MAT 322/523 Lec 16 M Mar 25

Recall:

- k-dim volume

$$V : \{(\vec{x}_1, \dots, \vec{x}_k) \mid \vec{x}_j \in \mathbb{R}^n\} \longrightarrow \mathbb{R}_{\geq 0}$$

$$V(\vec{x}_1, \dots, \vec{x}_k) = \sqrt{\det(X^T X)}$$

$$X := \begin{pmatrix} | & & | \\ \vec{x}_1 & \cdots & \vec{x}_k \\ | & & | \end{pmatrix}.$$

- M k-mfd in $\mathbb{R}^n$,

$\alpha : A \to \mathbb{R}^n$: $M = \alpha(A)$

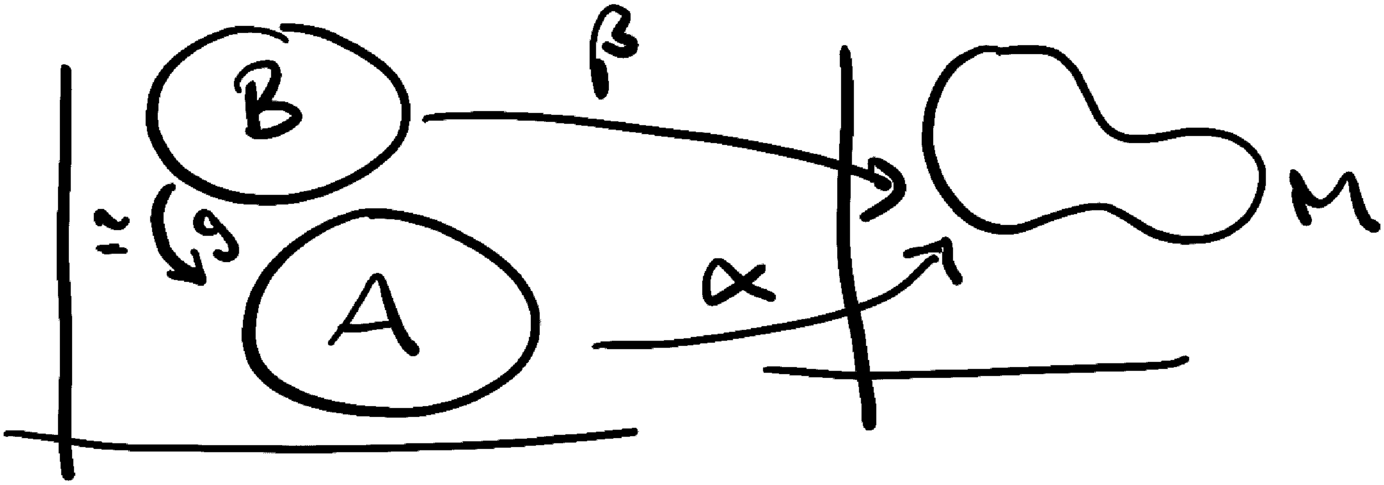
[actually M does not need to be assume to be a mfd. Just a set is ok.]

$f : M \to \mathbb{R}$ cts

$$\int_{M,\alpha} f \, dV := \int_A (f \circ \alpha) V(D\alpha).$$

Moreover if $g: B \to A$ diffeo and $\beta := \alpha \circ g$, then

$$\int_{M,\beta} f \, dV = \int_{M,\alpha} f \, dV$$



$\boxed{\text{Munkres §25}}$ Assume M compact.

<u>Def</u>: Let M be a k-mfd in $\mathbb{R}^n$, $f: M \to \mathbb{R}$ cts. Let $C = \operatorname{supp} f$ and suppose $\alpha: U \to V$ is a coord chart so that $C \subset V$.

Def
$$\int_M f \, dV := \int_{\operatorname{int} U} (f \circ \alpha) V(D\alpha).$$

Here $\operatorname{int} \mathcal{U} = \mathcal{U}$ if $\mathcal{U} \overset{\text{open}}{\subset} \mathbb{R}^k$

$\operatorname{int} \mathcal{U} = \mathcal{U} \cap \mathbb{H}^k_+$ if $\mathcal{U} \overset{\text{open}}{\subset} \mathbb{H}^k$ but not open in $\mathbb{R}^k$.

Rmk: • M is assumed to be compact, so C is compact $\to \alpha^{-1}(C)$ compact

• Can assume $\mathcal{U}$ is bounded

• The function $F := (f \circ \alpha) V(D\alpha)$ is continuous on $\mathcal{U}$ & vanishes outside $\alpha^{-1}(C)$, so F is bounded.

cpt

$\rightsquigarrow$ F integrable.

