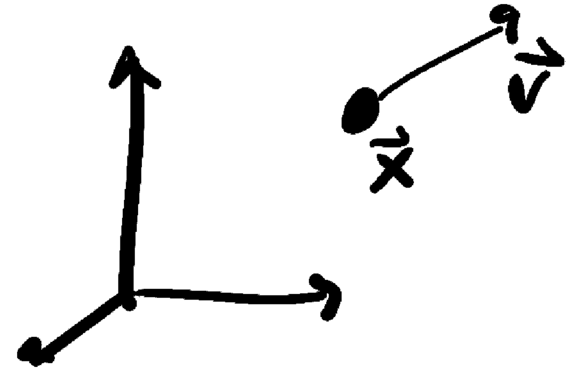
MAT 322/523 Lec 19 W Apr 3

<u>Recall</u>: • Tangent space of $\mathbb{R}^n$

$T_{\vec{x}}\mathbb{R}^n \ni (\vec{x}, \vec{v})$



• $\alpha : \mathbb{R}^k \to \mathbb{R}^n$ smooth

$\alpha_* : T_{\vec{x}}\mathbb{R}^k \to T_{\alpha(\vec{x})}\mathbb{R}^n$ pushforward of α

$(\vec{x}, \vec{v}) \mapsto (\vec{x}, D\alpha(\vec{x}) \cdot \vec{v})$

• Tangent vector field in $\mathbb{R}^n$:

$F : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$

$\vec{x} \mapsto (\vec{x}, f(\vec{x}))$

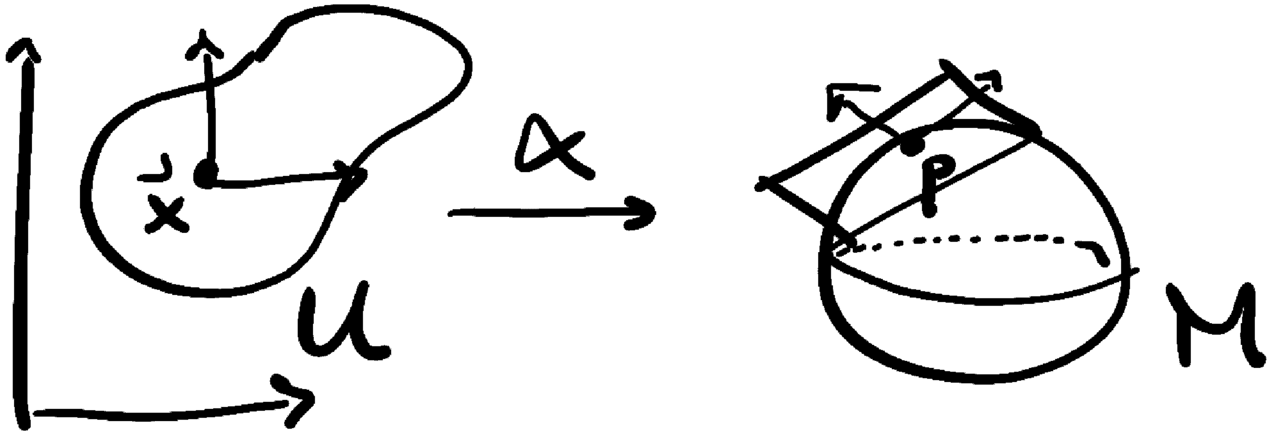
where $f : \mathbb{R}^n \to \mathbb{R}^n$ smooth

<u>Def</u>: Let M be a k-mfd in $\mathbb{R}^n$. Let $\alpha : U \to V$, $U \overset{\text{open}}{\subset} \mathbb{R}^k$, $V \overset{\text{open}}{\subset} M$ be a coord chart at $\vec{p} \in M$.

Let $\vec{p} = \alpha(\vec{x})$, $\vec{x} \in \mathcal{U}$.

The tangent space of M at p is defined as

$$T_pM := \alpha_*(T_x\mathbb{R}^k)$$



<u>Rmk</u>:
- T_pM is a linear subspace of $T_p\mathbb{R}^n$
- If $\{e_i\}_{i=1}^k$ is a basis for $T_{\vec{x}}\mathbb{R}^k$ then T_pM is spanned by

$$(\vec{p}, D\alpha(\vec{x})\cdot e_j) \quad j = 1, \dots, k$$

$$= \left(\vec{p}, \frac{\partial \alpha}{\partial x_j}\right)$$ & it's a basis because $\operatorname{rank} D\alpha = k$

Def: M k-mfd in $\mathbb{R}^n$

Define

$$\boxed{TM = \bigcup_{\vec{p} \in M} T_p M}$$

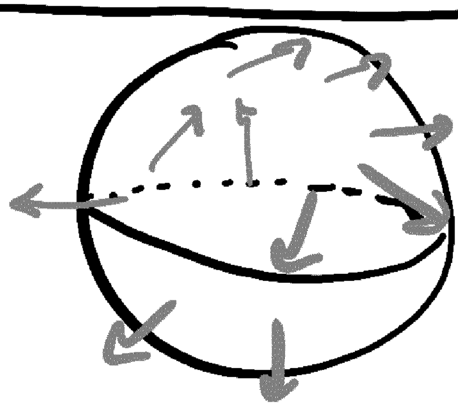
It's called the <u>tangent bundle</u> of M

Rmk: • TM can be shown to be a smooth $2k$-mfd, and it has the structure of a "vector bundle".

