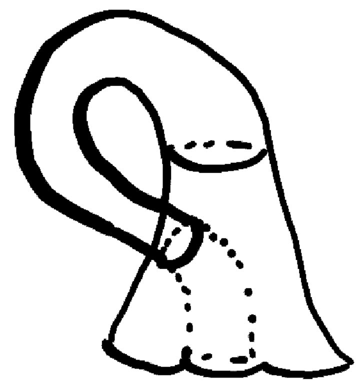
Manifolds in $\mathbb{R}^n$ (§23)

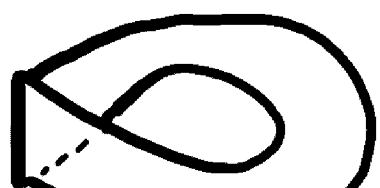
One of the most important classes of spaces in mathematics.

They are spaces that "locally look like" $\mathbb{R}^n$.


sphere


torus


Klein bottle


Möbius strip

There's an abstract definition of manifolds using topological spaces (more general than metric spaces) that we might see later.

<u>Def</u>: Let $k > 0$. Suppose $M \subset \mathbb{R}^n$ has the property that $\forall \vec{p} \in M$ $\exists$ set $V \overset{\text{open}}{\subset} M$, $\vec{p} \in V$

$U \overset{\text{open}}{\subset} \mathbb{R}^k$

and a C^r-diffeo $\alpha : U \to V$.

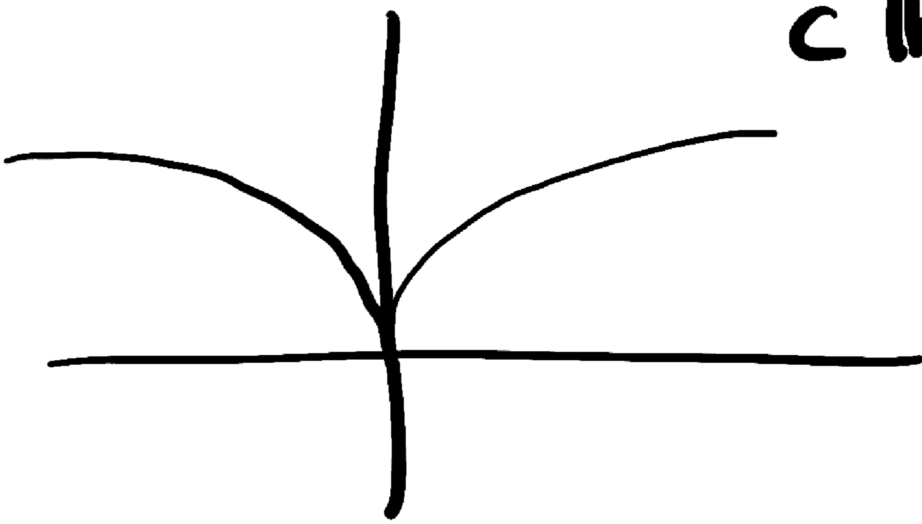
- α injective
- $\operatorname{rank} D\alpha(\vec{a}) = k \quad \forall \vec{a} \in U$

- M is called a k-manifold without boundary, of class C^r.
- α is called a coordinate chart on M at $\vec{p} \in M$.

<u>Rmk</u>. • The number k is unique, and is called the dimension of M.

- we usually only care about manifolds of class C^∞ "smooth manifold".
- Usually "manifold" means w/o ∂.

Ex: Consider $M = \{(t^3, t^2) \mid t \in \mathbb{R}\} \subset \mathbb{R}^2$



Intuitively this is not a smooth mfd since at $(0,0)$ it looks like

which does not look like $\mathbb{R}$.