Lma: If the support of f above can be covered by a single coord chart, the integral $\int_M f \, dV$ is well-defined & indep of the coord chart.

Proof: First a preliminary result:

If $W \overset{\text{open}}{\subset} U$: $\alpha(W) \supset \operatorname{supp} f$

then

$$(*) \quad \int_{\operatorname{int} W} (f \circ \alpha) V(D\alpha) = \int_{\operatorname{int} U} (f \circ \alpha) V(D\alpha)$$

because the integrand vanishes outside W.

Now let $\alpha_i : U_i \to V_i$ be two coord charts so that $V_i \supset \operatorname{supp} f$.

Let $W := V_0 \cap V_1$, $W_i := \alpha_i^{-1}(W)$

then $W \supset \operatorname{supp} f$, so by $(*)$

$$\int_{\operatorname{int} W_i} (f \circ \alpha_i) V(D\alpha_i) = \int_{\operatorname{int} U_i} (f \circ \alpha_i) V(D\alpha_i)$$

We also know $\alpha_1^{-1} \circ \alpha_0 : \operatorname{int} W_0 \to \operatorname{int} W_1$ is a diffeo, so

$$\int_{\operatorname{int} U_0} (f \circ \alpha_0) V(D\alpha_0) = \int_{\operatorname{int} W_0} (f \circ \alpha_0) V(D\alpha_0)$$

$$\boxed{=} \int_{\text{int } W_1} (f \circ \alpha_1) V(D\alpha_1) = \int_{\text{int } U_1} (f \circ \alpha_1) V(D\alpha_1)$$

Thm from last time

$\square$

To get rid of the assumption $V \supset \text{supp } f$ we use a partition of unity.

<u>Lma</u>: M compact k-mfd in $\mathbb{R}^n$.

Given a cover $M = \bigcup_{i \in J} V_i$,

$\alpha_i : U_i \to V_i$ $i \in J$ (coord charts)

there's a <u>finite</u> collection $\{\phi_i : \mathbb{R}^n \to \mathbb{R}\}_{i=1}^{\ell}$ such that

(1) $\phi_i(\vec{x}) \geq 0$ for all $\vec{x}$

(2) Given i, $\text{supp } \phi_i$ is compact and $\exists$ coord chart $\alpha_j : U_j \to V_j$:

$\text{supp } \phi_i \cap M \subset V_j$

(3) $\sum_{i=1}^{\ell} \phi_i(\vec{x}) = 1$ for all $\vec{x}$.

Proof: M is compact so we can choose a finite subcover

$$M = \bigcup_{j=1}^{\ell} V_j \ , \ \alpha_j : U_j \to V_j$$

For each V_j, choose $A_j \overset{\text{open}}{\subset} \mathbb{R}^n$ so that $A_j \cap M = V_j$. Then

$$M = \bigcup_{j=1}^{\ell} A_j . \ A_j \overset{\text{open}}{\subset} \mathbb{R}^n .$$

Then $\{\phi_i\}_{i=1}^{\ell}$ is a p.o.u. wrt this open cover of M. $\square$

Def: M cpt k-mfd in $\mathbb{R}^n$ $f : M \to \mathbb{R}$ cts. $\{\phi_i\}_{i=1}^{\ell}$ p.o.u. wrt to all coord charts. Define

$$\int_M f \, dV := \sum_{i=1}^{\ell} \int_M \phi_i f \, dV.$$

Also def the k-dim volume of M by

$$\operatorname{Vol} M := \int_M 1 \, dV$$

<u>Lma</u>: The definition is indep. of the p.o.u. chosen.

<u>Proof</u>: If $\{\psi_i\}_{i=1}^{\ell}$ is another p.o.u. wrt to coord charts on M, then

$$\int_M f \, dV = \sum_{i=1}^{\ell} \int_M \phi_i f \, dV$$

$$= \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} \int_M \psi_j \phi_i f \, dV$$

change sum order

$$\boxed{=} \sum_{j=1}^{\ell} \sum_{i=1}^{\ell} \int_M \phi_i \psi_j f \, dV$$

$$= \sum_{j=1}^{\ell} \int_M \psi_j f \, dV \qquad \square$$

<u>Thm</u>: M cpt k-mfd in $\mathbb{R}^n$

$f, g : M \to \mathbb{R}$ cts.