Def: A <u>tangent vector field</u> on a k-mfd M in $\mathbb{R}^n$ is a cts/smooth function

$$F : M \to TM \text{ such that}$$

$$F(p) \in T_{\vec{p}} M \text{ for all } \vec{p} \in M.$$



We now return to tensors.

$$\mathcal{L}^k(V) = \left\{ \begin{array}{c} \text{multilinear maps} \\ V^k \to \mathbb{R} \end{array} \right\}$$

<u>Def</u>: $A \overset{\text{open}}{\subset} \mathbb{R}^n$. A <u>k-tensor field</u> in A is a smooth function ω on A s.t. $\omega(\vec{x}) \in \mathcal{L}^k(T_{\vec{x}}\mathbb{R}^n)$. Meaning

$$\omega(\vec{x}) : (T_{\vec{x}}\mathbb{R}^n)^k \longrightarrow \mathbb{R}$$

$$\omega(\vec{x})\big((\vec{x}, v_1), \dots, (\vec{x}, v_k)\big) \in \mathbb{R}$$

<u>Def</u>: $A \overset{\text{open}}{\subset} \mathbb{R}^n$. A <u>differential form</u> in A is a smooth k-tensor field ω in A. $\omega(\vec{x}) \in \mathcal{A}^k(T_{\vec{x}}\mathbb{R}^n)$.

<u>Rmk</u>: Usually we will just call ω a "k-form".

Both of these notions extend to k-manifolds:

ω is a function on M so that $\omega(\vec{p}) \in A^k(T_{\vec{p}}M)$.

<u>Def</u> $\{e_i\}_{i=1}^n$ basis for $\mathbb{R}^n$. Then $\{(\vec{x}, e_i)\}_{i=1}^n$ basis for $T_{\vec{x}}\mathbb{R}^n$.

Define 1-forms on $\mathbb{R}^n$ by

$$\tilde{\phi}_i(\vec{x})(\vec{x}, e_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

<u>Rmk</u>: • $\tilde{\phi}_i(\vec{x}) \in \mathcal{L}^1(T_{\vec{x}}\mathbb{R}^n)$

• More modern notation:

if $\vec{x} = (x_1, \dots, x_n)$ coords in $\mathbb{R}^n$, then denote a basis of $\mathbb{R}^n$ by $\{\partial_{x_i}\}_{i=1}^n$

so $\{(\vec{x}, \partial_{x_i})\}_{i=1}^n$ basis for $T_{\vec{x}}\mathbb{R}^n$.

Then the dual basis of $\mathcal{L}^1(T_{\vec{x}}\mathbb{R}^n) = (T_{\vec{x}}\mathbb{R}^n)^*$ is denoted by $\{(dx_i)_{\vec{x}}\}_{i=1}^n$.

Sometimes its cumbersome to notationally keep track of at which point we are, so we might drop it and simply say that

$\{dx_i\}_{i=1}^n$ is the basis of $\mathcal{L}^1(T_{\vec{x}}\mathbb{R}^n)$ dual to $\{\partial_{x_i}\}_{i=1}^n$.

- Furthermore: Instead of writing $(dx_i)_{\vec{x}}(\vec{x},\vec{v})$ we might write $\boxed{dx_i(\vec{v})}$ instead, while keeping it implicit that

$$\left| \begin{array}{l} dx_i \in \mathcal{L}^1(T_{\vec{x}}\mathbb{R}^n) \\ \vec{v} \in T_{\vec{x}}\mathbb{R}^n \end{array} \right. .$$

Def: For $I=(i_1,\dots,i_k)$ $i_1<\dots<i_k$ define

$$dx_I := dx_{i_1}\wedge\dots\wedge dx_{i_k}.$$

$\{dx_I\}_I$ forms a basis of $\mathcal{L}^k(T_{\vec{x}}\mathbb{R}^n)$

Rmk: By definition

- $dx_i(\partial_j)=\begin{cases}0 & i\neq j\\ 1 & i=j\end{cases}$

- $dx_i(\vec{v})=v_i=$ i-th component of $\vec{v}$.

- $dx_I(\vec{v}_1,\dots,\vec{v}_k)=\det\begin{pmatrix}| & & |\\ v_1 & \cdots & v_k\\ | & & |\end{pmatrix}$

Lma: Any k-form ω on $A\overset{\text{open}}{\subset}\mathbb{R}^n$ can be uniquely written as

$$\omega=\sum_{\substack{I\\ \text{ascending}}} b_I\, dx_I$$

where $b_I : A \to \mathbb{R}$ smooth.
"component functions"

Moreover ω is smooth iff every b_I is.

<u>Lma</u>: Let ω, η be k-forms, and let θ be an l-form. If ω, η, θ are smooth then so is

$$a\omega + b\eta \text{ and } \omega \wedge \theta$$

<u>Def</u>: $A \overset{\text{open}}{\subset} \mathbb{R}^n$. A smooth function $f : A \to \mathbb{R}$ is sometimes called a scalar field or a "0-form".

$f, g : A \to \mathbb{R}$ we may use the def $f \wedge g := f \cdot g$
