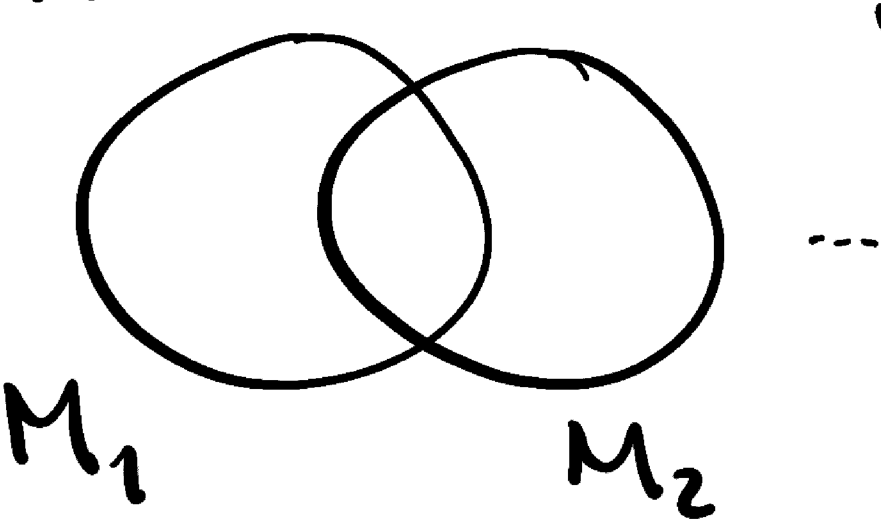
Recall:

Def integrals over a manifold via partitions of unity:

$$\int_M f\, dV = \sum_{i=1}^{\ell} \int_{M_i} \phi_i f\, dV$$

$$M = \bigcup_{i=1}^{N} M_i$$



In order to integrate vector fields etc on manifolds (going for Green's, Stokes' theorem, etc) we need to discuss multilinear algebra & differential forms.

$\boxed{\text{§26 Munkres}}$

Def: V vector space.

$f: V^k \to \mathbb{R}$ is <u>linear in the i-th variable</u> if

$$f(v_1, \dots, av_i + bw_i, \dots, v_k)$$

$$= af(v_1, \dots, v_i, \dots, v_k) + bf(v_1, \dots, w_i, \dots, w_k)$$

f is <u>multilinear</u> if it's linear in every variable.

Def: A <u>k-tensor</u> is a multilinear map $f: V^k \to \mathbb{R}$.

We let $\mathcal{L}^k(V) = \{k\text{-tensors on } V\}$.

Rmk $\mathcal{L}^1(V) = V^*$ dual space.

Thm: $\mathcal{L}^k(V)$ equipped w/ operations

$$(f+g)(v_1,\dots,v_k) := f(v_1,\dots,v_k) + g(v_1,\dots,v_k)$$

$$(cf)(v_1,\dots,v_k) := c f(v_1,\dots,v_k)$$

is a vector space.

Proof: Exercise; it's easy. $\square$

Lma: Let $e_1,\dots,e_n$ be a basis for V. If $f,g \in \mathcal{L}^k(V)$ are k-tensors such that

$$f(e_{i_1},\dots,e_{i_k}) = g(e_{i_1},\dots,e_{i_k})$$

for any $\{i_1,\dots,i_k\} \subset \{1,\dots,n\}$, then $f=g$.

Proof: Let $v_1,\dots,v_k \in V$ & write

$v_i = \sum_{j=1}^{n} a_j^i e_j$. Then

$$f(v_1,\dots,v_k) = f\left(\sum_{j_1=1}^{n} a_{j_1}^1 e_{j_1},\dots,\sum_{j_k=1}^{n} a_{j_k}^k e_{j_k}\right)$$

$$= \sum_{j_1=1}^{n} a^1_{j_1} f\left(e_{j_1}, \sum_{j_2=1}^{n} a^2_{j_2} e_{j_2}, \dots, \sum_{j_k=1}^{n} a^k_{j_k} e_{j_k}\right)$$

