if

$F \overset{\text{closed}}{\subset} X$ and $A \subset F$ then

$\bar{A} \subset F.$ $\square$

<u>Def</u>: Let $A \subset (X,d)$

① The <u>interior</u> of A is

$$\operatorname{int}(A) := \bigcup_{U \overset{\text{open}}{\subset} A} U$$

[its the largest open set contained in A]

② The <u>exterior</u> of A is

$$\operatorname{ext}(A) := \operatorname{int}(X \setminus A)$$

③ The <u>boundary</u> of A is

$$\partial A = X \setminus (\operatorname{int}(A) \cup \operatorname{ext}(A))$$

<u>Exercise</u>: Show that ∂A consists of those points $x \in X$ such that

$B(x;\varepsilon) \cap \operatorname{int}(A) \neq \emptyset \quad \forall \varepsilon > 0$ <u>and</u>

$B(x;\varepsilon) \cap \text{ext}(A) \neq \emptyset \quad \forall \varepsilon > 0.$

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§Limits and continuation

Let (X, d_X) and (Y, d_Y) be metric spaces.

<u>Def</u>: A function $f: X \longrightarrow Y$ is continuous at $x_0 \in X$ if for every $V \overset{\text{open}}{\subset} Y$ containing $f(x_0)$ there's some $U \overset{\text{open}}{\subset} X$ containing x_0 such that $f(U) \subset V$.

f is continuous if it is continuous at every $x_0 \in X$.

<u>Rmk</u> 1 Alternative formulation of continuity: f is continuous iff

$$V \overset{\text{open}}{\subset} Y \implies f^{-1}(V) \overset{\text{open}}{\subset} X.$$

② Can formulate continuity at x_0 in terms of the metrics:

For any $\varepsilon > 0$ there's a corresponding $\delta > 0$ such that

$d_Y(f(x), f(x_0)) < \varepsilon$ whenever $d_X(x, x_0) < \delta$.

<u>Thm</u> (X, d_X), (Y, d_Y) metric spaces. The following functions are continuous:

① Constant functions

$$c : X \longrightarrow Y, \quad x \longmapsto y_0$$

, $y_0 \in Y$ fixed point

② Identity function $id_X : X \longrightarrow X$, $x \longmapsto x$

③ If $f : X \to Y$ is continuous, then

$$f|_A : A \to Y, \quad a \mapsto f(a)$$

is continuous

④ If $f: X \to Y$, $g: Y \to Z$ are continuous, then so is $g \circ f$.

Proof:* ① For every $x \in X$ it suffices to pick X as the open set U because $c(X) = \{y_0\} \subset V$ for any $V \overset{\text{open}}{\subset} Y$ containing y_0.
Alternatively for any $V \overset{\text{open}}{\subset} Y$ containing y_0 we have $c^{-1}(V) = X \overset{\text{open}}{\subset} X$.

② For $V \overset{\text{open}}{\subset} X$, $f^{-1}(V) = V \overset{\text{open}}{\subset} X$.

③ For $V \overset{\text{open}}{\subset} Y$, $f^{-1}(V) \overset{\text{open}}{\subset} X$
then $(f|_A)^{-1}(V) = f^{-1}(V) \cap A \overset{\text{open}}{\subset} A$

④ Exercise

$\square$
