MAT 322/523 Lec 18 M Apr 1

Recall: • A k-tensor on a vector sp V is a multilinear map

$$\{f : V^k \longrightarrow \mathbb{R}\} = \mathcal{L}^k(V)$$

- $f \in \mathcal{L}^k(V)$, $g \in \mathcal{L}^l(V)$, then $f \otimes g \in \mathcal{L}^{k+l}(V)$

$$(f \otimes g)(v_1, \dots, v_{k+l}) = f(v_1, \dots, v_k)\, g(v_{k+1}, \dots, v_{k+l})$$

- $f \in \mathcal{L}^k(V)$ is alternating if $f^\sigma = (-1)^\sigma f$, where $\sigma \in S_k$

$$f^\sigma(v_1, \dots, v_k) = f(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

Symmetric if $f^\sigma = f$ $\forall \sigma \in S_k$.

$\mathcal{A}^k(V) \subset \mathcal{L}^k(V)$ subset of alternating tensors.

<u>Lma</u>: $\mathcal{A}^k(V) \subset \mathcal{L}^k(V)$ is a linear subspace.

<u>Proof</u>: It's clear that $(cf)^\sigma = cf^\sigma$.

$$
\begin{aligned}
(f+g)^\sigma&(v_1,\ldots,v_k) \\
&= (f+g)(v_{\sigma(1)},\ldots,v_{\sigma(k)}) \\
&= f(v_{\sigma(1)},\ldots,v_{\sigma(k)}) + g(v_{\sigma(1)},\ldots,\sigma(k)) \\
&= (f^\sigma + g^\sigma)(v_1,\ldots,v_k).
\end{aligned}
$$

$\square$

We have $\dim \mathcal{L}^k(V) = n^k$, but what's $\dim \mathcal{A}^k(V)$?

<u>Thm</u> $\dim \mathcal{A}^k(V) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$

<u>Proof</u>: We first have that

$f \in \mathcal{A}^k(V)$ is completely determined by $f(e_{i_1}, \ldots, e_{i_k})$

$$i_1 < i_2 < \cdots < i_k$$

($\{e_i\}_{i=1}^n$ basis of V.)

This is the case because if $(e_{j_1}, \ldots, e_{j_k})$ is a k-tuple w/ indices not in ascending order, we can always arrange them to be (at the expense of a sign)

$$\left[\begin{aligned} \text{e.g. } f(e_2, e_3, e_1) &= -f(e_2, e_1, e_3) \\ &= f(e_1, e_2, e_3) \end{aligned}\right]$$

Therefore a basis for $\mathcal{A}^k(V)$ is

$\{(e_{i_1}, \ldots, e_{i_k}) \mid i_1 < \cdots < i_k\}$, and this set has $\binom{n}{k}$ elements.

$\square$

Wedge product (§28)

Problem: $f \in A^k(V)$, $g \in A^\ell(V)$

then $f \otimes g \notin A^{k+\ell}(V)$ in general.

E.g. $f \in A^2(V)$

$$f(v_1, v_2) := \begin{vmatrix} | & | \\ v_1 & v_2 \\ | & | \end{vmatrix} \text{ is}$$

an alternating 2-tensor.

$$(f \otimes g)(v_1, v_2, v_3, v_4)$$

$$= f(v_1, v_2) f(v_3, v_4)$$

$$= \begin{vmatrix} | & | \\ v_1 & v_2 \\ | & | \end{vmatrix} \begin{vmatrix} | & | \\ v_3 & v_4 \\ | & | \end{vmatrix}$$

But this isn't equal to

$-(f \otimes g)(v_1, v_3, v_2, v_4)$ in general.

$$= - \begin{vmatrix} | & | \\ v_1 & v_3 \\ | & | \end{vmatrix} \begin{vmatrix} | & | \\ v_2 & v_4 \\ | & | \end{vmatrix}$$

$$\left[\begin{array}{l} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \\[2ex] \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0 \neq -1 \end{array} \right]$$

Instead we must take the "alternalization" of the tensor product:

<u>Def</u>: Alternalization is the linear map

$$A : \mathcal{L}^k(V) \longrightarrow \mathcal{A}^k(V)$$

$$f \longmapsto \sum_{\sigma \in S_k} (-1)^\sigma f^\sigma .$$

<u>Def</u> We define the <u>wedge product</u> as a map

$$\wedge : A^k(V) \times A^\ell(V) \to A^{k+\ell}(V)$$

$$(f, g) \mapsto f \wedge g.$$

<u>Lma</u> The wedge prod satisfies:

(1) $f \wedge (g \wedge h) = (f \wedge g) \wedge h$

(2) $(cf) \wedge g = c(f \wedge g) = f \wedge (cg)$

(3) If $f, g \in A^k(V)$ then

$(f+g) \wedge h = f \wedge h + g \wedge h$

$h \wedge (f+g) = h \wedge f + h \wedge g$

(4) $f \wedge g = (-1)^{k\ell} g \wedge f$

(5) $\{e_i\}_{i=1}^n$ basis for V

$\{\phi_i\}_{i=1}^n$ dual basis

$\{\psi_I\}_{I \text{ ascending}}$ basis for $A^k(V)$

then if $I = \{i_1 < \cdots < i_k\}$

$$\psi_I = \phi_{i_1} \wedge \cdots \wedge \phi_{i_k}$$

(6) $T: V \to W$ linear, then

$$T^*(f \wedge g) = T^*f \wedge T^*g$$

recall $T^*f(v_1, \ldots, v_k) = f(T(v_1), \ldots, T(v_k))$

<u>Rmk</u>:
- Property (4) implies $f \wedge f = 0$ if k^2 is odd.
- $\psi_I(\vec{v}_1, \ldots, \vec{v}_k) = \det\begin{pmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{pmatrix}$

Tangent vectors and vector fields

$\boxed{\S 29}$

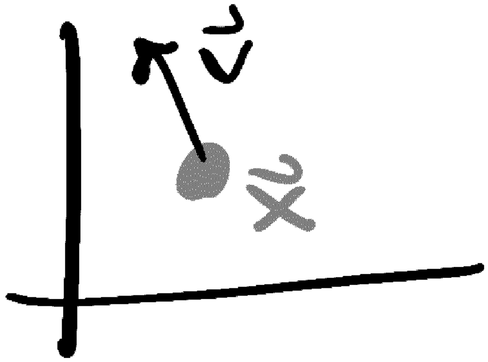
Have already seen vector fields in calculus, but we'll now see it in more detail.

Def: Given $\vec{x} \in \mathbb{R}^n$, a <u>tangent vector</u> (to $\mathbb{R}^n$) at $\vec{x}$ is a pair $(\vec{x}, \vec{v})$ where $\vec{v} \in \mathbb{R}^n$.

$T_{\vec{x}}\mathbb{R}^n := \{\text{set of all tangent vectors at } \vec{x}\}$

Ex: Both $\vec{x}, \vec{v} \in \mathbb{R}^n$ but they play different roles.

$\vec{x}$ = point; $\vec{v}$ = vector

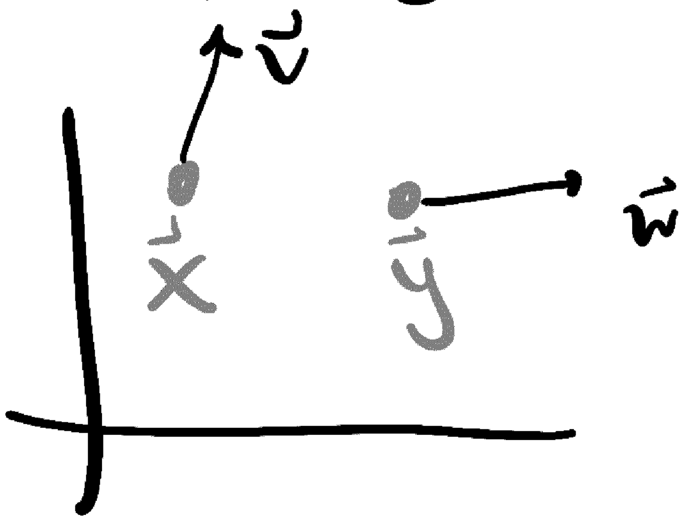


Rmk: $T_{\vec{x}}\mathbb{R}^n$ is just $\{\vec{x}\} \times \mathbb{R}^n$ & so it's a vector space:

* $(\vec{x}, \vec{v}) + (\vec{x}, \vec{w}) = (\vec{x}, \vec{v} + \vec{w})$

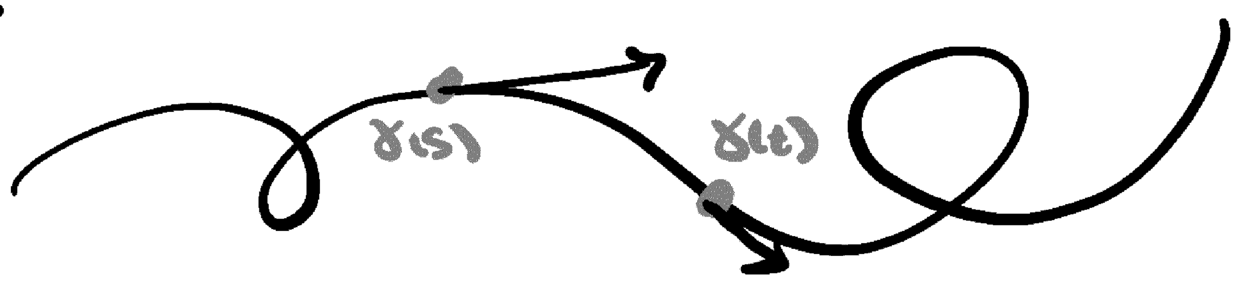
* $c(\vec{x},\vec{v}) = (\vec{x}, c\vec{v})$.

- We do not attempt to define addition between elements $(\vec{x},\vec{v})$ and $(\vec{y},\vec{w})$ where $\vec{x} \neq \vec{y}$



<u>Def</u>: Let $\gamma : (a,b) \longrightarrow \mathbb{R}^n$ be a smooth map. The <u>velocity vector</u> of the path γ at time t is

$$(\gamma(t), D\gamma(t)) \in T_{\gamma(t)}\mathbb{R}^n.$$



Rmk: If $n=3$ we of course have

$$\gamma(t) = (x(t), y(t), z(t))$$

and hence

$$D\gamma(t) = \left(\frac{dx}{dt}(t), \frac{dy}{dt}(t), \frac{dz}{dt}(t)\right)$$

More generally:

Def. Let $A \overset{\text{open}}{\subset} \mathbb{H}^k$, $\alpha : A \to \mathbb{R}^n$ smooth. Define

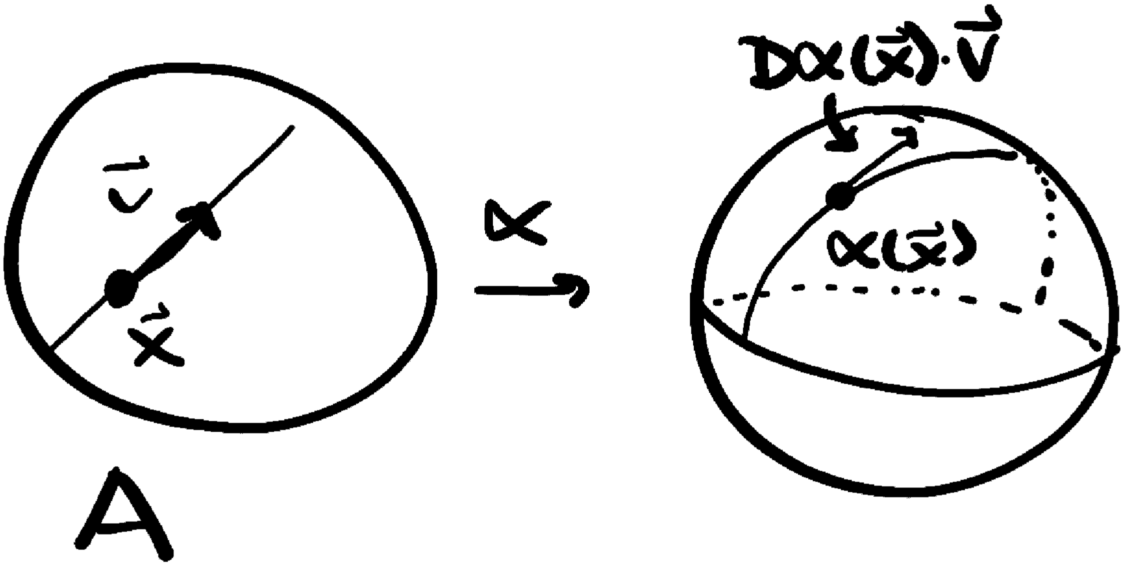
$$\alpha_* : T_{\vec{x}}\mathbb{R}^k \to T_{\alpha(\vec{x})}\mathbb{R}^n$$

"pushforward of α" $\quad (\vec{x}, \vec{v}) = (\alpha(\vec{x}), D\alpha(\vec{x})\cdot\vec{v})$

Rmk: The vector $\alpha_*(\vec{x}, \vec{v})$ is the velocity vector of the curve

$$\gamma(t) = \alpha(\vec{x} + t\vec{v}).$$

at time $\underline{t=0}$.