More precisely: $\alpha : \mathbb{R} \longrightarrow M$

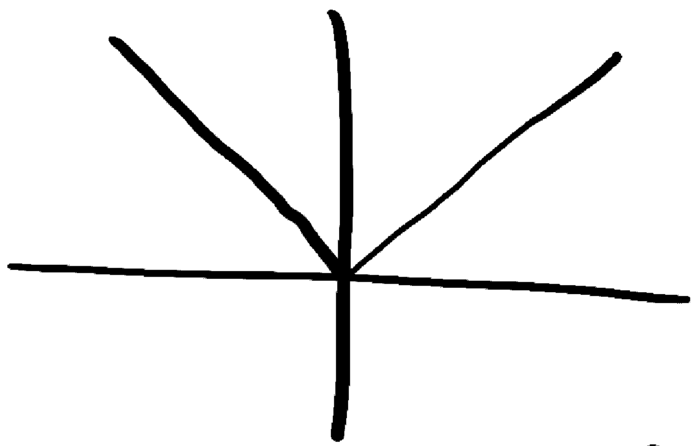
$$t \longmapsto (t^3, t^2).$$

$$D\alpha(t) = \begin{pmatrix} 3t^2 & 2t \end{pmatrix}$$

Has rank 1 iff $t \neq 0$. So $\operatorname{rk} D\alpha(0) = 0$, meaning that α isn't a C^r-diffeo for $r > 1$

M is the image of a continuous fcn so it <u>is</u> a C^0-manifold.

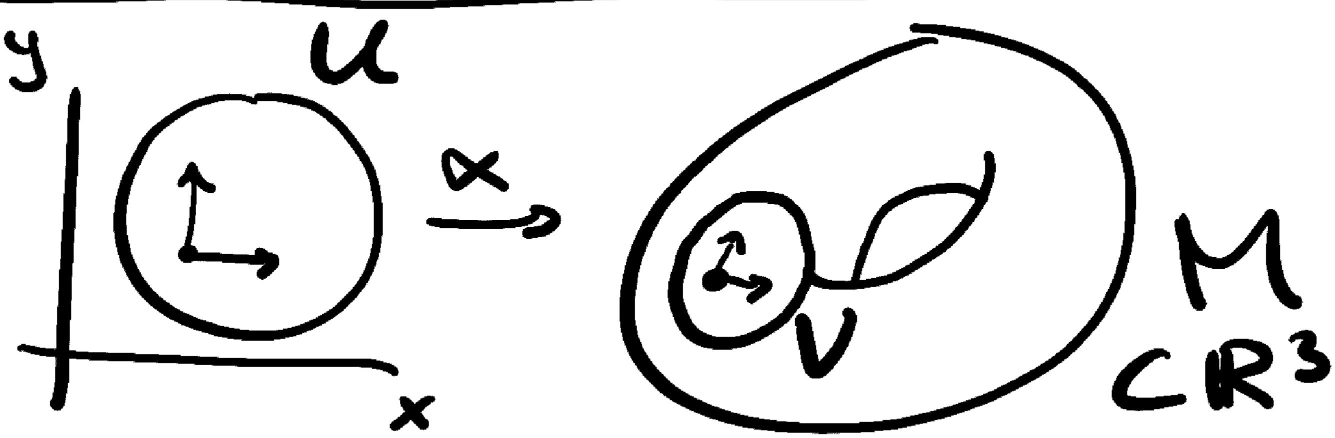
<u>Ex</u>: Let $M = \{(t, |t|) \mid t \in \mathbb{R}\}$



Can <u>not</u> find a C^r-coordinate chart at $(0,0) \in M$, because of the corner, as above.