$$= \cdots = \sum_{j_k=1}^{n} \cdots \sum_{j_1=1}^{n} a^1_{j_1} \cdots a^k_{j_k} f(e_{j_1}, \dots, e_{j_k})$$

$$= \sum_{j_k=1}^{n} \cdots \sum_{j_1=1}^{n} a^1_{j_1} \cdots a^k_{j_k} \, g(e_{j_1}, \dots, e_{j_k})$$

$$= \cdots = g\left(\sum_{j_1=1}^{n} a^1_{j_1} e_{j_1}, \dots, \sum_{j_k=1}^{n} a^k_{j_k} e_{j_k}\right)$$

$$= g(v_1, \dots, v_k)$$

$\square$

Thm. Let $\{e_i\}_{i=1}^{n}$ be a basis for V. Let $I := (i_1, \dots, i_k)$, $i_\ell \in \{1, \dots, n\}$ $\forall \ell$, $J := (j_1, \dots, j_k)$, $j_\ell \in \{1, \dots, n\}$ $\forall \ell$, def

$$\phi_I(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 0 & I \neq J \\ 1 & I = J \end{cases}.$$

Then $\{\phi_I\}_I$ forms a basis for $\mathcal{L}^k(V)$.

<u>Rmk</u>: • The dimension of $\mathcal{L}^k(V)$ is n^k.

• For $k=1$, V^* has the basis $\{\phi_1, \dots, \phi_n\}$ where

$$\phi_i(e_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}.$$

<u>Def</u>: If $f: V^k \to \mathbb{R}$ and $g: V^\ell \to \mathbb{R}$ are tensors (of possibly different orders) then define a $(k+\ell)$-tensor $f \otimes g \in \mathcal{L}^{k+\ell}(V)$ by

$$(f \otimes g)(v_1, \dots, v_{k+\ell}) = f(v_1, \dots, v_k)\, g(v_{k+1}, \dots, v_{k+\ell})$$

<u>Rmk</u>: It's clear from the def that $f \otimes g$ is multilinear.

Thm. If f, g, h are tensors, then

(1) $(f \otimes g) \otimes h = f \otimes (g \otimes h)$

(2) $(cf) \otimes g = c(f \otimes g) = f \otimes (cg)$

(3) If $f, g \in \mathcal{L}^k(V)$ then

$$(f+g) \otimes h = f \otimes h + g \otimes h$$

$$h \otimes (f+g) = h \otimes f + h \otimes g$$

(4) If $\{e_i\}_{i=1}^n$ is a basis for V, then if $I = (i_1, \dots, i_k)$

$$\phi_I = \phi_{i_1} \otimes \cdots \otimes \phi_{i_k}$$

Rmk By def it's clear that

$$f \otimes g \neq g \otimes f \text{ in general.}$$

Def: Let $T: V \to W$ be a linear map. The <u>dual map</u> is

$T^*: \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ defined

by
$$(T^*f)(v_1,\dots,v_k) := f(T(v_1),\dots,T(v_k))$$

In other words, T^* is the composition of $T\times\cdots\times T$ and f.

$$\begin{tikzcd} V^k \arrow[r, "T^k"] \arrow[dr, "T^*f"'] & W^k \arrow[d, "f"] \\ & \mathbb{R} \end{tikzcd}$$

<u>Thm</u>: $T: V \to W$ linear w/ dual $T^*: \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ then

(1) $T^*(af+bg) = aT^*f + bT^*g$

(2) $T^*(f\otimes g) = T^*f \otimes T^*g$

(3) If $S: W \to X$ linear, then
$$(S\circ T)^* = T^* \circ S^*.$$

---

§27 <u>Alternating tensors</u>

Crash course on permutations

Def A permutation of the set $\{1,..,k\}$ is a bijection

$$\phi:\{1,..,k\}\longrightarrow\{1,..,k\}$$

Let $S_k := \left\{\begin{matrix}\text{permutations}\\ \text{of } \{1,..,k\}\end{matrix}\right\}$. It forms a group w/ composition as the group operation.

