MAT 322/523 Lec 15 W Mar 20

Recall: Smooth k-mfd in $\mathbb{R}^n$ is $M \subset \mathbb{R}^n$: at every pt it looks like $\mathbb{R}^k$:

$$\forall \vec{x} \in M \quad \exists \; U \overset{\text{open}}{\subset} \mathbb{R}^k, \; V \overset{\text{open}}{\subset} M, \; \alpha: U \to V \text{ diffeo}$$

Coord. chart

A k-mfd with boundary is as above but we have $U \overset{\text{open}}{\subset} \mathbb{H}^k$

$$\mathbb{H}^k = \{(x_1, \dots, x_k) \mid x_k \geq 0\}.$$

Boundary of a manifold (§24)

Def. M k-mfd in $\mathbb{R}^n$, $\vec{p} \in M$.
We say that $\vec{p} \in M$ is an interior point if $\exists$ coord chart at $\vec{p}$

$$\alpha: U \to V \text{ where } U \overset{\text{open}}{\subset} \mathbb{R}^k$$

Otherwise, we say that $\vec{p}$ is a boundary point.

∂M = set of bdry pts

$M \setminus \partial M$ = set of interior pts

Lma: M k-mfd in $\mathbb{R}^n$, $\alpha: U \to V$ coord chart at $\vec{p} \in M$.

(a) If $U \overset{\text{open}}{\subset} \mathbb{R}^k$, $\vec{p} \in M \setminus \partial M$

(b) If $U \overset{\text{open}}{\subset} \mathbb{H}^k$, $\vec{p} = \alpha(\vec{x}_0)$ for $\vec{x}_0 \in \mathbb{H}^k_+$

then $\vec{p} \in M \setminus \partial M$

(c) If $U \overset{\text{open}}{\subset} \mathbb{H}^k$, $\vec{p} = \alpha(\vec{x}_0)$ for $\vec{x}_0 \in \partial \mathbb{H}^k$

then $\vec{p} \in \partial M$

Thm: If M is a k-mfd with boundary in $\mathbb{R}^n$. Then ∂M is a $(k-1)$-mfd without boundary in $\mathbb{R}^n$.

Proof: Let $\vec{p} \in \partial M$. $\alpha : \mathcal{U} \to U$ coord chart, $\mathcal{U} \overset{\text{open}}{\subset} \mathbb{H}^k$ & $\alpha(\vec{x}_0) = \vec{p}$ for $\vec{x}_0 \in \partial \mathbb{H}^k$

By the previous lemma, every pt in $\mathcal{U} \cap \mathbb{H}^k_+$ maps to $M \setminus \partial M$.

Therefore

$$\alpha\left(\mathcal{U} \cap (\partial \mathbb{H}^k)\right) \subset \partial M.$$

$\mathcal{U}' := \mathcal{U} \cap (\partial \mathbb{H}^k)$, then

$$\alpha|_{\mathcal{U}'} : \mathcal{U}' \to V',$$

$V' := V \cap \partial M$ is

a coordinate chart, where

$\mathcal{U}' = \mathcal{U} \cap (\partial \mathbb{H}^k) \subset \mathbb{R}^{k-1}$

$V' \subset \partial M$. $\square$

Now we consider a very useful theorem that we can use to construct manifold.

Thm: Let $U \overset{\text{open}}{\subset} \mathbb{R}^n$, $f: U \to \mathbb{R}$ smooth. Let $M := f^{-1}(0)$

$$N := f^{-1}([0,\infty))$$

If $M \neq \emptyset$ and $\operatorname{rk} Df(\vec{x}) = 1$ $\forall \vec{x} \in M$

then N is an n-mfd in $\mathbb{R}^n$ and $\partial N = M$.

Proof: We construct coord charts at every pt.

If $\vec{p} \in N \setminus \partial N$ it means by definition $f(\vec{p}) > 0$.