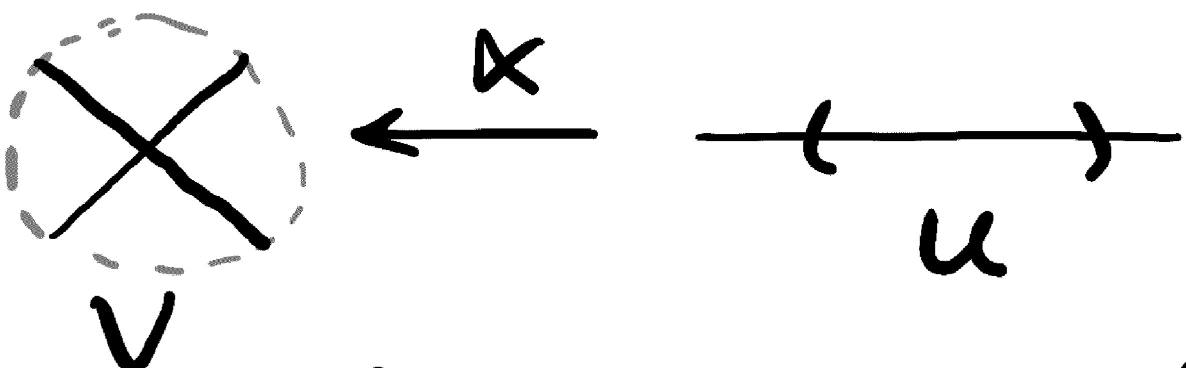
However, M is again a C^0-manifold since it's the image of $t \mapsto (t, |t|)$ which is continuous.

<u>Ex</u>:

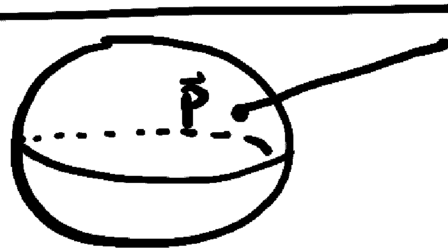


That α is a diffeo means that $\frac{\partial \alpha}{\partial x}$ and $\frac{\partial \alpha}{\partial y}$ are lin. independent. These vectors span a 2-dimensional tangent plane of M.

<u>Ex</u>: $M = \infty \subset \mathbb{R}^2$ is not a manifold (not even C^0-mfd) because of the self-intersection:

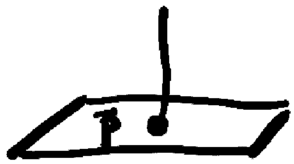


can not find a continuous function that is a diffeo. (can't def it so it has a continuous inverse)

<u>Ex</u> $M =$ 

"the lollipop"

Also not a manifold. Locally at $\vec{p}$:



can not find some open set in $\mathbb{R}^2$ or $\mathbb{R}$ that is diffeo to a nghd in M at $\vec{p}$.

Def: $S \subset \mathbb{R}^k$ (any subset; not necessarily open)

$f: S \to \mathbb{R}^n$

is of class C^r if $\exists\, U \overset{\text{open}}{\subset} \mathbb{R}^k : S \subset U$

and $g: U \to \mathbb{R}^k$ class C^r

$: g|_S = f$

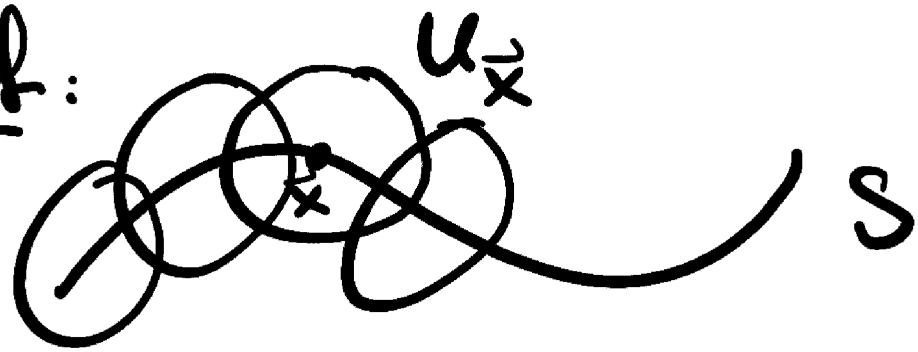
Lma: $S \subset \mathbb{R}^k$ subset. $f: S \to \mathbb{R}^k$ fcn

If $\forall x \in S\ \exists\, U_x \overset{\text{open}}{\subset} \mathbb{R}^k$ &

$g_x : U_x \to \mathbb{R}^n$ class C^r

$: g_x|_{U_x \cap S} = f$ then f is of class C^r.