S_k = symmetric group.

finite group w/ $k!$ elements.

Def $i\in\{1,..,k-1\}$, then the i-th elementary permutation $e_i\in S_k$ is defined as

$$e_i(j)=\begin{cases} i+1 & j=i \\ i & j=i+1 \\ j & \text{else}\end{cases}$$

<u>Fact</u>: Any permutation is a composition of elementary ones.

<u>Def</u>: If $\sigma = e_{i_1} \circ \cdots \circ e_{i_m}$, define

$$\operatorname{sgn} \sigma := (-1)^m$$

<u>Lma</u>: (1) $\operatorname{sgn}(\sigma \circ \tau) = \operatorname{sgn} \sigma \cdot \operatorname{sgn} \tau$

(2) $\operatorname{sgn} \sigma^{-1} = \operatorname{sgn} \sigma$

(3) If τ is the perm

$$\begin{pmatrix} 1 & \cdots & p & \cdots & q & \cdots & k \\ 1 & \cdots & q & \cdots & p & \cdots & k \end{pmatrix}$$

(it exchanges $p, q \in \{1, \dots, k\}$ & keeps every other number fixed)

then $\operatorname{sgn} \tau = -1$

Back to tensors:

Def: $f \in \mathcal{L}^k(V)$, $\sigma \in S_k$ def $f^\sigma \in \mathcal{L}^k(V)$ by

$$f^\sigma(v_1, \dots, v_k) := f(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

Def $f \in \mathcal{L}^k(V)$ is said to be

- **symmetric** if $f^e = f$ for all elementary permutations $e \in S_k$.
  In other words
  $$f(v_1, \dots, v_i, v_{i+1}, \dots, v_k) = f(v_1, \dots, v_{i+1}, v_i, \dots, v_k) \quad \forall i$$
- **alternating** if $f^e = -f$ for all elementary permutations $e \in S_k$.
  In other words
  $$f(v_1, \dots, v_i, v_{i+1}, \dots, v_k) = -f(v_1, \dots, v_{i+1}, v_i, \dots, v_k) \quad \forall i$$

Let $\mathcal{A}^k(V) := \{$alternating k-tensors on $V\}$.

We use the convention

$$\mathcal{A}^1(V) = \mathcal{L}^1(V).$$

<u>Ex</u>: Let $V = \mathbb{R}^n$. The basis vectors of $\mathcal{L}^k(\mathbb{R}^n)$ are

$$I = (i_1, \dots, i_k)$$

$$\phi_I(\vec{x}_1, \dots, \vec{x}_k) = \phi_{i_1}(\vec{x}_1) \cdots \phi_{i_k}(\vec{x}_k)$$

$$= x_{i_1 1} \cdots x_{i_k k}$$

where $\vec{x}_j = \begin{pmatrix} x_{1j} \\ \vdots \\ x_{nj} \end{pmatrix}$.

So eg. 2-tensors on $\mathbb{R}^n$ are spanned by the basis vectors

$$\phi_{ij}(\vec{x}_1, \vec{x}_2) = x_{i1}\, x_{j2}$$

The tensor $f := \phi_{ij} - \phi_{ji}$ is alternating &

$$f(\vec{x}_1, \vec{x}_2) = x_{i1}x_{j2} - x_{j1}x_{i2}$$

$$= \begin{vmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{vmatrix}$$

Similarly

$$g(\vec{x}, \vec{y}, \vec{z}) := \begin{vmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{vmatrix}$$

defines a 3-tensor & one can check

$$g = \underbrace{\phi_{ijk} + \phi_{jki} + \phi_{kij}}_{\text{Even perms of } (i,j,k)} - \underbrace{\phi_{jik} - \phi_{ikj} - \phi_{kji}}_{\text{odd perms of } (i,j,k)}$$

$$= \sum_{\sigma \in S_3} (\operatorname{sgn} \sigma)\, \Phi_{\sigma(i)\sigma(j)\sigma(k)}$$

This formula can in fact be used to compute determinants of an $(n \times n)$-matrix.