$$\int_M (af+bg)\, dV = a\int_M f\, dV + b\int_M g\, dV$$

<u>Def</u>: M k-mfd in $\mathbb{R}^n$. A subset $D \subset M$ is said <u>to have measure 0</u> ($\mu_M(D) = 0$) if D can be covered w/ countably many coord charts $\alpha_i : U_i \to V_i$ such that if $D_i = \alpha^{-1}(D \cap V_i)$ then $\mu_{\mathbb{R}^k}(D_i) = 0$ for all i

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<u>Rmk</u>: Equiv def:

$$\mu_M(D) = 0 \iff \forall \text{ coord charts } \alpha : U \to V \quad \mu_{\mathbb{R}^k}(\alpha^{-1}(V \cap D)) = 0.$$

Thm: M cpt k-mfd in $\mathbb{R}^n$. $f: M \to \mathbb{R}$ cts. Suppose $\alpha_i : A_i \to M_i$, $i \in \{1, \dots, N\}$ coord charts:

(1) $M_i \cap M_j = \emptyset \quad i \neq j$

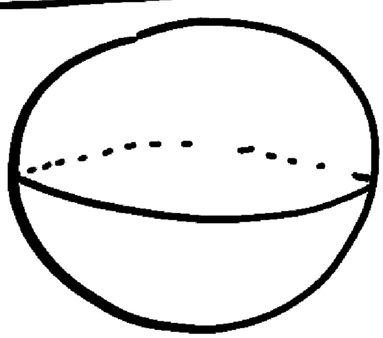
(2) $M = K \cup \bigcup_{i=1}^{N} M_i$ where $\mu_n(K) = 0$.

Then

$$\int_M f \, dV = \sum_{i=1}^{N} \int_{A_i} (f \circ \alpha_i) V(D\alpha_i)$$

Ex:

Calc vol of →



Cover by $\boxed{M_+} \cup \boxed{M_-} \cup \boxed{E}$

upper/lower hemisphere

equator

E has measure 0 in M

$$M_\pm = \{(x,y,z) \mid z = \pm\sqrt{1-x^2-y^2}\}$$

$$\alpha_\pm : \boxed{B^2(1)} \longrightarrow M_\pm$$
$$(x,y) \longmapsto (x, y, \pm\sqrt{1-x^2-y^2})$$

coord charts.

open 2d disk in $\mathbb{R}^2$

$$D\alpha_\pm = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \mp x(1-x^2-y^2)^{-\frac{1}{2}} & \mp y(1-x^2-y^2)^{-\frac{1}{2}} \end{bmatrix}$$

$$V(D\alpha_\pm) = \sqrt{\det((D\alpha_\pm)^T D\alpha_\pm)}$$

write $z = \sqrt{1-x^2-y^2}$

$$(D\alpha_\pm)^T D\alpha_\pm =$$

$$\begin{pmatrix} 1 & 0 & \frac{\mp x}{z} \\ 0 & 1 & \frac{\mp y}{z} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\mp x}{z} & \frac{\mp y}{z} \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{x^2}{z^2} & \frac{xy}{z} \\ \frac{xy}{z} & 1 + \frac{y^2}{z^2} \end{pmatrix}$$

$$V(D\alpha_\pm) = \sqrt{\left(1 + \frac{x^2}{z^2}\right)\left(1 + \frac{y^2}{z^2}\right) - \frac{x^2y^2}{z^2}}$$

$$= \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}$$

$$\int_M 1 \, dV = \int_{A_+} \alpha_+ V(D\alpha_+) + \int_{A_-} \alpha_- V(D\alpha_-)$$

$$= \int_{B^2(1)} \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} + \int_{B^2(1)} \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}$$

$$= 2 \int_{\{x^2+y^2<1\}} \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} \quad (\dagger)$$

Change to polar coords

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$z^2 = 1 - x^2 - y^2 = 1 - r^2$$

$$x^2 + y^2 = r^2$$

Jacobian det $= r$

$$(\dagger) = 2 \int_{r=0}^{1} \int_{\theta=0}^{2\pi} r \cdot \sqrt{1 + \frac{r^2}{1-r^2}}$$

$$= 4\pi \underbrace{\int_0^1 \frac{r}{\sqrt{1-r^2}}}_{=1} = 4\pi$$

$$\boxed{\begin{aligned} \text{Set } \quad u &= 1 - r^2 \\ du &= -2r\,dr \end{aligned}}$$

Conclusion:

$$\int_{S^2} 1 \, dV = 4\pi.$$