Proof:



$\{U_{\vec{x}}\}_{\vec{x}\in S}$ open cover of $S\subset\mathbb{R}^k$.

Pick partition of unity $\{\varphi_{\vec{x}}\}_{\vec{x}\in S}$

and define

$$g:\bigcup_{\vec{x}\in S} U_{\vec{x}} \longrightarrow \mathbb{R}^n$$

$$g=\sum_{\vec{x}\in S}\varphi_{\vec{x}}\, g_{\vec{x}}$$ then g is

a fcn of class C^r def on

$$U:=\bigcup_{\vec{x}\in S} U_{\vec{x}} \supset S$$

$\square$

Def: Let $\mathbb{H}^k:=\{(x_1,\dots,x_k)\mid x_k\geq 0\}$

upper half-space in $\mathbb{R}^k$

$\partial\mathbb{H}^k=\{(x_1,\dots,x_k)\mid x_k=0\}$.

$\mathbb{H}^k_+ := \{(x_1, \dots, x_k) \mid x_k > 0\}$.

<u>Lma</u>: Let $U \overset{\text{open}}{\subset} \mathbb{H}^k$ and assume U is not open in $\mathbb{R}^k$.

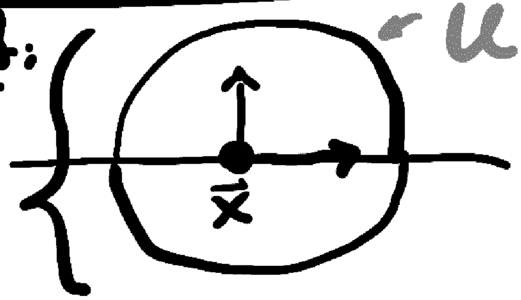
$\alpha : U \to \mathbb{R}^n$ Class C^r.

Let $\alpha' : U' \to \mathbb{R}^n$ be a C^r extension of α, $U' \overset{\text{open}}{\subset} \mathbb{R}^k$.

Then $D\alpha'(\vec{x})$ only depends on α & is independent of extension α'.

(Therefore we can henceforth write $D\alpha(\vec{x})$ without ambiguity)

<u>Proof</u>: U'



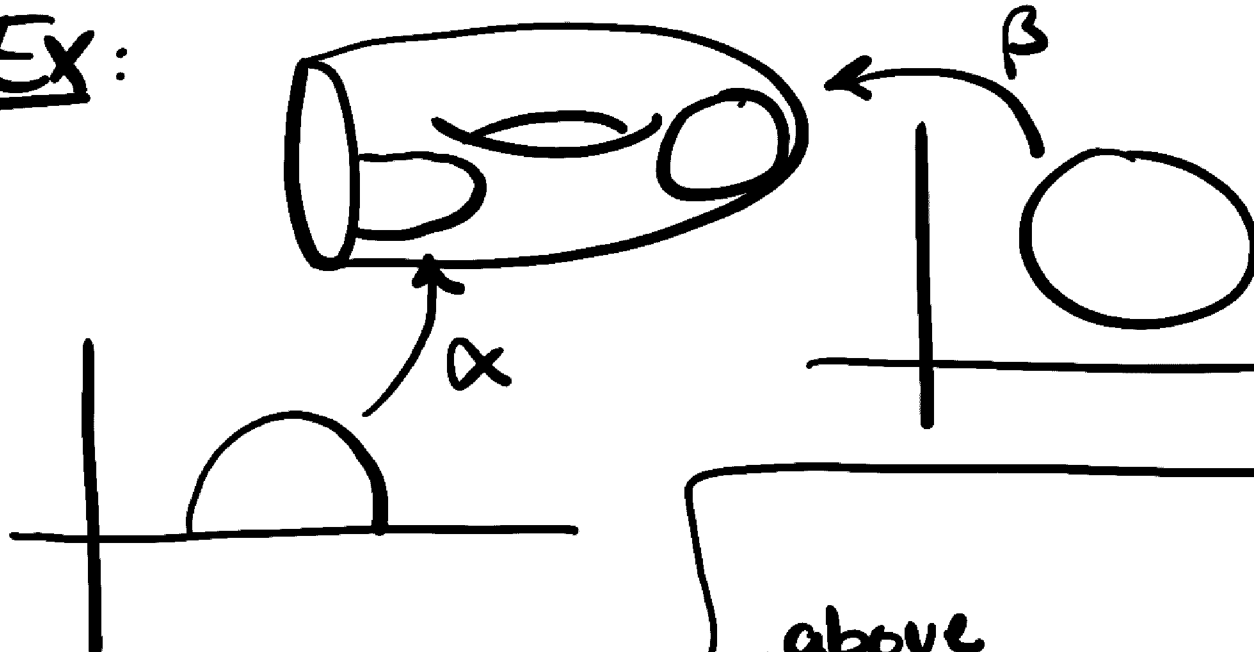
Only pt that needs to be checked is $\vec{x} \in \partial\mathbb{H}^k$