We have

$N \setminus \partial N = f^{-1}((0,\infty))$ & this is open (since f cts & $(0,\infty)$ open in $\mathbb{R}$)

So $\alpha: N \setminus \partial N \xrightarrow{id} N \setminus \partial N$ is
$\vec{x} \longmapsto \vec{x}$

a coord chart around any pt $\vec{x} \in N \setminus \partial N$.

Now if $\vec{p} \in M = \partial N = f^{-1}(0)$ & since $Df(\vec{p}) \neq 0$, at least one $\frac{\partial f}{\partial x_i}(\vec{p}) \neq 0$. Define

$$F(\vec{x}) := (x_1, \dots, x_{n-1}, f(\vec{x}))$$

$$\Rightarrow DF(\vec{x}) = \begin{pmatrix} I_{n-1} & 0 \\ * & D_n f(\vec{x}) \end{pmatrix}$$

non-singular. So by inverse fcn thm $\exists$ nghd $A \overset{\text{open}}{\subset} \mathbb{R}^n$, $\vec{p} \in A$ and $B \overset{\text{open}}{\subset} \mathbb{R}^k$:

$$F|_A : A \to B \text{ diffeo.}$$

Then $(F|_{A \cap M})^{-1} : B \cap \mathbb{H}^k \to A \cap M$

is a coord chart around $\vec{p} \in M$.

$\square$

<u>Def</u>: Let $r > 0$

$B^n(r) := \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| \le r\}$

n-ball (centered at $\vec{0} \in \mathbb{R}^n$) of radius r

$S^{n-1}(r) := \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| = r\}$

$(n-1)$-sphere of radius r.

<u>Cor</u> $B^n(r)$ is an n-mfd in $\mathbb{R}^n$ and $S^{n-1}(r) = \partial B^n(r)$.

<u>Proof</u>: Apply the theorem to the function $f(\vec{x}) = r^2 - \|\vec{x}\|^2$.

<u>Note</u> $f(\vec{x}) \ge 0 \iff r^2 - \|\vec{x}\|^2 \ge 0$

$\iff r \ge \|\vec{x}\|$

Also

$$Df(\vec{x}) = \begin{pmatrix} -2x_1 & \cdots & -2x_n \end{pmatrix}$$

$$\left[f(x_1, \dots, x_n) = r^2 - \sum_{i=1}^{n} x_i^2 \right]$$

So $Df(\vec{x}) \neq \vec{0}$ for any pt in $S^{n-1}(r)$.

$\square$

Before defining integrals of fcns on manifolds, we need to take a closer look at volumes.

<u>Volume of parallelipeds</u> (§21)

Let's recall some linear algebra:

<u>Lma</u> (Gram-schmidt)

$W \subset \mathbb{R}^n$ dim k linear subspace. Then $\exists$ orthonormal basis of $\mathbb{R}^n$ so that first k vectors form a basis for W.

Assembling the Gram-Schmidt ON-basis into a matrix yields:

<u>Cor</u>: $W \subset \mathbb{R}^n$ dim k linear subspace $\exists$ orthogonal transformation $h: \mathbb{R}^n \to \mathbb{R}^n$ so that $h(W) = \mathbb{R}^k \times \{\vec{0}\}$

<u>Thm</u> (k-dim volume) The k-dim volume is the function

$$V: \{(\vec{x}_1, \dots, \vec{x}_k) \mid \vec{x}_i \in \mathbb{R}^n\} \longrightarrow \mathbb{R}_{\geq 0}$$

$$V(\vec{x}_1, \dots, \vec{x}_k) = \sqrt{\det(X^T X)}$$

$$X := \begin{pmatrix} | & & | \\ \vec{x}_1 & \cdots & \vec{x}_k \\ | & & | \end{pmatrix}.$$

It's the unique function satisfying:

(1) If $h: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal, then

$$V(h(\vec{x}_1), \dots, h(\vec{x}_k)) = V(\vec{x}_1, \dots, \vec{x}_k)$$

(2) If $\vec{y}_1, \ldots, \vec{y}_k$ are vectors in the subspace $\mathbb{R}^k \times \{\vec{0}\} \subset \mathbb{R}^n$

$$\vec{y}_i = \begin{pmatrix} \vec{z}_i \\ \vec{0} \end{pmatrix}, \text{ then}$$

$$V(\vec{y}_1, \ldots, \vec{y}_k) = \left| \det(\vec{z}_1 \cdots \vec{z}_k) \right|$$

(3) $V(\vec{x}_1, \ldots, \vec{x}_k) = 0 \iff \vec{x}_1, \ldots, \vec{x}_k$ lin dep

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Volume of manifolds (§22)

Def: Assume M k-mfd in $\mathbb{R}^n$ such that $M = \alpha(A)$, $\alpha : A \to \mathbb{R}^n$, $A \overset{\text{open}}{\subset} \mathbb{R}^n$

Then its volume is

$$\operatorname{vol} M := \int_A V(D\alpha), \text{ provided}$$

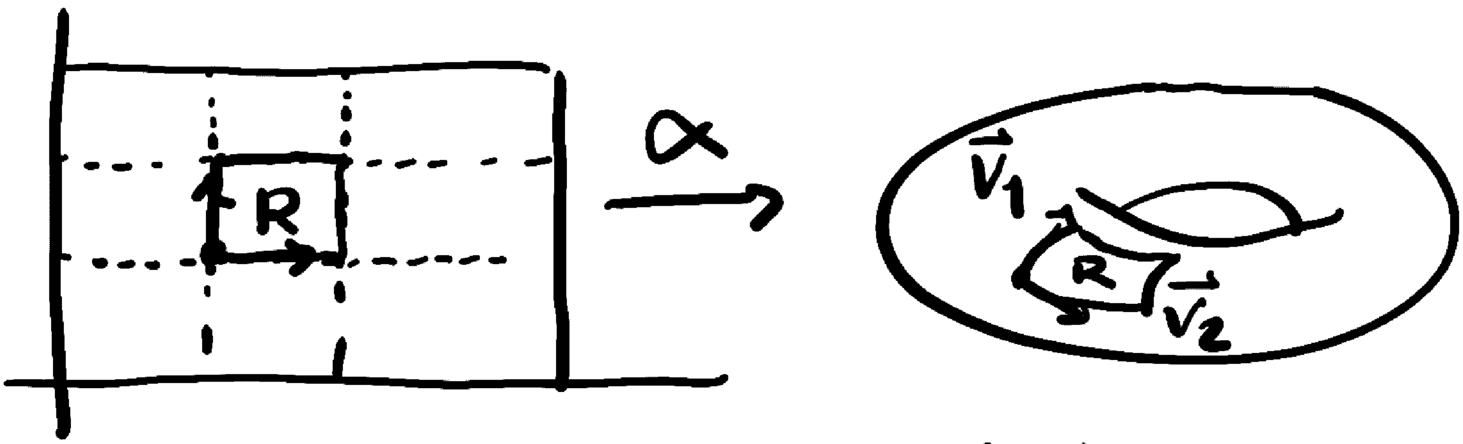
that it exists.

Suppose $A = \operatorname{int}(Q)$ Q rectangle

P partition of Q &

$$R = [a_1, a_1+h_1] \times \cdots \times [a_k, a_k+h_k]$$

one of its subrectangles.