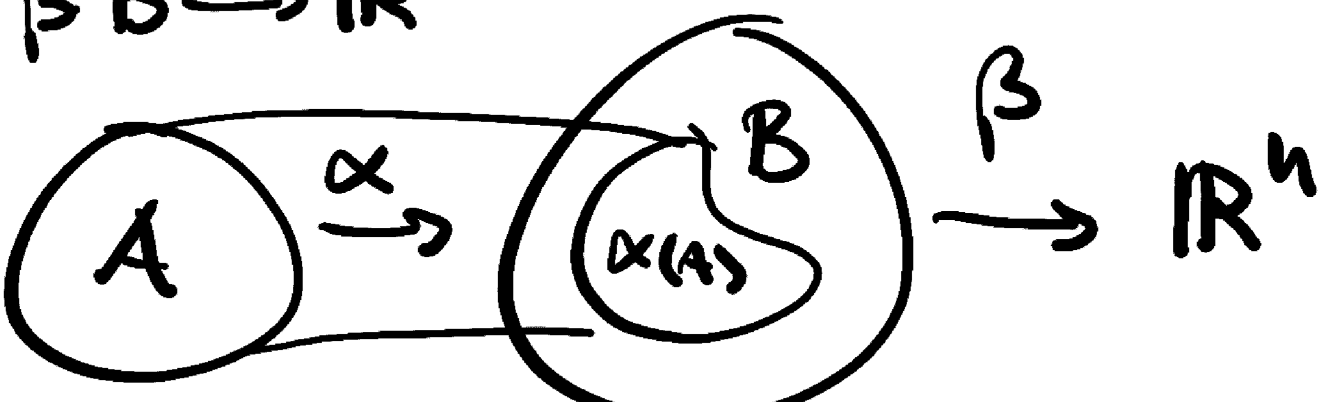
$$D\gamma(0) = D\alpha(\vec{x}+t\vec{v})\Big|_{t=0}\cdot\vec{v}$$
$$= D\alpha(\vec{x})\vec{v}$$

Another consequence of the chain rule is the following:

Lma: $A \overset{\text{open}}{\subset} \mathbb{H}^k$, $\alpha: A \to \mathbb{R}^m$ smooth

$B \overset{\text{open}}{\subset} \mathbb{R}^m$: $\alpha(A) \subset B$

$\beta: B \longrightarrow \mathbb{R}^n$



then
$$(\beta\circ\alpha)_* = \beta_*\circ\alpha_*$$

<u>Proof</u>: Let $\vec{y} := \alpha(\vec{x})$

$$(\beta\circ\alpha)_*(\vec{x},\vec{v})$$

$$= \Big((\beta\circ\alpha)(\vec{x}),\, D(\beta\circ\alpha)((\beta\circ\alpha)(\vec{x}))\cdot\vec{v}\Big)$$

$$= \Big(\beta(\vec{y}),\, D\beta(\beta(\vec{y}))\,D\alpha(\vec{y})\cdot\vec{v}\Big)$$

$$= \beta_*\Big(\vec{y},\, D\alpha(\vec{y})\cdot\vec{v}\Big)$$

$$= \beta_*\Big(\alpha(\vec{x}),\, D\alpha(\alpha(\vec{x}))\cdot\vec{v}\Big)$$

$$= \beta_*\Big(\alpha_*(\vec{x},\vec{v})\Big) = (\beta_*\circ\alpha_*)(\vec{x},\vec{v})$$

$\square$

$$\begin{tikzcd}
A \arrow[rr, "\beta\circ\alpha"] \arrow[dr, "\alpha"'] & & \mathbb{R}^n \\
& B \arrow[ur, "\beta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
T_{\vec{x}}\mathbb{R}^k \arrow[rr, "\beta_*\circ\alpha_*"] \arrow[dr, "\alpha_*"'] & & T_{\beta(\vec{y})}\mathbb{R}^n \\
& T_{\vec{y}}\mathbb{R}^m \arrow[ur, "\beta_*"'] &
\end{tikzcd}$$

Def: A tangent vector field in $A \overset{\text{open}}{\subset} \mathbb{R}^n$ is a continuous map

$$F : A \to \mathbb{R}^n \times \mathbb{R}^n$$

so that $F(\vec{x}) \in T_{\vec{x}}\mathbb{R}^n \ \forall \vec{x} \in A.$

It has the form

$$F(\vec{x}) = (\vec{x}, f(\vec{x}))$$

$$f : A \to \mathbb{R}^n$$