$$\frac{\partial \alpha}{\partial x_k} = \lim_{h \to 0} \frac{\alpha(\vec{x} + he_k) - \alpha(\vec{x})}{h}$$

suffices to calculate limit by $h \searrow 0$ $\square$

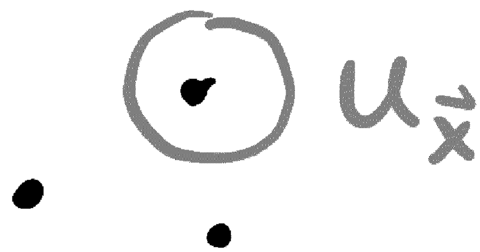
<u>Def</u>: Let $k > 0$. A <u>smooth manifold with boundary</u> is a subset $M \subset \mathbb{R}^n$: $\forall p \in M$ $\exists\, U$ open in either $\boxed{\mathbb{R}^k \text{ or } \mathbb{H}^k}$, $V \overset{\text{open}}{\subset} M$ & a diffeo $\alpha : U \to V$.

<u>Ex</u>:



<u>Def</u>: We extend the above def to $k=0$ by defining a 0-mfd to be a discrete collection of points in $\mathbb{R}^n$.

__Discrete__ means $\forall \vec{x} \in M$ $\exists U_{\vec{x}} \overset{\text{open}}{\subset} \mathbb{R}^n : U_{\vec{x}} \cap M = \{\vec{x}\}$



__Lma__: M mfd in $\mathbb{R}^n$, $\alpha: U \to V$ coord chart at $\vec{p} \in M$. If $U_0 \overset{\text{open}}{\subset} U$ then $\alpha|_{U_0} : U_0 \to V_0$ $(V_0 := \alpha|_{U_0}(U_0))$ is also a coord chart.

__Proof__: $\alpha: U \to V$ diffeo means

$\alpha^{-1}: V \to U$ smooth

so $V_0 = \alpha|_{U_0}(U_0)$ is open since

$V_0 = [\alpha^{-1}]^{-1}(U_0)$, and U_0 open

$\Rightarrow V_0$ open.

Restrictions preserve $\overset{\text{inj}}{+ C^\infty}$ $\square$

Coordinate charts have "smooth overlaps".

<u>Thm</u>: M k-mfd in $\mathbb{R}^n$.

$\alpha_0 : U_0 \to V_0$
$\alpha_1 : U_1 \to V_1$

coordinate charts on M (maybe at different points)

and assume $V_0 \cap V_1 \neq \emptyset$

Let $W_i := \alpha_i^{-1}(W)$. Then

$\alpha_1^{-1} \circ \alpha_0 : W_0 \to W_1$ is a diffeo