First order approx of the vectors $\vec{v}_i$ is

$$\vec{v}_i = D\alpha(\vec{a}) \cdot h_i \vec{e}_i = \frac{\partial \alpha}{\partial x_i} \cdot h_i$$

It's therefore reasonable to consider the volume of the parallelopiped spanned by $\vec{v}_i$. Its volume is

$$V(\vec{v}_1, \dots, \vec{v}_k) = V\left(\frac{\partial \alpha}{\partial x_1}, \dots, \frac{\partial \alpha}{\partial x_k}\right) \cdot \begin{pmatrix} h_1 \\ \vdots \\ h_k \end{pmatrix}$$

$$= V(D\alpha(\tilde{a}))\cdot \mathrm{vol}(R)$$

Summing over all R etc, it approxes

$$\int_A V(D\alpha).$$

<u>Def</u>: Assume M k-mfd in $\mathbb{R}^n$ such that $M = \alpha(A)$, $\alpha : A \to \mathbb{R}^n$, $A \overset{\text{open}}{\subset} \mathbb{R}^k$

$f : M \to \mathbb{R}$ cts.

$$\int_M f \, dV := \int_A (f \circ \alpha) V(D\alpha)$$

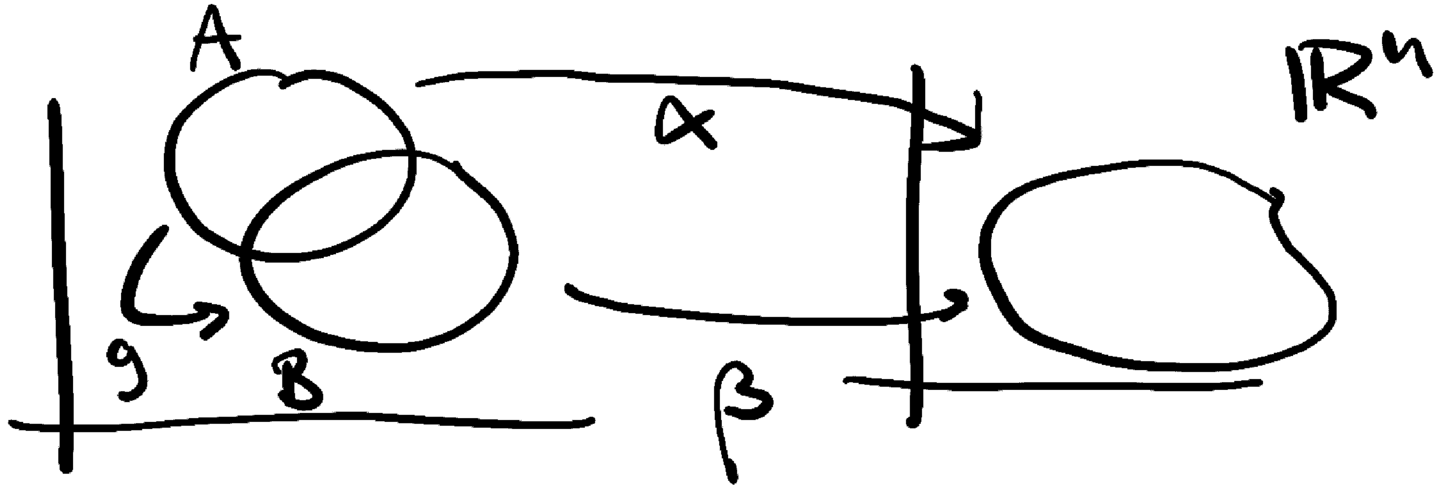
provided it exists.

Integrals with respect to volume are invariant under coord changes:

<u>Thm</u>: Let $A, B \overset{\text{open}}{\subset} \mathbb{R}^k$, $g : A \to B$ diffeo

$\beta : B \to \mathbb{R}^n$ class C^r, $Y := \beta(B)$.

Let $\alpha := \beta \circ g : A \to \mathbb{R}^n$.



If $f: Y \to \mathbb{R}$ continuous then

$$\int_A (f \circ \alpha) V(D\alpha) = \int_B (f \circ \beta) V(D\beta)$$

& one integral exists iff the other does.

<u>Rmk</u>:

Sometimes to emphasize that we use the map $\beta: B \to \mathbb{R}^n$ we will use the notation

$$\int_{Y,\beta} f\, dV := \int_B (f \circ \beta) V(D\beta)$$
